Elastic scattering of electrons by helium at intermediate and high energies

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(Received 19 December 1978; revised manuscript received 16 January 1980)

A detailed study of the differential cross section for the elastic scattering of electrons by helium at intermediate and high energies has been made. In this study a theoretical procedure, recently suggested by one of the authors (J.N.D.), has been followed. The calculated differential cross sections in the energy range 100–200 eV appear to be in very good agreement with experiment and better than the existing theoretical results. At higher energies, the present results are also better except at very small angles where the eikonal-Born-series results of Byron and Joachain are better. The scattering amplitudes obtained during this calculation have also been presented.

I. INTRODUCTION

For the theoretical computation of the differential cross sections for the elastic scattering of electrons by atoms at intermediate and high energies there exist, at present, some methods[1-5] which are considered to be quite successful. Of these methods the extended polarization potential method of Labahn and Callaway[1] (LC) and the eikonal Born series (EBS) method of Byron and Joachain[2] are particularly noteworthy. Basically the extended polarization potential method is a variation of the method of polarized orbitals, which was initiated as a low-energy method, and it applies best at low and intermediate energies. The EBS method, on the contrary, is basically a high-energy method. It projects an inverse power series in energy. Calculation shows that for the electron-helium elastic scattering the EBS results are very good for energies $E>400$ eV. The results for $200<E<400$ eV are also good. But the results for $E<200$ eV are not so good. For still lower energies (for example, <100 eV) the results are very bad. On the other hand the LC method gives quite good results in the energy range 50–200 eV except at large angles. At higher energies the results are bad both at small and at large angles. The most unfavorable point regarding this method is that it is too complicated.

Recently a simple computational procedure has been suggested by Das.[6] It is not more difficult than the second Born calculation and yields very good results for electron-helium elastic scattering at 100 eV, an intermediate energy. It may be noted in this connection that the success of a similar method for potential scattering was previously observed by Das[7] [see also Figs. 1(a)–1(c) of this paper] when calculations were done for some typical potentials for which exact results are known[22] through other means. In this paper we propose to study in detail the elastic scattering of electrons by helium at various intermediate and high energies. This will help us in having an assessment of the above computational method. It is also expected to give a broad view of the accuracy of various experimental and theoretical results. In Sec. II of this paper we include a brief description of the theoretical method of Das, which is followed in the present study. In Sec. III we present the computed results. There we have also compared our results with other theoretical and experimental results in different tables and graphs. This is followed by a discussion in Sec. IV. Finally in Sec. V we have made some concluding remarks.

II. COMPUTATIONAL METHOD OF DAS

In this section we describe the computational method of Das[6,7] in some more details. In Sec. II A we review the potential scattering problem and in Sec. II B we outline the corresponding method for the scattering of electrons by atoms, especially for the elastic collisions.

A. The potential scattering problem

We begin by pointing out that in the method of Das one treats the problem in the momentum space. This appears to be quite appropriate. In momentum space one has a function $f(\vec{p},\vec{k}_i)$, the off-shell "amplitude" which satisfies the Fredholm integral equation[7]

$$f(\vec{p},\vec{k}_i)=f_B(\vec{p}-\vec{k}_i)+\frac{1}{2\pi^2}\int\frac{f_B(\vec{p}-\vec{q})f(\vec{q},\vec{k}_i)}{q^2-k^2-i\eta}d^3q\,. \qquad (1)$$

The scattering amplitude $f(\vec{k}_f,\vec{k}_i)$ is obtained from $f(\vec{p},\vec{k}_i)$ by replacing $\vec{p}$ by $\vec{k}_f$.

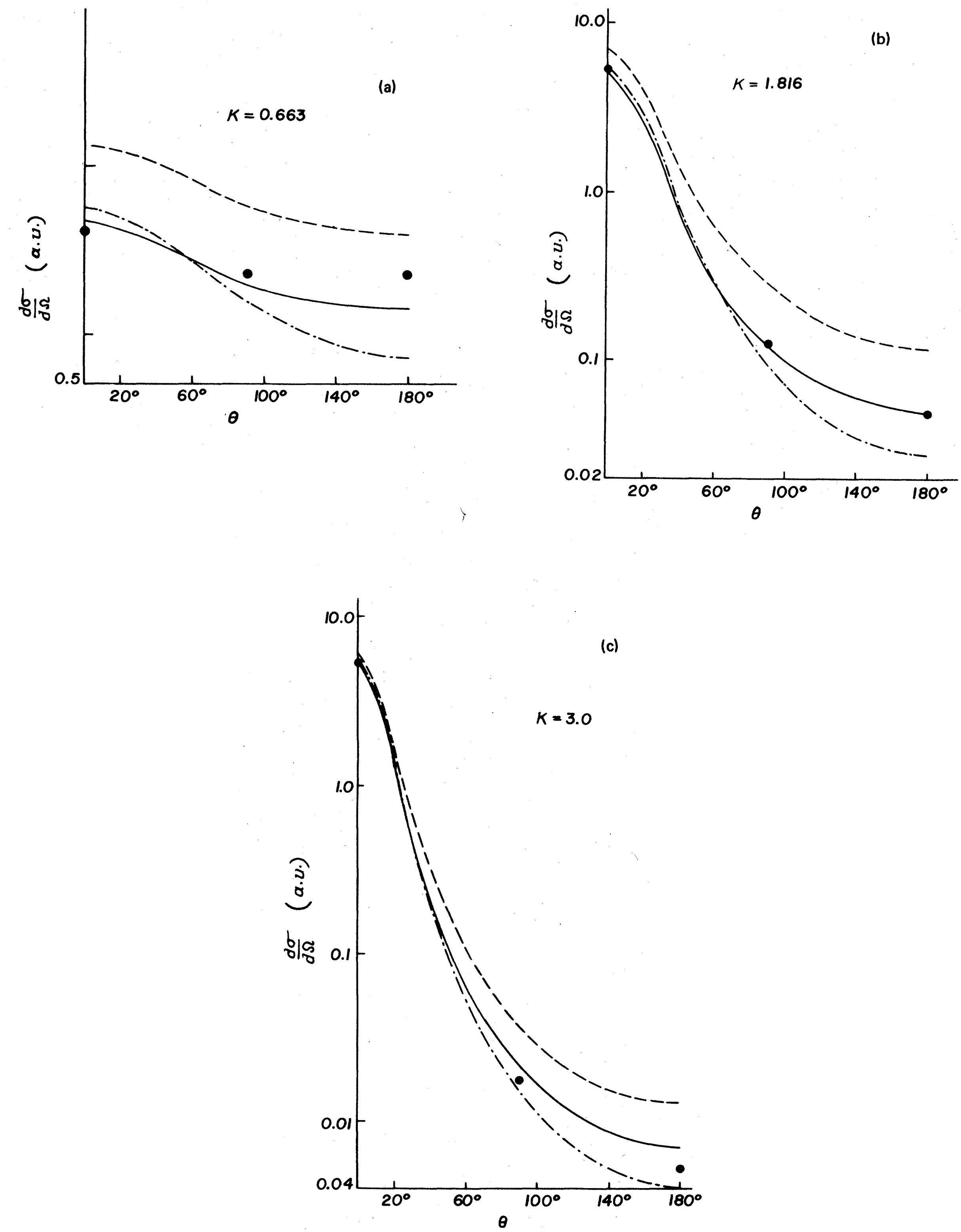

FIG. 1. Comparison of the differential cross sections (a.u.) obtained by various calculational methods for scattering by the potential $U=-2.365\,e^{-r}/r$, —— Das (Ref. 7), -·-·- first Born, ---- second Born, • exact results (Ref. 17).

Now to solve for the scattering amplitude one has to solve Eq. (1) first. To do this one may choose some trial function $f_t(\vec{p}, \vec{k}_i)$ in some suitable function space $\mathfrak{F}$ for the right-hand side of Eq. (1). The resultant value of the right-hand side is considered as output. In general the input trial function contains some parameters. These parameters are estimated by making the "distance" between the input and the output functions a minimum. In defining the distance between two elements in the function space $\mathfrak{F}$ one may use an L_2 norm.

Thus the distance

$$\mathrm{dis}(f^{(\text{out})}, f^{(\text{in})}) \equiv \| f^{(\text{out})} - f^{(\text{in})} \| = \left(\int | f^{(\text{out})} - f^{(\text{in})} |^2 d^3p \right)^{1/2} . \quad (2)$$

Two very interesting properties of the functions $f \in \mathfrak{F}$ are[7]

$$f(\vec{p}, \vec{k}_i) \underset{p\to\infty}{\sim} c(k) f_B(\vec{p} - \vec{k}_i) \quad (3a)$$

and

$$f(\vec{p}, \vec{k}_i) \underset{p\to 0}{\sim} d(k) f_B(\vec{k}_i) , \quad (3b)$$

where $c(k)$ and $d(k)$ are some energy-dependent complex quantities.

All these suggest that for the simplest computation following Das one may choose

$$f_t(\vec{p}, \vec{k}_i) \equiv f^{(\text{in})}(\vec{p}, \vec{k}_i) = [a(k) + ib(k)] f_B(\vec{p} - \vec{k}_i) . \quad (4)$$

At this point another simplification is in order. Both the input and the output functions vanish as fast as $f_B(\vec{p})$ does when $p \to \infty$ and that we are ultimately interested in the mass-shell values of the output function (e.g., for $\vec{p} = \vec{k}_f$). So as a measure of the distance between the input and the output functions one may fix $\vec{p} = \vec{k}_f$ and integrate over the directions of $\vec{k}_f$ only Thus in place of (2) one may choose as an approximation for the norm the following:

$$\| f^{(\text{out})} - f^{(\text{in})} \| \simeq \left(\int | f^{(\text{out})}(\vec{k}_f, \vec{k}_i) - f^{(\text{in})}(\vec{k}_f, \vec{k}_i) |^2 d\Omega_f \right)^{1/2} . \quad (5)$$

The calculation may further be simplified by replacing the integral in (5) by suitable sums. This procedure has been followed in calculating the scattering amplitude in Ref. 7 for two particular potentials. One of these is the Yukawa potential $-2.365 e^{-r}/r$ and the other is the Gaussian potential $-e^{-r^2}$. The results obtained appear to be highly encouraging. For the shorter-range Gaussian potential the results seem to be very good. In the calculation of Das of Ref. 7 only three points were used in the sum corresponding to the integral (5). With the use of additional points the results improve a little bit more. Results of such a calculation with 18 points for the sum are presented here in Figs. 1(a)–1(c) for the potential $U(r) = -2.365 e^{-r}/r$. The figures are for three typical values of the momentum $k = 0.663$, $k = 1.816$, and $k = 3.0$, representing a small, medium, and a high value and for which exact results are known.[22] Comparison with the first and second Born results and the exact results show that our results for the differential cross sections are really good and much better than the second Born results, particularly for the intermediate momentum value $k = 1.816$.

The calculation may still be improved by taking a more general expression for the input trial function

$$f_t(\vec{p}, \vec{k}_i) = (a + ib) f_B(\vec{p} - \vec{k}_i) + \sum_r \alpha_r \varphi_r(\vec{p}, \vec{k}_i) , \quad (6)$$

where $\alpha_r = a_r + ib_r$ and $a + ib$ are complex energy-dependent parameters and φ_r's are suitable known functions. The parameters may again be estimated by minimizing the distance between the input and the output functions. Finally, the output function with the above estimated parameters is to be taken to represent the scattering amplitude.

B. The electron-atom collision problem

The scattering of electrons by an atom may be described in terms of two scattering amplitudes. One of these is the direct scattering amplitude f_d and the other is the exchange amplitude f_{ex}. Now for scattering at high or intermediate energies the effect of exchange is small. The exchange amplitude is usually calculated in the Ochkur approximation.[8] The problem of determining accurately the cross sections then reduces to obtain the direct amplitude with reasonable accuracy. This may be simply obtained following the method of Das. We now outline the method of Das for this case.

The direct scattering amplitude may be written as

$$f_d = 2\pi^2 T_{(d)} , \quad (7)$$

where

$$T_{(d)} = \langle \Phi_f | V | \Psi_i^+ \rangle , \quad (8)$$

and where $|\Phi_f\rangle$ is the free wave function for the final state, V is the interaction potential, and $|\Psi_i^+\rangle$ is the scattering-state wave function which approaches asymptotically the free wave function of the initial state. The scattering state $|\Psi_i^+\rangle$ satisfies the integral equation

$$|\Psi_i^+\rangle = |\Phi_i\rangle + \frac{1}{E - H_0 + i\eta} V |\Psi_i^+\rangle . \qquad (9)$$

Now multiplying this equation by V from the left and then taking the scalar product with $|\Phi_n\rangle$, an arbitrary state of H_0, we get

$$\langle\Phi_n|V|\Psi_i^+\rangle = \langle\Phi_n|V|\Phi_i\rangle + \left\langle\Phi_n\right| V \frac{1}{E - H_0 + i\eta} V \left|\Psi_i^+\right\rangle . \qquad (10)$$

From this it follows that

$$T_{(d)ni} = T^B_{(d)ni} + \sum_I \int d^3p_I\, T^B_{(d)nI} \frac{1}{E - E_I + i\eta} T_{(d)Ii} , \qquad (11)$$

where $T_{(d)n\,m}$ stands in general for the off-shell T-matrix element for the transition (direct) from the state "m" to the state "n" and $T^B_{(d)n\,m}$, the corresponding Born term. Equation (11) is an infinite set of coupled integral equations for the relevant T-matrix elements. Writing $|\vec{p}_n, \xi_n\rangle$ to designate the state n more explicitly in which $\vec{p}_n$ stands for the momentum of the free electron and ξ_n stands for the bound-state energy of the atom, we have

$$T_{(d)ni}(\vec{p}_n - \vec{p}_i\,;\, \xi_n, \xi_i) = T^B_{(d)ni}(\vec{p}_n - \vec{p}_i\,;\, \xi_n, \xi_i) + \sum_I \int d^3p_I\, T^B_{(d)nI}(\vec{p}_n - \vec{p}_I\,;\, \xi_n, \xi_I) \frac{1}{E - p_I^2/2 - \xi_I + i\eta} T_{(d)Ii}(\vec{p}_I - \vec{p}_i\,,\, \xi_I, \xi_i) . \qquad (12)$$

Equation (12) is the generalization, for the electron-atom collision, of Eq. (1) of the potential scattering case. In this case also one may try to solve the equation approximately following a procedure similar to the one described in Sec. II A. Thus one may start by choosing a trial set of T-matrix elements in some suitable function space $\mathfrak{F}$. With these trial input functions one then calculates the output functions from the right-hand side of Eq. (12). As usual the trial input functions contain some variational parameters. These are then estimated by minimizing the distance between the input T-matrix elements and the output T-matrix elements, where by distance we may mean the L_2 norm

$$\| T^{(\mathrm{out})}_{(d)} - T^{(\mathrm{in})}_{(d)} \| = \left(\sum_n \int | T^{(\mathrm{out})}_{(d)ni} - T^{(\mathrm{in})}_{(d)ni} |^2 d^3p_n \right)^{1/2} . \qquad (13)$$

Let us now consider the electron-helium elastic-scattering problem as an application. The asymptotic behavior of the relevant T-matrix element may be obtained by making p_n very large and noting that

$$T^B_{(d)nI}(\vec{p}_n - \vec{p}_I\,;\, \xi_n, \xi_I)$$

vanishes faster than

$$T^B_{(d)ni}(\vec{p}_n - \vec{p}_i\,;\, \xi_n, \xi_i)$$

for all $I \neq i$. One obtains in this way

$$T_{(d)ni}(\vec{p}_n - \vec{p}_i\,;\, \xi_n, \xi_i) \sim [a'(E) + ib'(E)]\, T^B_{(d)ni} , \qquad (14a)$$

where

$$a' + ib' = 1 + \int d^3p_I \frac{T_{(d)Ii}(\vec{p}_I - \vec{p}_i\,;\, \xi_i, \xi_i)}{E - p_I^2/2 - \xi_i + i\eta} . \qquad (14b)$$

Thus the simplest input trial functions one may choose are

$$T_{(d)ni} = [a(E) + ib(E)]\, T^B_{(d)ni} . \qquad (15)$$

With this choice of the input T-matrix elements, one obtains the input and the output scattering amplitudes as

$$f^{(\mathrm{in})}_{(d)} = af_{B_1} + ibf_{B_1} , \qquad (16a)$$

and

$$f^{(\mathrm{out})}_{(d)} = (f_{B_1} + af_{B_2R} - bf_{B_2I}) + i(af_{B_2I} + bf_{B_2R}) , \qquad (16b)$$

where f_{B_1} is the first Born (direct) and $f_{B_1} + f_{B_2}$ is the second Born (direct) on-shell amplitudes, f_{B_2R} and f_{B_2I} being the real and imaginary parts of f_{B_2}.

As in the case of potential scattering we may take as an approximate measure of the distance between the input and the output functions the quantity

$$\left(\int | f^{(\mathrm{out})}_d - f^{(\mathrm{in})}_d |^2 d\Omega_f \right)^{1/2} ,$$

in which the integration is taken over the scattering angles. To make the calculation still simpler one may replace the integral by a sum and put the minimization problem as

$$\sum \sin\theta_j \; | f^{(\mathrm{out})}_d - f^{(\mathrm{in})}_d |^2_{\theta=\theta_j} = \min . \qquad (17)$$

After solving this minimization problem one obtains the minimizing values of a and b which we denote by $a_{\min}$ and $b_{\min}$. One may then take

$$f_d=(f_{B_1}+a_{\min}f_{B_2R}-b_{\min}f_{B_2I})+i(a_{\min}f_{B_2I}+b_{\min}f_{B_2R}) \quad (18)$$

as the final result for the direct scattering amplitude. The differential cross section is then $|f_d-f_{ex}|^2$. The calculation may be further improved by making a better choice of the initial trial function. An order of accuracy of the calculation is also obtained from the minimized value of $\| f_d^{(out)}-f_d^{(in)}\|^2$.

Similar steps may be followed in the inelastic collision cases and suitable initial trial functions may be obtained from the asymptotic behavior of the equations corresponding to Eq. (12).

III. NUMERICAL COMPUTATION: RESULTS

In a computation in electron-helium elastic scattering, which follows the method of Das, the first thing to be done is to obtain the first and second Born results in simple forms. So we use for the wave function of the ground state of the helium atom the same simple form as was used by Byron and Joachain,[2] together with their chosen value for Δ (the mean excitation energy), viz., $\Delta=1.3$ a.u. We also take for the exchange amplitude the Ochkur approximation result. Consequently our first Born and the simplified second Born results for the direct scattering amplitude and the result for the exchange amplitude are the same as those of Byron and Joachain.[2] We prefer again with Byron and Joachain,[2] to compute the second-order terms by reducing these to integrals over single Feynman parameters and finally evaluating these integrals numerically. We used 64-point Gauss quadrature for this purpose. We have compared the results for integrals with those of Byron and Joachain for 300 eV and find good agreement in general. The small differences which are observed are estimated not to be significant. In our present study we consider an energy range of 50–700 eV and an angular range of 0–180°. We have estimated the variational parameters a and b from Eq. (17), the summation having been evaluated for angles ranging from 0 to 180° at 5° intervals. It was estimated that if one uses more points, the results will not significantly change. With these estimated values of the parameters, the direct part of the scattering amplitude is evaluated from Eq. (18). The calculated results for the differential cross sections are presented in Tables I and II, and Figs. 2 and 3. In Table I we present the differential cross sections for energies of 100, 200, 300, and 500 eV and for several scattering angles. In Table II we compare our results with various theoretical and experimental results. We have also included in this table the results for first Born, first Born with exchange included, and the second Born (simplified) results to give an idea how the present scheme improves upon these first and second Born results at different energies. Accuracy of the differential cross section does not necessarily imply accuracy of the scattering amplitude, so we have presented our computed scattering amplitude and compared it with those of first and second Born calculations in Table III. These results may be useful to other workers in many connections. In this table we have also included the exchange amplitude to present a detailed idea about the importance of exchange effects at different energies and different scattering angles. We have also studied the effect of the small variation of the mean excitation energy Δ about the adopted value 1.3 a.u. and find only a very few percents variation as is evident from Table IV. In Table V, the dependence of $a_{\min}$ and $b_{\min}$ on energy E is displayed. We have also compared our results with various theoretical and experimental results in several graphs. In Fig. 2, the variation of the differential cross section with angles has been compared for several energies ranging from 100 to 500 eV. Finally, dependence of the differential cross sections on energy, for fixed scattering angles, has been compared in Figs. 3(a)–3(c).

TABLE I. Differential cross sections (10^{-16} cm^2/sr) for the elastic scattering of electrons by helium in the energy range 100–500 eV.

θ (deg) \ E (eV)	100	200	300	500
5	0.589	0.354	0.266	0.197
10	0.421	0.225	0.171	0.134
20	0.225	0.124	0.093 3	0.061 1
30	0.141	0.0748	0.048 9	0.025 7
40	0.0966	0.0441	0.025 4	0.011 5
50	0.0675	0.0263	0.013 9	0.005 82
60	0.0475	0.0164	0.008 25	0.003 29
70	0.0341	0.0108	0.005 25	0.002 05
80	0.0251	0.0076	0.003 60	0.001 38
90	0.0192	0.0056	0.002 63	0.000 99
120	0.0106	0.0029	0.001 35	0.000 50
150	0.0077	0.0021	0.000 95	0.000 34
180	0.0070	0.0019	0.000 85	0.000 30

IV. COMPARISON WITH AVAILABLE RESULTS

The comparison that is made in Table II and Fig. 2 between our computed differential cross section for the electron-helium elastic scattering and those of other theories and experiments shows that our results are quite good over the whole energy range

TABLE II. Comparison of the theoretical and experimental results for differential cross sections (10^{-16} cm^2/sr). The symbols of these tables correspond to the theoretical and experimental results as follows—Theoretical: B_1—first Born, BE—first Born with exchange included, B_2—second Born with exchange included, BJ—Byron and Joachain, Pr—present calculation. Experimental: J—Jansen *et al.* (Ref. 10), H—Hughes *et al.* (Ref. 19), V—Vriens *et al.* (Ref. 18), B—Bromberg (Ref. 11), KV—Kurepa and Vušković (Ref. 15), CR—Crooks and Rudd (Ref. 14), JFH—Jost *et al.* (Ref. 13), SRG—Sethuraman *et al.* (Ref. 16), MP—McConkey and Preston (Ref. 9), RTS—Register *et al.* (Ref. 20), O—Oda *et al.* (Ref. 17), W—Williams (Ref. 21).

A. For the elastic scattering of electrons by helium for intermediate energies.

Energy (eV)	Scattering angle (deg)	Theory B_1	BE	B_2	LC	BJ	Pr	Experiment[a] J	H	V†	B	W	KV†	CR	JFH	SRG	MP	RTS
100	10	0.163	0.294	1.004	0.448	0.767	0.421	0.476		0.484			0.497	0.682	0.610			0.487
	30	0.099	0.174	0.358	0.154	0.201	0.141	0.154	0.108	0.139		0.151	0.149	0.197		0.157	0.148	0.145
	50	0.049	0.082	0.159	0.067	0.065	0.068	0.060				0.064	0.060	0.075	0.069		0.043	0.055
	90	0.014	0.021	0.051		0.016	0.019		0.012			0.026	0.019	0.022	0.019	0.016	0.011	0.017
	120	0.0071	0.010	0.030		0.010	0.011		0.0086				0.012		0.011	0.012		0.011
	150	0.0049	0.0069	0.022		0.0082	0.0077		0.0075				0.0097	0.011		0.010		0.010
200	10	0.152	0.208	0.464	0.314	0.375	0.225	0.302	0.325		0.314		0.345	0.539	0.420			
	30	0.063	0.084	0.133	0.070	0.081	0.075	0.079	0.060		0.077	0.068	0.082	0.091		0.074		
	50	0.022	0.028	0.050	0.028	0.025	0.026	0.025	0.018		0.025	0.026	0.027	0.029				
	90	0.0044	0.0052	0.0126		0.0055	0.0056		0.0050				0.0059	0.0065	0.0056	0.0049		
	120	0.0021	0.0024	0.0070		0.0032	0.0029		0.0030				0.0034		0.0028	0.0027		
	150	0.0014	0.0016	0.0049		0.0024	0.0021		0.0025				0.0026	0.0024		0.0017		

B. For the elastic scattering of electrons by helium for high energies.

Energy (eV)	Scattering angle (deg)	Theory B_1	BE	B_2	LC	BJ	Pr	Experiment J	H	V†	B	JFH	SRG	O
300	10	0.142	0.176	0.302	0.260	0.239	0.171	0.220		0.217	0.227	0.319		
	30	0.043	0.052	0.076	0.046	0.049	0.049	0.047		0.037	0.049	0.061	0.042	
	50	0.012	0.014	0.023	0.010	0.013	0.014	0.013			0.013	0.015	0.012	
	90	0.0021	0.0023	0.0053			0.0026						0.0025	
	120	0.0010	0.0011	0.0028			0.0014						0.0011	
	150	0.0006	0.0007	0.0020			0.0010						0.0008	
500	10	0.125	0.142	0.191		0.160	0.134	0.154	0.120		0.160	0.190		0.160
	30	0.024	0.026	0.036		0.026	0.026	0.026	0.022		0.026	0.028		0.026
	50	0.0053	0.0057	0.0092		0.0056	0.0058	0.0057	0.0057		0.0057	0.0059	0.0049	0.0057
	90	0.00079	0.00083	0.00176			0.00099		0.0006				0.00070	0.00087
	120	0.00035	0.00036	0.00087			0.00050		0.0005				0.00026	0.00035
	150	0.00024	0.00024	0.00062			0.00034						0.00018	

[a]The † indicates that the data have been renormalized to the data of Chamberlain *et al.* (Ref. 12) at 5°, except the 200 eV results of Kurepa and Vušković.

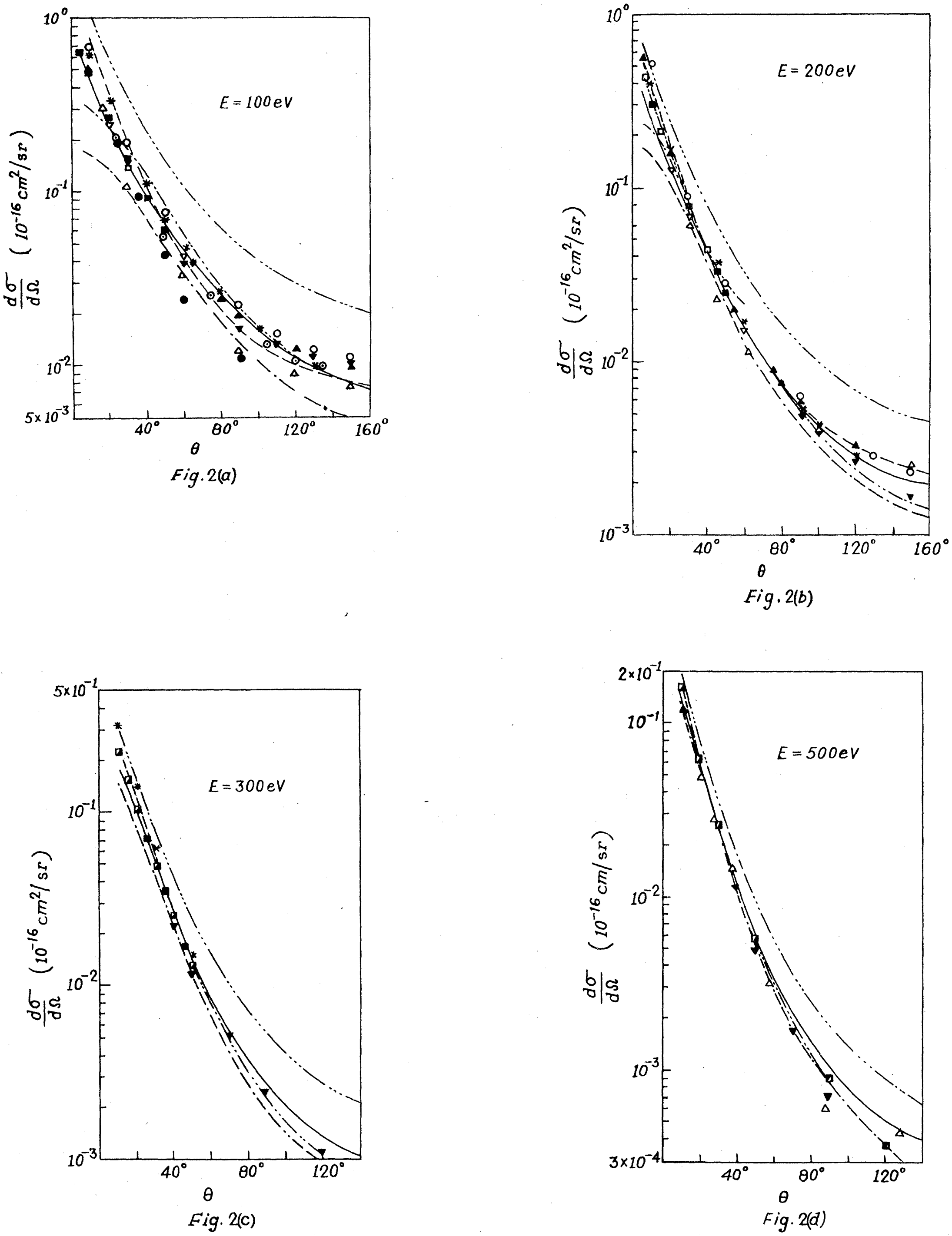

FIG. 2. Comparison of the calculated and observed differential cross section (10^{-16} cm^2/sr) for the elastic scattering of electrons by helium. Theoretical: Curves—first Born without exchange —•—•; first Born with exchange —••—••—; second Born with exchange —•••—•••; Labahn and Callaway (Ref. 1) (1969) —•—•—•; Byron and Joachain (Ref. 2) (1973) ----; present calculation ——. Experimental: ■ Jansen *et al.* (Ref. 10) (1976); △ Hughes *et al.* (Ref. 19) (1932); □ Chamberlain *et al.* (Ref. 12) (1970); ▽ Williams (Ref. 21) (1969); ◪ Bromberg (Ref. 11) (1969, 1974); ○ Crooks and Rudd (Ref. 14) (1972); ● McConkey and Preston (Ref. 9) (1975); ▼ Sethuraman *et al.* (Ref. 16) (1974); ▲ Kurepa and Vusković (Ref. 15) (1975); ◩ Oda *et al.* (Ref. 17) (1972); * Jost *et al.* (Ref. 13) (1973); ⊙ Register *et al.* (Ref. 20) (1979).

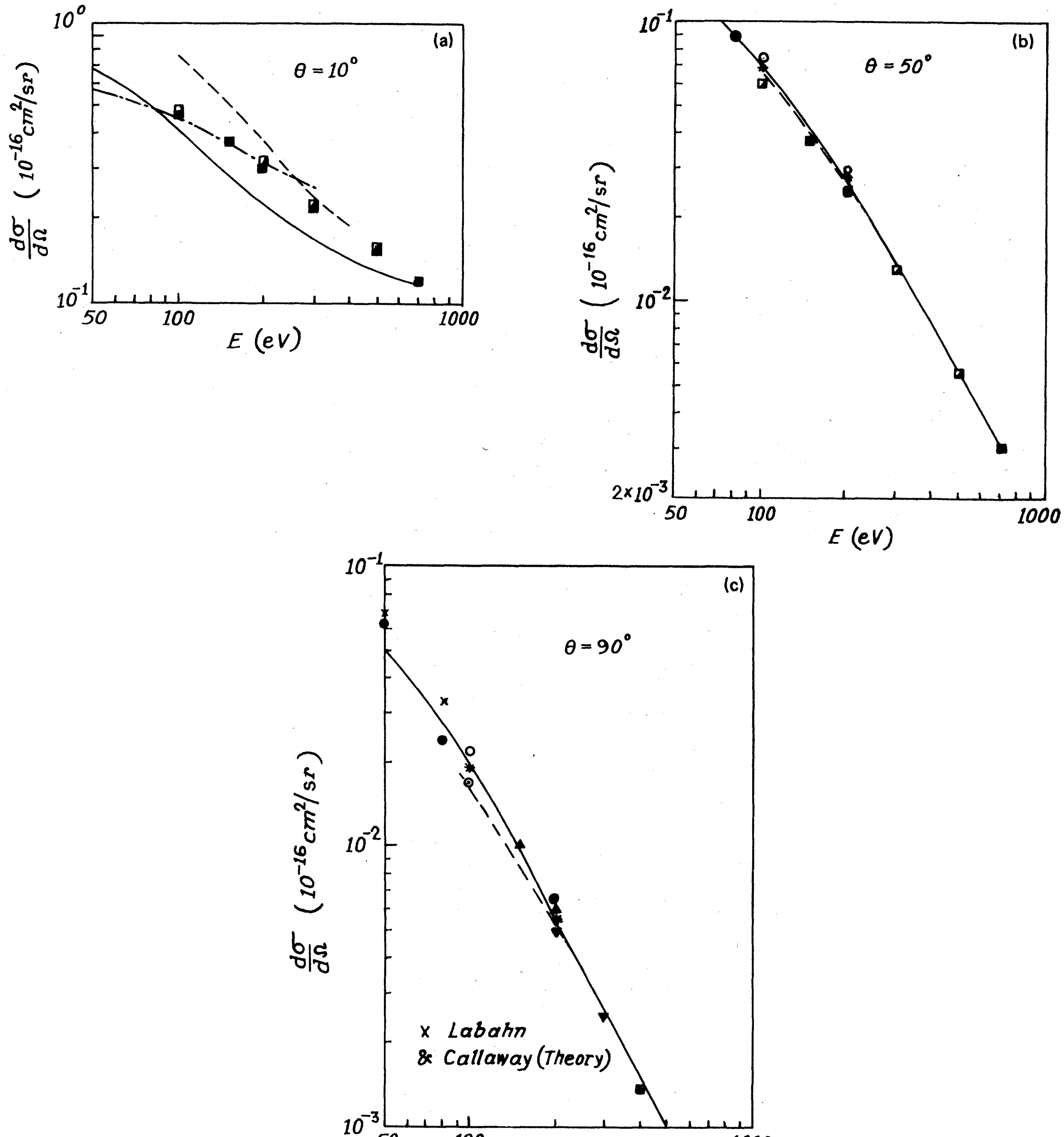

FIG. 3. Variation of the calculated and observed differential cross section (10^{-16} cm^2/sr) with energy for the elastic scattering of electrons by helium. For identification of the theoretical curves and the experimental points see the figure caption under Fig. 2.

considered, viz., 50–700 eV. Now the energy range 50–200 eV may be considered as an intermediate-energy range and the range 200–700 eV may be considered as a high-energy range.

In the intermediate-energy range, defined as above, our results are excellent except at the lower end, e.g., 50 eV, where small discrepancies are observed between our results and those of the theoretical results of Labahn and Callaway and the experimental results of McConkey and Preston.[9] We do not expect our results for the differential cross section for such a low energy as 50 eV to be very good. Because at this energy the exchange effect is quite considerable, and the

TABLE III. The scattering amplitude (in a.u.) for the elastic scattering of electrons by helium for intermidiate and high energies.

	100 eV				300 eV			
θ (deg)	First Born (direct) f_{B_1}	Exchange f_{ex}	Second Born (direct) $f_{B_1}+f_{B_2}$	Present (direct)	First Born (direct) f_{B_1}	Exchange f_{ex}	Second Born (direct) $f_{B_1}+f_{B_2}$	Present (direct)
5	0.7849	−0.2691	1.6683 + 0.9127i	0.7374 + 1.249i	0.7706	−0.0878	1.0493 + 0.6492i	0.7102 + 0.6671i
10	0.7637	−0.2605	1.442 + 0.8322i	0.6675 + 1.029i	0.7125	−0.0799	0.8555 + 0.4516i	0.6494 + 0.4354i
20	0.6900	−0.2305	1.084 + 0.6161i	0.5937 + 0.6686i	0.5455	−0.0577	0.6233 + 0.2448i	0.5270 + 0.2361i
30	0.5950	−0.1924	0.8510 + 0.4372i	0.5456 + 0.4542i	0.3931	−0.0380	0.4546 + 0.1732i	0.3815 + 0.1707i
50	0.4170	−0.1231	0.5822 + 0.2629i	0.4014 + 0.2828i	0.2037	−0.0162	0.2451 + 0.1244i	0.1870 + 0.1211i
70	0.2948	−0.0781	0.4288 + 0.2070i	0.2658 + 0.2261i	0.1261	−0.0077	0.1543 + 0.0990i	0.1000 + 0.0937i
90	0.2199	−0.0522	0.3332 + 0.1833i	0.1743 + 0.1955i	0.0864	−0.0043	0.1068 + 0.0817i	0.0605 + 0.0757i
120	0.1595	−0.0329	0.2514 + 0.1635i	0.1003 + 0.1666i	0.0590	−0.0023	0.0734 + 0.0658i	0.0354 + 0.0598i
150	0.1325	−0.0249	0.2129 + 0.1526i	0.0686 + 0.1508i	0.0478	−0.0016	0.0596 + 0.0578i	0.0259 + 0.0521i

	500 eV			
θ (deg)	First Born (direct) f_{B_1}	Exchange f_{ex}	Second Born (direct) $f_{B_1}+f_{B_2}$	Present (direct)
5	0.7569	−0.0515	0.8855 + 0.4710i	0.7020 + 0.4605i
10	0.6674	−0.0443	0.7311 + 0.2885i	0.6338 + 0.2752i
20	0.4510	−0.0272	0.4924 + 0.1559i	0.4420 + 0.1519i
30	0.2914	−0.0154	0.3233 + 0.1169i	0.2804 + 0.1144i
50	0.1376	−0.0053	0.1561 + 0.0822i	0.1209 + 0.0786i
70	0.0793	−0.0022	0.0908 + 0.0617i	0.0631 + 0.0578i
90	0.0533	−0.0012	0.0612 + 0.0487i	0.0389 + 0.0452i
120	0.0354	−0.0006	0.0408 + 0.0378i	0.0238 + 0.0347i
150	0.0290	−0.0004	0.0334 + 0.0326i	0.0181 + 0.0299i

Ochkur approximation result is not expected to be very good (however, we expect our results for the direct scattering amplitude to be sufficiently accurate). At 50-eV energy the exchange amplitude is $1\frac{1}{2}$ to 4 times larger in magnitude (but opposite in sign) compared to the real part of our computed direct amplitude but $1\frac{1}{2}$ to 3 times smaller in magnitude compared to the imaginary parts over the angular range considered. As a result we have the observed small discrepancies over the whole angular range. As the energy increases the exchange amplitude quickly falls off.

TABLE IV. Dependence of the differential cross section (a_0^2/sr) on Δ, the mean excitation energy ($E=200$ eV).

θ (deg.)	$\Delta=1.2$ (a.u.)	$\Delta=1.3$ (a.u.)
10	0.7784	0.8054
30	0.2661	0.2675
50	0.0932	0.0940
90	0.0197	0.0200
120	0.0103	0.0105

So that at 80-eV energy, the exchange amplitude becomes less in magnitude everywhere than the real part of the computed direct amplitude and becomes $2\frac{1}{2}$ to 4 times smaller in magnitude compared to the imaginary part. Consequently the results for the differential cross sections become excellent at 80 eV and for higher energies as well. Now in this intermediate-energy range our results agree closely with the theoretical results of Labahn and Callaway except at large angles. At large angles their results tend toward inaccuracy.

Let us next consider the behavior of our results for the differential cross section at high energies. Again comparison shows that in the high-energy range 200–700 eV our results agree nicely with the EBS results of Byron and Joachain,[2] except at very small angles. In this energy range the agreement of our results with the experimental results is

TABLE V. Dependence of vatiational parameters a_{min} and b_{min} on energy.

	100 eV	200 eV	300 eV	500 eV
a_{min}	0.5334	0.7145	0.7850	0.8527
b_{min}	0.8632	0.6627	0.5652	0.4586

again excellent except at small angles. At small angles our results are several precents less and the EBS results are several percents greater compared to the experimental results. At the upper end of this energy range the EBS results are better at small angles compared to ours but at large angles our results appear to be better. Regarding the accuracies of the various experimental results we have the feeling that the following groups of experimental results are likely to be most reliable: (i) all the results of Jansen *et al.*,[10] (ii) the results of McConkey and Preston,[9] except their large angle results for 100-eV energy, (iii) the results of Bromberg,[11] and of (iv) Chamberlain *et al.*,[12] (v) the large-angle results of Jost *et al.*[13] and of (vi) Crooks and Rudd,[14] (vii) the large-angle results of Kurepa and Vusković[15] except their 100-eV results, (viii) the high-energy results of Oda *et al.*[17] and of (ix) Sethuraman *et al.*,[16] and the most recent measurements of Register *et al.*[20] for 100-eV energy. Some of the other experimental results, perhaps, have large errors.

We now consider the curves in Fig. 3 which represent the variation of the cross section with energy for fixed scattering angle and show that the results of Byron and Joachain are expected to be very poor for energy less than 100 eV. The results of Labahn and Callaway, on the other hand, appear to be poor only at high energies (say for energy >300 eV). Our results appear to be good both at very high energies (say for energy >500 eV) and for intermediate energies (say <150 eV). In between these ranges our 10° results are poor. The curves, representing large angle (say ≥50°) scattering cross sections, show that our results are excellent for all energies >50 eV. At lower energies there are some errors. For these large scattering angles the results of Byron and Joachain are bad for $E = 100$ eV and expected to be very bad for $E < 100$ eV, where as the results of Labahn and Callaway are bad only for high energies.

In the present context let us see the behavior of the first and second Born (simplified) results for different energies. Comparison shows that the first Born differential cross sections are bad even at 300 eV, especially at small angles. Inclusion of exchange improves the results very much. In fact, the first Born results with exchange included are very good for energies $E > 200$ eV, except at small angles. The second Born (simplified) differential cross sections are bad even at 500 eV. However, the second Born results systematically improve as the energy increases. The behavior of the present computed results in relation to first Born and second Born results are more or less the same as previously observed by Das[7] for potential scattering. The main difference is in the value of wave number k for which results behave similarly regarding accuracies (compare Table II of Ref. 7 with Table III of the present article).

Now we look at Table III for the scattering amplitude. For energies of 50 or 100 eV, the present results differ widely from the first Born and second Born amplitudes. For example, at a scattering angle of 5° and energy 100 eV, the direct scattering amplitude for the first Born, second Born, and the present calculations are 0.785, $1.67 + 0.913i$, and $0.737 + 1.25i$ a.u., respectively. So the real part of the present result is approximately the same as the first Born result but less than half of the real part of the second Born result. The imaginary part of our result is approximately one and half times that of the second Born result. For larger angles the same is nearly true for the real parts, but for imaginary parts the differences between our results and those of second Born results become smaller. At higher energies, say 300 eV, the first Born and second Born amplitudes still appear to be very inaccurate. At these energies the present results differ from the corresponding first Born results by phases larger than $\frac{1}{4}\pi$ and from the corresponding second Born results, moduli larger than 50%. Even at an energy of 500 eV, the present results differ from first Born results by as large as $\frac{1}{4}\pi$ phases. At this energy our results and second Born results are quite close. The degree of closeness of our computed amplitudes and the second Born amplitude may be represented by the estimated values of the variational parameters a_{min} and b_{min}. The values of a_{min} and b_{min} are presented in Table V. For lower energies the estimated values of a are observed to be small (nearer to 0) and those of b are large (nearer to 1). Thus at $E = 50$ eV, $a_{min} \approx 0.1$ and $b_{min} \approx 1$. As energy increases a_{min} continuously increases to the value 1 and b_{min} continuously diminishes to the value 0. So in the limit our results approach the second Born results. That the second Born results are not accurate even at 300 eV is simply reflected from the estimates $a_{min} = 0.7850$ and $b_{min} = 0.5652$ at this energy. Minimum value of $\| f^{(out)} - f^{(in)} \|^2$ also gives a measure of accuracy of the computation. We have checked that in general these minimum values are within 5–10 % of $\| f \|^2$ reflecting that our results are generally within 5–10 % in error.

V. CONCLUDING REMARKS

The present study clearly indicates that the theoretical procedure suggested by Das is as successful in treating the problem of elastic scattering of electrons by a helium atom as it is in the case of potential scattering at intermediate and high energies. The procedure is especially suitable for intermediate energies. The same success may be expected in cases of elastic or inelastic scattering of electrons by other atoms. The method may be applied in many other problems of atomic or nuclear physics. In problems where there are no exchange effects, such as positron-helium or proton-helium scattering, the method is expected not only to yield very accurate results at intermediate energies but is expected to yield good results for low energies as well. In problems where there are exchange effects, accurate results may possibly be obtained for low energies by computing both the direct and the exchange amplitude following the procedure. Explicit calculations to verify these expectations are in progress.

ACKNOWLEDGMENT

The authors are thankful to Professor P. K. Ghosh and Professor C. K. Majumdar and Dr. M. R. Gupta for some useful discussions. One of the authors (JND) is thankful to the University Grants Commission for sanctioning a research grant.

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