Unitary eikonal approximation including second-order phase function for particle-atom collisions

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A scattering amplitude for particle-atom collisions is developed in the eikonal approximation, taking into consideration the second-order term in the phase function of the S matrix. A finite value of the average excitation energy of the atom in its intermediate states is retained in each order (except the first) of the perturbation series. For zero excitation energy the second-order phase function reduces to twice the second-order Wentzel-Kramers-Brillouin phase shift for a static potential. Our amplitude becomes the modified Glauber amplitude by putting the second-order phase function equal to zero. The new amplitude satisfies the unitarity theorem to all orders of the coupling constant. It is free of *all* the shortcomings of the conventional Glauber theory.

I. INTRODUCTION

A variety of eikonal approximations of the scattering amplitude for particle-atom collisions at intermediate energies have had certain achievements to their credit. The basic approach goes back to Glauber[1] who designed his theory in order to tackle the complex multiple-scattering processes involved in the high-energy collisions of particles from nuclei. It was introduced into the realm of particle-atom scattering by Franco,[2] and Birman and Rosendorff.[3] For particle-atom scattering the conventional Glauber theory suffers from a number of shortcomings, the most important of which are the following. (i) The logarithmic divergence of the imaginary part of the elastic amplitude in the forward direction because of the long-range electromagnetic forces. (ii) The excitation cross sections for states for which $(L_f - M_f) - (L_i - M_i)$ is an *odd* number are identically zero. Here L_f, M_f, and L_i, M_i are the angular momentum quantum numbers of the final and initial states, respectively. The calculation of the orientation parameter λ is, therefore, meaningless in this theory. On the other hand, the excitation cross sections for which $(L_f - M_f) - (L_i - M_i)$ is an *even* number turn out to be enhanced in the Glauber theory. (iii) It does not distinguish between scattering of positive and negative particles.

The first two problems can be removed by introducing an average excitation energy of finite value of the atom in its intermediate states. This has been worked out[4] recently in a consistent way by retaining only the leading term in inverse powers of the momentum k of the scattered particle in each order term of the amplitude. In spite of the finiteness of the excitation energy, the perturbation series can be summed. After making use of the usual eikonization process, the amplitude obtained, (called the modified Glauber amplitude MGA), resembles the conventional Glauber amplitude in character but is without the shortcomings of (i) and (ii). It has been proved[5] that the MGA satisfies the unitarity theorem (to all orders of the perturbation expansion). The MGA has been examined[5,6] by analyzing elastic e-H scattering for 50-200 eV. One interesting result obtained is that the average excitation energy is essentially independent of the incident energy. As to the agreement with the experimental data, it turns out to be very good, provided one adds to the MGA the real part of the second-order Born term $\mathrm{Re}f^{B2}$. This is a rather unsatisfactory situation from the theoretical point of view, as the addition of one single term to the MGA has no theoretical justification. The only reason given is to get a reasonable agreement with the experimental data of a certain kind (elastic scattering, for instance). That it is theoretically unjustified and inconsistent follows clearly from the fact that this amplitude (MGA $+ \mathrm{Re}f^{B2}$) violates the unitarity theorem.

It is the purpose of the present paper to derive a scattering amplitude for particle-atom collisions in the eikonal approximation which satisfies the unitarity theorem and does not suffer from the shortcomings (i)-(iii) mentioned above, with which the conventional Glauber amplitude is burdened. As has already been mentioned in Paper I (Reference 4) the technique developed there opens the way to the possibility of inclusion of the second (and higher) -order terms in the phase function of the S matrix. Here we are concerned with the second-order phase function only. In Sec. II an amplitude of the second order with fixed intermediate states is developed in the eikonal approximation which corresponds to the real part of the propagator in momentum representation. For elastic scattering it coincides with the real part of the second-order amplitude. It depends intimately on the second-order phase function. The latter is discussed in Sec. III. In Sec. IV the complete (including all orders) scattering amplitude is derived after summing over all intermediate states, keeping the average excitation energy at a finite value. This amplitude is unitary and reduces to the MGA by putting the second-order phase function equal to zero. A simple expression of Clebsch-Gordan coefficients for large angular momenta is derived in Appendix C.

II. THE SECOND-ORDER AMPLITUDE

We shall start with the evaluation of the scattering amplitude of the second order. We follow closely the first part of Sec. 4 of Paper I. We take the initial state of the atom to be the ground state. The axis of quantization z is taken along the direction of the initial beam $\vec{k}_i$.

The second-order amplitude is given by

$$f^{(2)}_{\sigma_f\sigma_i}=\left(\frac{\mu}{2\pi\hbar^2}\right)^2\sum_\sigma f^{(2)}_{\sigma_f\sigma\sigma_i}\,, \tag{1}$$

$$f^{(2)}_{\sigma_f\sigma\sigma_i}=\int e^{-i\vec{k}_f\cdot\vec{r}_2}\langle\sigma_f|V(\vec{r}_2)|\sigma\rangle G(\vec{r}_2,\vec{r}_1)\times\langle\sigma|V(\vec{r}_1)|\sigma_i\rangle e^{i\vec{k}_i\cdot\vec{r}_1}\,d\vec{r}_2 d\vec{r}_1\,. \tag{2}$$

Here σ_i, σ_f, and σ denote the quantum numbers of the initial, final, and intermediate states of the atom, respectively. The vectors $\vec{k}_i$ and $\vec{k}_f$ are the momenta of the scattered particle in the initial and final states, and G is the free-particle Green's function

$$G(\vec{r}_2,\vec{r}_1)=\frac{e^{ik_\sigma|\vec{r}_2-\vec{r}_1|}}{|\vec{r}_2-\vec{r}_1|}\,, \tag{3}$$

with k_σ being the momentum of the particle in the intermediate state σ. The interaction V is the Coulomb potential between the scattered particle and the electrons and nucleus of the atom. μ is the reduced mass of the system.

Making use of the angular momentum expansion of the matrix elements, the Green's function, and the plane waves we obtain

$$f^{(2)}_{\sigma_f\sigma\sigma_i}=(4\pi)^{3/2}ik_\sigma\sum_{l_2l_1l_0,L_2'L_1'}(2l_2+1)^{1/2}i^{l_0-l_2}Y_{l_2}^{-M_f}(\theta_s,\varphi_s)\times\Re^{L_2'L_1',l_2l_1l_0}_{\sigma_f\sigma\sigma_i}(k_fk_\sigma k_i)\prod_{j=1}^{2}(2L_j'+1)^{1/2}C(l_jL_j';l_{j-1};00)\,C(l_jL_j'l_{j-1};-M_jM_j-M_{j-1}) \tag{4}$$

since the angular integrals are easily worked out. The C's are the Clebsch-Gordan coefficients, $M_0=M_i$, $M_2=M_f$, and $M_1=M_\sigma$ are the magnetic quantum numbers of the atom for the states σ_i, σ_f, and σ, respectively. L_1' and L_2' are the angular momenta exchanged between the scattered particle and the atom at the two vertices. θ_s, φ_s denote the polar and azimuthal angles of the outgoing particle. The radial function is given by the expression

$$\Re^{L_2'L_1',l_2l_1l_0}_{\sigma_f\sigma\sigma_i}(k_fk_\sigma k_i)=\int_0^\infty\int_0^\infty j_{l_2}(k_fr_2)f^{(L_2')}_{\sigma_f\sigma}(r_2)\,g_{l_1}(k_\sigma r_2;k_\sigma r_1)f^{(L_1')}_{\sigma\sigma_i}(r_1)j_{l_0}(k_ir_1)r_2^2r_1^2\,dr_2dr_1\,, \tag{5}$$

where the functions $f^{(L')}_{\sigma_2\sigma_1}$ are defined by the expansion

$$\langle\sigma_j|V(\vec{r}_j)|\sigma_{j-1}\rangle=\sum_{L_j'}f^{(L_j')}_{\sigma_j\sigma_{j-1}}(r_j)(Y_{L_j'}^{M_j-M_{j-1}}(\Omega_j))^* \tag{6}$$

and j_l is the lth spherical Bessel function. g_l is the radial Green's function, defined by

$$g_l(k_\sigma r_2;k_\sigma r_1)=\begin{cases}j_l(k_\sigma r_2)h_l^{(1)}(k_\sigma r_1) & \text{for } r_2<r_1\,,\\ j_l(k_\sigma r_1)h_l^{(1)}(k_\sigma r_2) & \text{for } r_1<r_2\,,\end{cases} \tag{7}$$

where

$$h_l^{(1)}(\rho)=j_l(\rho)+in_l(\rho) \tag{7'}$$

is the first spherical Hankel function, and n_l is the spherical Neumann function.

Making use of the definition of g_l, Eqs. (7) and (7′) the radial function becomes

$$\mathcal{R}^{L_2'L_1';l_2l_1l}_{\sigma_f\sigma\sigma_i}(k_fk_\sigma k_i)=\int_0^\infty j_{l_2}(k_fr_2)f^{(L_2')}_{\sigma_f\sigma}(r_2)j_{l_1}(k_\sigma r_2)r_2^2dr_2\int_0^\infty j_{l_1}(k_\sigma r_1)f^{(L_1')}_{\sigma\sigma_i}(r_1)j_{l_0}(k_ir_1)r_1^2dr_1$$
$$+i\int_0^\infty j_{l_2}(k_fr_2)f^{(L_2')}_{\sigma_f\sigma}(r_2)n_{l_1}(k_\sigma r_2)r_2^2dr_2\int_0^{r_2} j_{l_1}(k_\sigma r_1)f^{(L_1')}_{\sigma\sigma_i}(r_1)j_{l_0}(k_ir_1)r_1^2dr_1$$
$$+i\int_0^\infty j_{l_2}(k_fr_2)f^{(L_2')}_{\sigma_f\sigma}(r_2)j_{l_1}(k_\sigma r_2)r_2^2dr_2\int_{r_2}^\infty n_{l_1}(k_\sigma r_1)f^{(L_1')}_{\sigma\sigma_i}(r_1)j_{l_0}(k_ir_1)r_1^2dr_1\,. \tag{8}$$

The first term in this expression (the real part of $\mathcal{R}$) has been dealt with in detail in Paper I. It gives rise to the second-order term of the modified Glauber amplitude. The contribution of the remaining two terms (called the j_ln_l terms in Paper I) of Eq. (8) to the second-order amplitude will be discussed here. We shall denote the corresponding amplitude by $Rf^{(2)}_{\sigma_f\sigma\sigma_i}$. The reason for this notation is the following: If one writes the amplitude in momentum representation then the real part of the propagator $(q^2-k_n^2-i\epsilon)^{-1}$ leads to the same amplitude. This is easily verified. Also, for elastic scattering this amplitude coincides with the real part of the second-order amplitude.

Taking advantage of the function

$$J^{(m_1m_2)}_{l_1l_2}(x)=\sum_{l_3} i^{l_3}C(l_1l_2l_3;00)C(l_1l_2l_3;m_1m_2)j_{l_3}(x) \tag{9}$$

which was introduced in Paper I, we find from Eqs. (8) and (4)

$$Rf^{(2)}_{\sigma_f\sigma\sigma_i}=-(4\pi)^{3/2}k_\sigma\sum_{l_2l_1,L_2'L_1'}(2l_2+1)^{1/2}i^{-l_2}Y^{-M_f}_{l_2}(\theta_s,\varphi_s)(2L_1'+1)^{1/2}(2L_2'+1)^{1/2}C(l_2L_2'l_1;00)C(l_2L_2'l_1;\,-M_fM_f-M_\sigma)$$
$$\times\left(\int_0^\infty j_{l_2}(k_fr_2)f^{(L_2')}_{\sigma_f\sigma}(r_2)j_{l_1}(k_\sigma r_2)r_2^2dr_2\int_{r_2}^\infty n_{l_1}(k_\sigma r_1)f^{(L_1')}_{\sigma\sigma_i}(r_1)J^{(-M_\sigma M_\sigma)}_{l_1L_1'}(k_ir_1)r_1^2dr_1\right.$$
$$\left.+\int_0^\infty j_{l_2}(k_fr_2)f^{(L_2')}_{\sigma_f\sigma}(r_2)n_{l_1}(k_\sigma r_2)r_2^2dr_2\int_0^{r_2} j_{l_1}(k_\sigma r_1)f^{(L_1')}_{\sigma\sigma_i}(r_1)J^{(-M_\sigma M_\sigma)}_{l_1L_1'}(k_ir_1)r_1^2dr_1\right). \tag{10}$$

So far everything is exact. We shall now perform the eikonization of the above amplitude by introducing the appropriate approximate expressions of the functions $J^{(m_1m_2)}_{l_1l_2}$, j_l, and n_l. First the function $J^{(m_1m_2)}_{l_1l_2}$. It has been proved in the Appendix of Paper I that for $l_1\gg l_2$ and $x\gg1$, $J^{(m_1m_2)}_{l_1l_2}$ behaves in the following way:

$$J^{(m_1m_2)}_{l_1l_2}(x)=i^{l_1+m_2}\left(\frac{4\pi}{2l_2+1}\right)^{1/2}e^{im_2\varphi}Y^{m_2*}_{l_2}(\theta,\varphi)\begin{cases}j_{l_1}(x) & \text{for } (l_2-m_2)\text{ even},\\ in_{l_1}(x) & \text{for } (l_2-m_2)\text{ odd},\end{cases} \tag{11}$$

where θ is defined by $\sin\theta=(l_1+\frac{1}{2})/x$.

We thus get, for the amplitude, after summing over L_1' according to Eq. (6),

$$Rf^{(2)}_{\sigma_f\sigma\sigma_i}=-(4\pi)^2k_\sigma i^{M_\sigma}\sum_{l_2l_1,L_2'}(2l_2+1)^{1/2}i^{l_1-l_2}Y^{-M_f}_{l_2}(\theta_s,\varphi_s)(2L_2'+1)^{1/2}C(l_2L_2'l_1;00)C(l_2L_2'l_1;\,-M_fM_f-M_\sigma)$$
$$\times\left(\int_0^\infty j_{l_2}(k_fr_2)f^{(L_2')}_{\sigma_f\sigma}(r_2)\,j_{l_1}(k_\sigma r_2)r_2^2dr_2\int_{r_2}^\infty n_{l_1}(k_\sigma r_1)r_1^2dr_1\langle\sigma e^{-iM_\sigma\varphi}|V(\vec{b}+\hat{k}_iz_1;\zeta)|\sigma_i\rangle\right.$$
$$\times\begin{cases}j_{l_1}(k_ir_1)\text{ for }(L_\sigma-M_\sigma)\text{ even}\\ in_{l_1}(k_ir_1)\text{ for }(L_\sigma-M_\sigma)\text{ odd}\end{cases}+\int_0^\infty j_{l_2}(k_fr_2)f^{(L_2')}_{\sigma_f\sigma}(r_2)n_{l_1}(k_\sigma r_2)r_2^2dr_2$$
$$\times\int_0^{r_2}j_{l_1}(k_\sigma r_1)r_1^2dr_1\langle\sigma e^{-iM_\sigma\varphi}|V(\vec{b}+\hat{k}_iz_1;\vec{\zeta})|\sigma_i\rangle$$
$$\left.\times\begin{cases}j_{l_1}(k_ir_1)\text{ for }(L_\sigma-M_\sigma)\text{ even}\\ in_{l_1}(k_ir_1)\text{ for }(L_\sigma-M_\sigma)\text{ odd}\end{cases}\right), \tag{12}$$

where $b=(l_1+\frac{1}{2})/k_\sigma$ is the magnitude of the impact parameter and φ is the corresponding azimuthal angle of $\vec{b}$.

Next, the large l, large k with $l<kr$ expansion of $j_l(kr)$ and $n_l(kr)$ is given by[7]

$$\left.\begin{matrix} j_l(kr) \\ \\ n_l(kr) \end{matrix}\right\} = D(k,z,b)\left\{\begin{matrix} \cos\alpha_l \\ \\ \sin\alpha_l \end{matrix}\right. , \tag{13}$$

with

$$\alpha_l = k(z - b\cos^{-1}b/r) - \pi/4$$

and

$$z=(r^2-b^2)^{1/2}. \tag{13'}$$

The amplitude D is a smoothly decreasing function with a weak dependence on l. It is discussed in detail in Sec. III. We are now ready to discuss the integrals in Eq. (12) in more detail. Let us initially concentrate on the first integral.

A. The contribution of the first integral in Eq. (12)

As is seen from Eq. (12) we have to distinguish between two cases, (1) $(L_\sigma - M_\sigma)$ even (even intermediate state) and (2) $(L_\sigma - M_\sigma)$ odd (odd intermediate state).

1. $(L_\sigma - M_\sigma)$ even

The product of the Bessel function and the Neumann function under the integral sign over r_1 is, according to Eq. (13),

$$n_{l_1}(k_\sigma r_1)j_{l_1}(k_i r_1) = \tfrac{1}{2}D^2\{\sin[\alpha_{l_1}(k_\sigma)+\alpha_{l_1}(k_i)] + \sin[\alpha_{l_1}(k_\sigma)-\alpha_{l_1}(k_i)]\}. \tag{14}$$

As $\partial\alpha_l/\partial k = z$ for fixed l and r_1, we have that $[\alpha_{l_1}(k_\sigma)-\alpha_{l_1}(k_i)]$ is equal to $(k_\sigma - k_i)z$. Thus

$$\int_{r_2}^{\infty} n_{l_1}(k_\sigma r_1)\langle\sigma e^{-iM_\sigma\varphi}|V|\sigma_i\rangle j_{l_1}(k_i r_1) r_1^2 dr_1 = \tfrac{1}{2}\int_{z_2}^{\infty} D^2\langle\sigma e^{-iM_\sigma\varphi}|V|\sigma_i\rangle\{\sin[\alpha_{l_1}(k_\sigma)+\alpha_{l_1}(k_i)] + \sin(k_\sigma - k_i)z_1\} r_1 z_1 dz_1. \tag{15}$$

By Eq. (13′) we define γ_l by $\alpha_l = k\gamma_l - \pi/4$. Then

$$\sin[\alpha_{l_1}(k_\sigma)+\alpha_{l_1}(k_i)] = -\cos[k_\sigma\gamma_{l_1}(k_\sigma)+k_i\gamma_{l_1}(k_i)].$$

As k_σ, k_i are large, and γ_l is a smooth, slowly varying function of the momentum, the first part of the above integral is evaluated with the help of the stationary-phase method.[8] The dominant contribution comes from the lower limit as there are no other critical points in the region of integration. We get

$$\int_{z_2}^{\infty} D^2\langle\sigma e^{-iM_\sigma\varphi}|V|\sigma_i\rangle \sin[\alpha_{l_1}(k_\sigma)+\alpha_{l_1}(k_i)] r_1 z_1 dz_1 = -\mathrm{Re}\exp\{-i[k_\sigma\gamma_{l_1}(k_\sigma)+k_i\gamma_{l_1}(k_i)]\}\frac{a_0 e^{-i\pi/2}}{2k\partial\gamma_l/\partial z}, \tag{16}$$

with

$$a_0 = D^2(z_2)\langle\sigma e^{-iM_\sigma\varphi}|V(\vec{b}+\hat{k}_i z_2;\vec{\zeta})|\sigma_i\rangle r_2 z_2. \tag{16'}$$

k is an average of k_σ and k_i. As $\partial\gamma_l/\partial z = z^2/r^2$ for fixed l and k, the integral, Eq. (15) becomes

$$\int_{r_2}^{\infty} n_{l_1}(k_\sigma - r_1)\langle\sigma e^{-iM_\sigma\varphi}|V|\sigma_i\rangle j_{l_1}(k_i r_1) r_1^2 dr_1 = \frac{r_2^3}{4kz_2} D^2(z_2)\langle\sigma e^{-iM_\sigma\varphi}|V(\vec{b}+\hat{k}_i z_2;\vec{\zeta})|\sigma_i\rangle\cos[\alpha_{l_1}(k_\sigma)+\alpha_{l_1}(k_i)] + \tfrac{1}{2}\int_{z_2}^{\infty} r_1 z_1 D^2\langle\sigma e^{-iM_\sigma\varphi}|V|\sigma_i\rangle\sin(k_\sigma - k_i)z_1 dz_1. \tag{17}$$

At this point it is worthwhile to introduce the function

$$H_{\sigma_1\sigma_2}(\vec{b},z)=\langle\sigma_1 e^{-iM_1\varphi}\,|\,V(\vec{b}+\hat{k}_i z;\vec{\zeta})\,|\,e^{-iM_2\varphi}\sigma_2\rangle e^{-i(k_1-k_2)z}, \tag{18}$$

which was defined in Paper I. By Eq. (6), the matrix element $\langle\sigma_1 e^{-iM_1\varphi}|V|e^{-iM_2\varphi}\sigma_2\rangle$ has definite parity with respect to the transformation $z\to -z$. It is even (odd) when $[(L_1-M_1)-(L_2-M_2)]$ is even (odd). We can therefore express the second term of Eq. (17) by the above function

$$\tfrac{1}{2}\int_{z_1}^{\infty} r_1 z_1 D^2\langle\sigma e^{-iM_\sigma\varphi}\,|\,V\,|\,\sigma_1\rangle\sin(k_\sigma-k_i)z_1\,dz_1=\tfrac{1}{2}i\int_{z_2}^{\infty} r_1 z_1 D^2 H^{(o)}_{\sigma\sigma_i}(\vec{b},z_1)\,dz_1\,, \tag{19}$$

where $H^{(o)}$ is the odd part of H.

Our next task is the evaluation of the sum over l_1. Let us start with the first term of Eq. (17). For $j_{l_1}(k_\sigma r_1)$ we again make use of Eq. (13). Furthermore, we notice that it is permissible to take the function D and the matrix element of V outside the summation, because (i) both functions are smooth, slowly varying functions of the impact parameter $\vec{b}$, (ii) The triangular relation $\Delta(l_2L_2'l_1)$ and $l_1\gg L_1'$ assures $l_1\approx l_2$. We are therefore, according to Eq. (12), in need of the sum

$$C^{(1)}_{l_2L_2'}(z_2)=\sum_{l_1} i^{l_1}C(l_2L_2'l_1;00)C(l_2L_2'l_1;\,-M_fM_f-M_\sigma)\cos\alpha_{l_1}(k_\sigma)\cos[\alpha_{l_1}(k_\sigma)+\alpha_{l_1}(k_i)]\,. \tag{20}$$

We show in Appendix A that for l_1 large the above function is given by

$$\begin{aligned}C^{(1)}_{l_2L_2'}(z_2)=\tfrac{1}{2}i^{\,l_2+M_f-M_\sigma}\left(\frac{\pi}{2L_2'+1}\right)^{1/2}e^{i(M_f-M_\sigma)\varphi}\\ \times\{[Y^{M_f-M_\sigma}_{L_2'}(\theta,\varphi)]^*[e^{i\alpha_{l_2}(k_i)}+(-1)^\Delta e^{-i\alpha_{l_2}(k_i)}]\\ +(-1)^{L_2'}[Y^{M_f-M_\sigma}_{L_2'}(3\theta,\varphi)][e^{i[\alpha_{l_2}(k_i)+2\alpha_{l_2}(k_\sigma)]}+(-1)^\Delta e^{-i[\alpha_{l_2}(k_i)+2\alpha_{l_2}(k_\sigma)]}]\}\,,\end{aligned} \tag{21}$$

where $\Delta=L_2'-(M_f-M_\sigma)$ and $\cos\theta=z_2/r_2$. Hence, the sum over l_1 of the first term of Eq. (17) of the first integral in Eq. (12), for even intermediate states becomes

$$\frac{1}{4k}\int_0^\infty \cos\alpha_{l_2}(k_f)f^{(L_2')}_{\sigma_f\sigma}(r_2)\,D^4(z_2)\langle\sigma e^{-iM_\sigma\varphi}\,|\,V(\vec{b}+\hat{k}_iz_2;\vec{\zeta})\,|\,\sigma_i\rangle C^{(1)}_{l_2L_2'}(z_2)r_2^4\,dz_2\,. \tag{22}$$

According to Eq. (21), this expression factorizes into eight different terms, six of which are negligibly small because $[\alpha_l(k)+\pi/4]$ is proportional to k. The remaining two terms depend on the difference of two α_l's which is k independent because $[\alpha_l(k_1)-\alpha_l(k_2)]=(k_1-k_2)z$. Therefore Eq. (22) becomes

$$\begin{aligned}\frac{1}{16k}i^{\,l_2+M_f-M_\sigma}\left(\frac{\pi}{2L_2'+1}\right)^{1/2}e^{i(M_f-M_\sigma)\varphi}\int_0^\infty r_2^4D^4(z_2)dz_2f^{(L_2')}_{\sigma_f\sigma}(r_2)[Y^{M_f-M_\sigma}_{L_2'}(\theta,\varphi)]^*[e^{-i(k_f-k_i)z_2}+(-1)^\Delta e^{i(k_f-k_i)z_2}]\\ \times\langle\sigma e^{-iM_\sigma\varphi}\,|\,V(\vec{b}+\hat{k}_iz_2;\vec{\zeta})\,|\,\sigma_i\rangle\,.\end{aligned} \tag{23}$$

Finally, the summation over L_2' in Eq. (12) is easily carried out. Remembering the expansion of the matrix element of V of Eq. (6), the summation of Eq. (23) over L_2' becomes

$$\begin{aligned}\frac{\sqrt{\pi}}{16k}i^{\,l_2+M_f-M_\sigma}\int_0^\infty r_2^4D^4dz_2\langle\sigma_fe^{-iM_f\varphi}\,|\,V(\vec{b}+\hat{k}_iz_2;\vec{\zeta})\,|\,e^{-iM_\sigma\varphi}\sigma\rangle\\ \times\langle\sigma e^{-iM_\sigma\varphi}\,|\,V(\vec{b}+\hat{k}_iz_2;\vec{\zeta})\,|\,\sigma_i\rangle[e^{-i(k_f-k_i)z_2}+(-1)^\Delta e^{i(k_f-k_i)z_2}]\,.\end{aligned} \tag{24}$$

Now this expression can be written in a more concise way by making use of the function $H_{\sigma_1\sigma_2}$ of Eq. (18). It is easily verified that, for even as well as odd values of (M_f-M_σ), we have

$$\langle\sigma_fe^{-iM_f\varphi}\,|\,V\,|\,e^{-iM_\sigma\varphi}\sigma\rangle\langle\sigma e^{-iM_\sigma\varphi}\,|\,V\,|\,\sigma_i\rangle(e^{-i(k_f-k_i)z}+(-1)^\Delta e^{i(k_f-k_i)z})=2(H_{\sigma_f\sigma}H_{\sigma\sigma_i})^{(e)}\,, \tag{25}$$

where the superscript (e) means the even part of the product with respect to z. This is due to the fact that L_2' is even (odd) when (L_f-L_σ) is even (odd); thus Δ is even (odd) if (L_f-M_f) is even (odd) because $(L_\sigma-M_\sigma)$ is assumed to be even. Expression (24) therefore becomes

$$\frac{\sqrt{\pi}}{8k}i^{\,l_2+M_f-M_\sigma}\int_0^\infty r^4D^4(H_{\sigma_f\sigma}H_{\sigma\sigma_i})^{(e)}dz\,. \tag{26}$$

It follows that the contribution of this term to the second-order amplitude Eq. (12) is

$$-\pi^{5/2} i^{M_f} \sum_{l_2=0}^{\infty} (2l_2+1)^{1/2} Y_{l_2}^{-M_f}(\theta_s \varphi_s) \int_{-\infty}^{\infty} r^4 D^4 H_{\sigma_f \sigma} H_{\sigma\sigma_i} dz . \tag{27}$$

The summation over l_1 of the second term of Eq. (17) is easily performed. According to Eq. (12) and Eq. (19) we are in need of the sum

$$\begin{aligned}\tfrac{1}{2} i \sum_{l_1} i^{l_1} C(l_2 L_2' l_1; 00) C(l_2 L_2' l_1; -M_f M_f - M_\sigma) \\ \times \int_0^{\infty} j_{l_2}(k_f r_2) f_{\sigma_f\sigma}^{(L_2')}(r_2) j_{l_1}(k_\sigma r_2) r_2^2 dr_2 \int_{z_2}^{\infty} r_1 z_1 D^2(z_1) H_{\sigma\sigma_i}^{(o)}(\vec{b}, z_1) dz_1 ,\end{aligned} \tag{28}$$

which becomes, after making use of Eq. (9)

$$\tfrac{1}{2} i \int_0^{\infty} j_{l_2}(k_f r_2) f_{\sigma_f\sigma}^{(L_2')} J_{l_2 L_2'}^{(-M_f M_f - M_\sigma)}(k_\sigma r_2) r_2^2 dr_2 \int_{z_2}^{\infty} r_1 z_1 D^2 H_{\sigma\sigma_i}^{(o)} dz_1 , \tag{28$'$}$$

as the r_1 integral is virtually independent of l_1 [for clarification, see the text after Eq. (19)]. Now, with the help of the l large, k large approximation of the function $J_{l_1 l_2}^{(m_1 m_2)}$ given by Eq. (11), and subsequent summation over L_2' this expression becomes

$$\begin{aligned} i^{l_2+M_f-M_\sigma+1}\sqrt{\pi} \int_0^{\infty} r_2 z_2 dz_2 j_{l_2}(k_f r_2) \langle \sigma_f e^{-iM_f\varphi} | V(\vec{b}+\hat{k}_i z_2; \vec{\zeta}) | e^{-iM_\sigma\varphi}\sigma\rangle \\ \times \begin{cases} j_{l_2}(k_\sigma r_2) \text{ for } (L_f - M_f) \text{ even} \\ i n_{l_2}(k_\sigma r_2) \text{ for } (L_f - M_f) \text{ odd} \end{cases} \int_{z_2}^{\infty} r_1 z_1 D^2 H_{\sigma\sigma_i}^{(o)} dz_1 ,\end{aligned} \tag{29}$$

where we have made explicit use of the fact that $(L_\sigma - M_\sigma)$ is even. Next, we make use of (i) Eq. (13) for the Bessel functions, (ii) the definition of the function $H_{\sigma_1\sigma_2}$, Eq. (18), (iii) the fact that under the integral sign the dominant term of $\cos\alpha_{l_2}(k_f)\cos\alpha_{l_2}(k_\sigma)$ and $\cos\alpha_{l_2}(k_f)\sin\alpha_{l_2}(k_\sigma)$ is $\frac{1}{2}\cos(k_f - k_\sigma)z$ and $-\frac{1}{2}\sin(k_f - k_\sigma)z$, respectively, in order to convert Eq. (29) into

$$\frac{\sqrt{\pi}}{2} i^{l_2+M_f-M_\sigma+1} \int_0^{\infty} r_2 z_2 D^2 H_{\sigma_f\sigma}^{(e)}(\vec{b}, z_2) dz_2 \int_{z_2}^{\infty} r_1 z_1 D^2 H_{\sigma\sigma_i}^{(o)}(\vec{b}, z_1) dz_1 . \tag{30}$$

In conclusion, the contribution of the first integral in Eq. (12) to the second-order amplitude $Rf_{\sigma_f\sigma\sigma_i}^{(2)}$ for even intermediate states is given by Eq. (27), and Eq. (30) summed over l_2,

$$-\pi^{5/2} i^{M_f} \sum_{l_2=0}^{\infty} (2l_2+1)^{1/2} Y_{l_2}^{-M_f}(\theta_s\varphi_s) \left(2\int_0^{\infty} r^4 D^4 (H_{\sigma_f\sigma} H_{\sigma\sigma_i})^{(e)} dz + 8ik_\sigma \int_0^{\infty} r_2 z_2 D^2 H_{\sigma_f\sigma}^{(e)} dz_2 \int_{z_2}^{\infty} r_1 z_1 D^2 H_{\sigma\sigma_i}^{(o)} dz_1 \right) . \tag{31}$$

2. $(L_\sigma - M_\sigma)$ odd

In this case, according to Eq. (12), we are in need of the product of $n_{l_1}(k_\sigma r_1)$ and $n_{l_1}(k_i r_1)$. By Eq. (13), we have

$$n_{l_1}(k_\sigma r_1) n_{l_1}(k_i r_1) = \tfrac{1}{2} D^2 \{ -\cos[\alpha_{l_1}(k_\sigma) + \alpha_{l_1}(k_i)] + \cos[\alpha_{l_1}(k_\sigma) - \alpha_{l_1}(k_i)] \} . \tag{32}$$

Putting this expression into the r_1 part of the first integral of Eq. (12) gives

$$\begin{aligned} i \int_{r_2}^{\infty} n_{l_1}(k_\sigma r_1) \langle \sigma e^{-iM_\sigma\varphi} | V | \sigma_i \rangle n_{l_1}(k_i r_1) r_1^2 dr_1 \\ = \tfrac{1}{2} i \int_{z_2}^{\infty} D^2 r_1 z_1 dz_1 \langle \sigma e^{-iM_\sigma\varphi} | V | \sigma_i \rangle \{ -\sin[k_\sigma \gamma_{l_1}(k_\sigma) + k_i \gamma_{l_1}(k_i)] + \cos(k_\sigma - k_i) z_1 \} ,\end{aligned} \tag{33}$$

where γ_l was defined after Eq. (15), and again use has been made of $\partial\alpha_l/\partial k = z$. As in the even case, the first term is evaluated by the stationary-phase method. We obtain

$$i\int_{r_2}^{\infty} n_{l_1}(k_\sigma r_1)\langle\sigma e^{-iM_\sigma\varphi}\,|\,V\,|\,\sigma_i\rangle n_{l_1}(k_i r_1)r_1^2 dr_1$$

$$=\frac{i}{4k}\frac{r_2^3}{z_2}D^2(z_2)\langle\sigma e^{-iM_\sigma\varphi}\,|\,V(\vec{b}+\hat{k}_i z_2;\vec{\zeta})\,|\,\sigma_i\rangle\sin[\alpha_{l_1}(k_\sigma)+\alpha_{l_1}(k_i)]+\tfrac{1}{2}i\int_{z_2}^{\infty}D^2 r_1 z_1 H^{(o)}_{\sigma\sigma_i}(\vec{b},z_1)dz_1\,, \quad (34)$$

where again we have taken advantage of the definition of $H_{\sigma_1\sigma_2}$, Eq. (18). As can be seen, the second term is identical with the corresponding term of Eq. (19), for even intermediate states.

Our next task is the summation over l_1 of the above expression. The first term of Eq. (34) yields [see the explanation after Eq. (19)], according to Eq. (12)

$$\frac{1}{4k}\int_0^{\infty}\cos\alpha_{l_2}(k_f)f^{(L_2')}_{\sigma_f\sigma}(r_2)D^4(z_2)\langle\sigma e^{-iM_\sigma\varphi}\,|\,V(\vec{b}+\hat{k}_i z_2;\vec{\zeta})\,|\,\sigma_i\rangle S^{(1)}_{l_2L_2'}(z_2)r_2^4 dz_2\,, \quad (35)$$

where the function $S^{(1)}_{l_2L_2'}$ is defined by the sum

$$S^{(1)}_{l_2L_2'}(z_2)=i\sum_{l_1} i^{l_1}C(l_2L_2'l_1;00)C(l_2L_2'l_1;\,-M_f M_f-M_\sigma)\cos\alpha_{l_1}(k_\sigma)\sin[\alpha_{l_1}(k_\sigma)+\alpha_{l_1}(k_i)]\,. \quad (36)$$

Next, we employ the large l_2 and k_σ, k_i approximation of $S^{(1)}_{l_2L_2'}$. It is equal to the expression of $C^{(1)}_{l_2L_2'}$ of Eq. (21) except that $\Delta=[L_2'-(M_f-M_\sigma)]$ is replaced by $(\Delta+1)$. The proof follows the same lines as for the function $C^{(1)}_{l_2L_2'}$ (see Appendix A). Obviously the contribution of the term proportional to $[Y^{M_f-M_\sigma}_{L_2'}(3\theta,\varphi)]^*$ of Eq. (21) is negligibly small because of the rapid oscillations of the trigonometric functions. The contribution of the first term of Eq. (21) to Eq. (35) with $\Delta\to\Delta+1$ is

$$\frac{1}{16k}i^{l_2+M_f-M_\sigma}\left(\frac{\pi}{2L_2'+1}\right)^{1/2}e^{i(M_f-M_\sigma)\varphi}$$

$$\times\int_0^{\infty}D^4 f^{(L_2')}_{\sigma_f\sigma}r_2^4 dz_2(e^{i\alpha_{l_2}(k_f)}+e^{-i\alpha_{l_2}(k_f)})[Y^{M_f-M_\sigma}_{L_2'}(\theta,\varphi)]^*[e^{i\alpha_{l_2}(k_i)}+(-1)^{\Delta+1}e^{-i\alpha_{l_2}(k_i)}]$$

$$\times\langle\sigma e^{-iM_\sigma\varphi}\,|\,V(\vec{b}+\hat{k}_i z_2;\vec{\zeta})\,|\sigma_i\rangle\,. \quad (37)$$

The dominant contribution of

$$(e^{i\alpha_{l_2}(k_f)}+e^{-i\alpha_{l_2}(k_f)})[e^{i\alpha_{l_2}(k_i)}+(-1)^{\Delta+1}e^{-i\alpha_{l_2}(k_i)}] \quad (38)$$

is

$$[e^{-i(k_f-k_i)z}+(-1)^{\Delta+1}e^{i(k_f-k_i)z}]\,, \quad (38')$$

again, because of the rapid oscillations of the other two terms. Hence, Eq. (37) summed over L_2' becomes, in consideration of Eqs. (6) and (12)

$$\frac{\sqrt{\pi}}{16k}i^{l_2+M_f-M_\sigma}\int_0^{\infty}D^4 r^4 dz\langle\sigma_f e^{-iM_f\varphi}\,|\,V(\vec{b}+\hat{k}_i z;\vec{\zeta})\,|\,e^{-iM_\sigma\varphi}\sigma\rangle$$

$$\times\langle\sigma e^{-iM_\sigma\varphi}\,|\,V(\vec{b}+\hat{k}_i z;\vec{\zeta})\,|\,\sigma_i\rangle[e^{-i(k_f-k_i)z}+(-1)^{\Delta+1}e^{i(k_f-k_i)z}]\,. \quad (39)$$

Here also as in the case of even intermediate states, the product of the two-matrix element and the trigonometric function is simply given by

$$2(H_{\sigma_f\sigma}H_{\sigma\sigma_i})^{(e)}\,.$$

Equation (39) therefore becomes

$$\frac{\sqrt{\pi}}{8k}i^{l_2+M_f-M_\sigma}\int_0^{\infty}D^4 r^4(H_{\sigma_f\sigma}H_{\sigma\sigma_i})^{(e)}dz\,. \quad (40)$$

The summation over l_1 of the second term in Eq. (34) is identical with the corresponding case of even intermediate states [see Eq. (30)].

It follows that the contribution of the first integral in Eq. (12) to the amplitude $Rf_{\sigma_f\sigma\sigma_i}$; for odd intermediate states is of the form

$$-\pi^{5/2}i^{M_f}\sum_{l_2}(2l_2+1)^{1/2}Y_{l_2}^{-M_f}(\theta_s\varphi_s)\left(2\int_0^\infty D^4r^4\,(H_{\sigma_f\sigma}H_{\sigma\sigma_i})^{(e)}dz+8ik\int_0^\infty r_2z_2D^2H_{\sigma_f\sigma}^{(e)}dz_2\int_{z_2}^\infty r_1z_1D^2H_{\sigma\sigma_i}^{(o)}dz_1\right), \tag{41}$$

which is identical with Eq. (31). In other words, the above expression is equally valid for even and odd intermediate states. This concludes the calculation of the contribution to the amplitude of the first integral in Eq. (12).

B. The contribution of the second integral in Eq. (12)

1. $(L_\sigma - M_\sigma)$ even

We are in need of the product $j_{l_1}(k_\sigma r_1)j_{l_1}(k_i r_1)$ which by Eq. (13) becomes

$$j_{l_1}(k_\sigma r_1)j_{l_1}(k_ir_1)=\tfrac{1}{2}D^2\{\cos[\alpha_{l_1}(k_\sigma)+\alpha_{l_1}(k_i)]+\cos(k_\sigma-k_i)z\}. \tag{42}$$

The r_1 integral of the second integral in Eq. (12) thus assumes the form

$$\tfrac{1}{2}\int_0^{z_2}r_1z_1D^2\langle\sigma e^{-iM_\sigma\varphi}|V|\sigma_i\rangle\{\cos[\alpha_{l_1}(k_\sigma)+\alpha_{l_1}(k_i)]+\cos(k_\sigma-k_i)z\}dz_1$$

$$=\tfrac{1}{2}\int_0^{z_2}r_1z_1D^2\langle\sigma e^{-iM_\sigma\varphi}|V|\sigma_i\rangle\sin[k_\sigma\gamma_{l_1}(k_\sigma)+k_i\gamma_{l_1}(k_i)]dz_1+\tfrac{1}{2}\int_0^{z_2}r_1z_1D^2H_{\sigma\sigma_i}^{(e)}dz_1. \tag{43}$$

As in the former case, here also the first term is evaluated by the stationary-phase method. The significant contribution comes from the upper limit of the integral. Equation (43) thus becomes

$$\frac{1}{4k}\frac{r_2^3}{z_2}D^2(z_2)\langle\sigma e^{-iM_\sigma\varphi}|V(\vec{b}+\hat{k}_iz_2;\vec{\zeta})|\sigma_i\rangle\sin[\alpha_{l_1}(k_\sigma)+\alpha_{l_1}(k_i)]+\tfrac{1}{2}\int_0^{z_2}r_1z_1D^2H_{\sigma\sigma_i}^{(e)}dz_1. \tag{44}$$

Therefore, the summation over l_1 of the second integral in Eq. (12) is given by

$$\frac{1}{4k}\int_0^\infty r_2^4D^4(z_2)\cos\alpha_{l_1}(k_f)f_{\sigma_f\sigma}^{(L_2')}(r_2)\langle\sigma e^{-iM_\sigma\varphi}|V(\vec{b}+\hat{k}_iz_2;\vec{\zeta})|\sigma_i\rangle C_{l_2L_2'}^{(2)}dz_2$$

$$+\tfrac{1}{2}\int_0^\infty r_2z_2j_{l_2}(k_fr_2)f_{\sigma_f\sigma}^{(L_2')}(r_2)N_{l_2L_2'}^{(-M_fM_f-M_\sigma)}(k_\sigma r_2)dz_2\int_0^{z_2}r_1z_1D^2H_{\sigma\sigma_i}^{(e)}dz_1, \tag{45}$$

where

$$C_{l_2L_2'}^{(2)}(z_2)=\sum_{l_1}i^{l_1}C(l_2L_2'l_1;00)C(l_2L_2'l_1;-M_fM_f-M_\sigma)\sin\alpha_{l_1}(k_\sigma)\sin[\alpha_{l_1}(k_\sigma)+\alpha_{l_1}(k_i)], \tag{46}$$

and

$$N_{l_1l_2}^{(m_1m_2)}(x)=\sum_{l_3}i^{l_3}C(l_1l_2l_3;00)C(l_1l_2l_3;m_1m_2)n_{l_3}(x); \tag{47}$$

the last expression was first introduced in Paper I. It is easily shown that the large-l, large-k approximation of $C_{l_2L_2'}^{(2)}$ is equal to the approximation of $C_{l_2L_2'}^{(1)}$, Eq. (21) except for the sign of the second term, i.e., $(-1)^{L_2'}$ is replaced by $(-1)^{L_2'+1}$ (see also Appendix A). As previously explained, this term is completely negligible in the calculation of the amplitude, therefore the two functions $C_{l_2L_2'}^{(1)}$ and $C_{l_2L_2'}^{(2)}$ are identical from this point of view. The eikonal approximation of $N_{l_1l_2}^{(m_1m_2)}$ has been given in Paper I; it is derived from the corresponding approximation of $J_{l_1l_2}^{(m_1m_2)}$, Eq. (11), by replacing j_{l_1} by n_{l_1} for (l_2-m_2) even, and replacing n_{l_1} by $-j_{l_1}$ for (l_2-m_2) odd.

Equation (45) therefore becomes, after summing over L_2',

$$\frac{\sqrt{\pi}}{8k}i^{l_2+M_f-M_\sigma}\int_0^\infty r_2^4D^4\cos\alpha_{l_2}(k_f)\langle\sigma_fe^{-iM_f\varphi}|V|e^{-iM_\sigma\varphi}\sigma\rangle\langle\sigma e^{-iM_\sigma\varphi}|V|\sigma_i\rangle[e^{i\alpha_{l_2}(k_i)}+(-1)^\Delta e^{-i\alpha_{l_2}(k_i)}]dz_2$$

$$+\frac{\sqrt{\pi}}{2}i^{l_2+M_f-M_\sigma}\int_0^\infty r_2z_2j_{l_2}(k_fr_2)\langle\sigma_fe^{-iM_f\varphi}|V|e^{-iM_\sigma\varphi}\sigma\rangle\{[1+(-1)^\Delta]n_{l_2}(k_\sigma r_2)-i[1-(-1)^\Delta]j_{l_2}(k_\sigma r_2)\}dz_2$$

$$\times\int_0^{z_2}r_1z_1D^2H_{\sigma\sigma_i}^{(e)}dz_1. \tag{48}$$

As before, we neglect the strongly oscillatory expressions in the first term of above expression. Thus, making use of the definition of $H_{\sigma_1\sigma_2}$, Eq. (18), and Eq. (25) this term assumes the form

$$\frac{\sqrt{\pi}}{8k}\, i^{l_2+M_f-M_\sigma}\int_0^\infty r^4 D^4 (H_{\sigma_f\sigma}H_{\sigma\sigma_i})^{(e)}dz\,, \tag{49}$$

which is identical with the corresponding term, Eq. (26) of the first integral.

As to the second term of Eq. (48), it becomes, in consideration of the eikonal approximation of the Bessel functions, Eq. (13)

$$\frac{\sqrt{\pi}}{2}\, i^{l_2+M_f-M_\sigma-1}\int_0^\infty r_2 z_2 D^2(z_2)\langle\sigma_f e^{-iM_f\varphi}|V|e^{-iM_\sigma\varphi}\sigma\rangle\cos\alpha_{l_2}(k_f)$$
$$\times[e^{i\alpha_{l_2}(k_\sigma)}-(-1)^\Delta e^{-i\alpha_{l_2}(k_\sigma)}]dz_2\int_0^{z_2} r_1 z_1 D^2(z_1)H^{(e)}_{\sigma\sigma_i}dz_1\,, \tag{50}$$

the dominant term of which, for the same reason as above, is given by

$$\frac{\sqrt{\pi}}{2}\, i^{l_2+M_f-M_\sigma-1}\int_0^\infty r_2 z_2 D^2 H^{(o)}_{\sigma_f\sigma}dz_2\int_0^{z_2} r_1 z_1 D^2 H^{(e)}_{\sigma\sigma_i}dz_1\,. \tag{50$'$}$$

Hence, it follows that the contribution of the second integral in Eq. (12) to the amplitude $Rf^{(2)}_{\sigma_f\sigma\sigma_i}$ for even intermediate states is, according to Eqs. (49) and (50′), of the form

$$-\pi^{5/2} i^{M_f}\sum_{l_2}(2l_2+1)^{1/2}Y_{l_2}^{-M_f}(\theta_s\varphi_s)\left(2\int_0^\infty r^4D^4(H_{\sigma_f\sigma}H_{\sigma\sigma_i})^{(e)}dz-8ik\int_0^\infty r_2z_2D^2H^{(o)}_{\sigma_f\sigma}dz_2\int_0^{z_2}r_1z_1D^2H^{(e)}_{\sigma\sigma_i}dz_1\right). \tag{51}$$

2. ($L_\sigma - M_\sigma$) odd

Here we are in need of

$$j_{l_1}(k_\sigma r_1)n_{l_1}(k_i r_1)=\tfrac{1}{2}D^2\{\sin[\alpha_{l_1}(k_\sigma)+\alpha_{l_1}(k_i)]-\sin(k_\sigma-k_i)z\}. \tag{52}$$

The derivation of the amplitude in this case is very similar to the $(L_\sigma - M_\sigma)$-even case. It is easily verified that the second term of Eq. (52) gives rise to exactly the same term as in the even case, namely, the term given by (50′). As to the first term, after applying the stationary-phase method and summing over l_1 it becomes

$$-\frac{1}{4k}\int_0^\infty r_2^4 D^4\cos\alpha_{l_2}(k_f)f^{(L_2')}_{\sigma_f\sigma}\langle\sigma e^{-iM_\sigma\varphi}|V|\sigma_i\rangle S^{(2)}_{l_2L_2'}(z_2)dz_2\,, \tag{53}$$

where

$$S^{(2)}_{l_2L_2'}(z)=i\sum_{l_1}i^{l_1}C(l_2L_2'l_1;00)C(l_2L_2'l_1;\,-M_fM_f-M_\sigma)\sin\alpha_{l_1}(k_\sigma)\cos[\alpha_{l_1}(k_\sigma)+\alpha_{l_1}(k_i)]. \tag{54}$$

The eikonal approximation of this function is identical with the corresponding approximation of $C^{(1)}_{l_2L_2'}$, Eq. (21), with the following modifications: (i) instead of Δ appears $(\Delta+1)$, (ii) the positive sign of the first term is replaced by a negative sign. The proof follows the same lines as for the function $C^{(1)}_{l_2L_2'}$ (see Appendix A). Therefore Eq. (53) becomes after summing over L_2', and taking into consideration only the nonoscillatory terms

$$\frac{\sqrt{\pi}}{16k}\, i^{l_2+M_f-M_\sigma}\int_0^\infty r^4D^4\langle\sigma_f e^{-iM_f\varphi}|V|e^{-iM_\sigma\varphi}\sigma\rangle\langle\sigma e^{-M_\sigma\varphi}|V|\sigma_i\rangle[e^{-i(k_f-k_i)z}+(-1)^{\Delta+1}e^{i(k_f-k_i)z}]dz$$
$$=\frac{\sqrt{\pi}}{8k}\, i^{l_2+M_f-M_\sigma}\int_0^\infty r^4D^4(H_{\sigma_f\sigma}H_{\sigma\sigma_i})^{(e)}dz\,, \tag{55}$$

which is identical with Eq. (49), the corresponding term for even intermediate states. It follows that the contributions of the second integral to the amplitude are the same, irrespective of the "parity" of the intermediate states.

To summarize, we have evaluated in the eikonal approximation the second-order scattering amplitude $Rf^{(2)}_{\sigma_f\sigma\sigma_i}$ (which corresponds to the contribution of the real part of the propagator). According to Eq. (12), Eqs. (31) and (41), Eq. (51), and the conclusion after Eq. (55), it is determined by the expression

$$\left(\frac{\mu}{2\pi\hbar^2}\right)^2 Rf^{(2)}_{\sigma_f\sigma\sigma_i} = -\left(\frac{\mu}{\hbar^2}\right)^2 \sqrt{2\pi}\, i^{M_f} \sum_l (l+\tfrac{1}{2})^{1/2} Y_l^{-M_f}(\theta_s\varphi_s)$$

$$\times\left(\int_0^\infty r^4 D^4 (H_{\sigma_f\sigma}H_{\sigma\sigma_i})^{(e)}dz + 2ik\int_0^\infty r_2 z_2 D^2 H^{(e)}_{\sigma_f\sigma}dz_2 \int_{z_2}^\infty r_1 z_1 D^2 H^{(o)}_{\sigma\sigma_1} dz_1\right.$$

$$\left. -2ik\int_0^\infty r_2 z_2 D^2 H^{(o)}_{\sigma_f\sigma}dz_2\int_0^{z_2} r_1 z_1 D^2 H^{(e)}_{\sigma\sigma_i}dz_1\right), \tag{56}$$

where $H_{\sigma_1\sigma_2}$ is defined by Eq. (18), and D is the "amplitude" of the Bessel functions, Eq. (13).

III. THE MODIFIED SECOND-ORDER PHASE FUNCTION

Let us write the complete second-order scattering amplitude. It is given by the sum of Eq. (56) and the second-order modified Glauber amplitude of Paper I, Eq. (4.18) [which can be derived from the first term of Eq. (8)]. We find

$$\left(\frac{\mu}{2\pi\hbar^2}\right)^2 f^{(2)}_{\sigma_f\sigma\sigma_i} = -\frac{\sqrt{2\pi}}{k} i^{M_f+1}\sum_l (l+\tfrac{1}{2})^{1/2} Y_l^{-M_f}(\theta_s\varphi_s)$$

$$\times\left(-\tfrac{1}{2}\langle\sigma_f e^{-iM_f\varphi}|\Lambda^{(1)}_{k_f-k_\sigma}|e^{-iM_\sigma\varphi}\sigma\rangle\langle\sigma e^{-iM_\sigma\varphi}|\Lambda^{(1)}_{k_\sigma-k_i}|\sigma_i\rangle + i\phi^{(2)}_{\sigma_f\sigma\sigma_i}\right), \tag{57}$$

where $\Lambda^{(1)}_{\Delta k}\equiv\Lambda_{\Delta k}$ is the modified Glauber phase function introduced in Paper I, Eq. (5.19), and

$$\phi^{(2)}_{\sigma_f\sigma\sigma_i} = -k\left(\frac{\mu}{\hbar^2}\right)^2\left(\int_0^\infty r^4 D^4(H_{\sigma_f\sigma}H_{\sigma\sigma_i})^{(e)}dz + 2ik\int_0^\infty r_2 z_2 D^2 H^{(e)}_{\sigma_f\sigma}dz_2\int_{z_2}^\infty r_1 z_1 D^2 H^{(o)}_{\sigma\sigma_i}dz_1\right.$$

$$\left.-2ik\int_0^\infty r_2 z_2 D^2 H^{(o)}_{\sigma_f\sigma}dz_2\int_0^{z_2} r_1 z_1 D^2 H^{(e)}_{\sigma\sigma_i}dz_1\right). \tag{58}$$

Obviously, $\phi^{(2)}_{\sigma_f\sigma\sigma_i}$ is the second-order phase function for given initial, intermediate, and final states. It will be shown below that in case of a static potential it becomes twice the second-order WKB phase shift.

To continue with Eq. (58), we have to find the explicit expression of D, the "amplitude" of the Bessel functions, Eq. (13). The large-l, large-k expression of the spherical Hankel function, $h_l^{(1)} = j_l + in_l$, valid for the *entire* range $l \leq kr$ has been given by Watson[9]

$$\left(\frac{2l}{\pi\cos\beta}\right)^{1/2} h_l^{(1)}(l/\cos\beta) = \tfrac{2}{3}e^{il\gamma_l}\tan\beta[e^{-i\pi/3}J_{-1/3}(x_\beta) + e^{i\pi/3}J_{1/3}(x_\beta)], \tag{59}$$

where

$$\cos\beta = l/kr,\quad x_\beta = \tfrac{1}{3}l\tan^3\beta,$$

and

$$\gamma_l = \tan\beta - \tfrac{1}{3}\tan^3\beta - \beta. \tag{59'}$$

Now the phases γ_l and α_l of Eq. (13′) are related to each other by $l\gamma_l = \alpha_l + \pi/4 - x_\beta$. We therefore obtain

$$h_l^{(1)}(l/\cos\beta) = (D - iD_s)e^{i\alpha_l}, \tag{60}$$

where the amplitudes are of the form

$$D = \frac{1}{3}\left(\frac{2\pi\cos\beta}{l}\right)^{1/2}\tan\beta\left[J_{-1/3}(x_\beta)\cos\left(x_\beta+\frac{\pi}{12}\right) + J_{1/3}(x_\beta)\sin\left(x_\beta - \frac{\pi}{12}\right)\right],$$

$$D_s = \frac{1}{3}\left(\frac{2\pi\cos\beta}{l}\right)^{1/2}\tan\beta\left[J_{-1/3}(x_\beta)\sin\left(x_\beta+\frac{\pi}{12}\right) - J_{1/3}(x_\beta)\cos\left(x_\beta-\frac{\pi}{12}\right)\right]. \tag{61}$$

For $x_\beta > 1$ we get, making use of the Hankel approximation of the Bessel functions

$$J_\nu(x) = (2/\pi x)^{1/2}\cos(x - \nu\pi/2 - \pi/4),$$

$$D = \frac{\cos\beta}{l(\sin\beta)^{1/2}} = \frac{1}{k\sqrt{rz}}, \tag{62}$$

$$D_s = 0\,. \tag{62'}$$

According to Eq. (59′), $x_\beta = 1$ corresponds to $z = z_0$, where $z_0 = b(3/l)^{1/3}$, thus we have $z_0 \ll b$. Equation (62) means that for $z > z_0$, D_s is negligibly small and D simply becomes the Debye amplitude.[7] Now for $z < z_0$ it is still justified to neglect D_s in comparison to D. As to D itself, whereas the Debye amplitude diverges when $z \to 0$, the Watson amplitude of Eq. (61) remains finite. Because $J_\nu = (x/2)^\nu/\Gamma(1+\nu)$ for small x, we get, as is easily verified, at $z = 0$

$$D^2 = \frac{a}{k^2 b z_0}\,, \tag{63}$$

with

$$a = \frac{2^{2/3}\pi(1+\sqrt{3})^2}{12\Gamma^2(\frac{2}{3})}\,1.07 = 1.81\,. \tag{63'}$$

The factor 1.07 has been added to account for the small contribution of D_s. It is a simple matter to show that the derivative of $D^2(z)$ for small z is negative.

We are in need of a simple interpolation formula for $D^2(z)$, valid for all $z \geq 0$ which will enable us to continue with the evaluation of the phase function, Eq. (58). From the above analysis of D we infer that a very reasonable and simple expression of D^2 should be of the form

$$D^2(z) = \frac{1}{k^2 r[z^2 + (z_0/a)^2]^{1/2}}\,. \tag{64}$$

Obviously, for $z > z_0$ this expression rapidly approaches the Debye amplitude, and for $z \to 0$ it coincides with the Watson value at $z = 0$.

The results of the present paper are probably very insensitive to the exact form of the interpolation formula. In any case, it is easy to derive a more exact expression for $D(z)$, if necessary.

We shall now make use of Eq. (64) to simplify the expression of the phase function, Eq. (58). Let us first deal with the last two integrals: for them it is sufficient to use the Debye amplitude [putting $z_0 = 0$ in Eq. (64)]. On the other hand, in the first integral the above amplitude with $z_0 \neq 0$ is used. This integral is converted by integration by parts into the double integral

$$-\int_0^\infty \frac{d}{dr}[r^3(H_{\sigma_f\sigma}H_{\sigma\sigma_i})^{(e)}]\frac{z\,dz}{r}\int_0^z r' D^4(z')\,dz'\,,$$

the free term vanishes because $(H_{\sigma_f\sigma}H_{\sigma\sigma_i})$ decreases sufficiently fast at infinity. It follows that the phase function, Eq. (58) becomes

$$\begin{aligned}\phi^{(2)}_{\sigma_f\sigma\sigma_i} = -k\left(\frac{\mu}{\hbar^2}\right)^2\bigg(&-\int_0^\infty \frac{d}{dr}[r^3(H_{\sigma_f\sigma}H_{\sigma\sigma_i})^{(e)}]\frac{z\,dz}{r}\int_0^z r' D^4(z')\,dz' \\ &+\frac{2i}{k^3}\int_0^\infty H^{(e)}_{\sigma_f\sigma}dz_2\int_{z_2}^\infty H^{(o)}_{\sigma\sigma_i}dz_1 - \frac{2i}{k^3}\int_0^\infty H^{(o)}_{\sigma_f\sigma}dz_2\int_0^{z_2} H^{(e)}_{\sigma\sigma_i}dz_1\bigg)\,,\end{aligned} \tag{65}$$

with D^2 given by Eq. (64). Now we have

$$\int_0^z r' D^4\,dz' = \frac{a}{2k^4 b_0 z_0}\left(\pi - \tan^{-1}\frac{2b_0 r z(z_0/a)}{b_0^2 r^2 - b^4 - z^2(z_0/a)^2}\right)\,, \tag{66}$$

where $b_0 = [b^2 - (z_0/a)^2]^{1/2}$. The $\tan^{-1}$ function varies from π to $\tan^{-1}\{2b_0(z_0/a)/[b_0^2 - (z_0/a)^2]\}$ when z changes from 0 to ∞. In the course of the derivation of the eikonal approximation of the second-order phase function from the two integrals in Eq. (12) we neglected several terms. One term originates from the integral from 0 to b [the expressions of the Bessel functions, Eq. (13) are valid only for $r > b$]. Two other terms arise from those integrands in which appear functions proportional to $\cos 2\alpha_t$ and $\cos\alpha_t \cos 3\alpha_t$ [see Eqs. (21) and (22)]. Now a detailed analysis shows[10] that these additional terms cancel the first term on the right-hand side of Eq. (66). In conclusion, we are left with the following expression for the second-order phase function:

$$\begin{aligned}\phi^{(2)}_{\sigma_f\sigma\sigma_i} = -k\left(\frac{\mu}{\hbar^2}\right)^2\bigg(&\frac{a}{2k^4 b_0 z_0}\int_0^\infty \frac{d}{dr}[r^3(H_{\sigma_f\sigma}H_{\sigma\sigma_i})^{(e)}]\frac{z\,dz}{r}\tan^{-1}\frac{2b_0 r z(z_0/a)}{b_0^2 r^2 - b^4 - z^2(z_0/a)^2} \\ &+\frac{2i}{k^3}\int_0^\infty H^{(e)}_{\sigma_f\sigma}dz_2\int_{z_2}^\infty H^{(o)}_{\sigma\sigma_i}dz_1 - \frac{2i}{k^3}\int_0^\infty H^{(o)}_{\sigma_f\sigma}dz_2\int_0^{z_2} H^{(e)}_{\sigma\sigma_i}dz_1\bigg)\,.\end{aligned} \tag{67}$$

In Sec. IV we shall see that we cannot neglect the finiteness of z_0 in spite of the fact that $(z_0/a) \ll b$. Before concluding this section, let us apply the above expression to a spherical symmetric, static potential $V(r)$. In this case $H_{\sigma_f\sigma} = H_{\sigma\sigma_i} = V$, and V is an even function of z. Thus the last two terms in Eq. (67) are identically zero. The second-order phase function thus becomes

$$\phi^{(2)} = -\left(\frac{\mu}{\hbar^2}\right)^2 \frac{a}{2k^3 b_0 z_0} \int_0^\infty \frac{d}{dr}(r^3V^2) \tan^{-1}\left(\frac{2b_0 rz(z_0/a)}{b_0^2 r^2 - b^4 - z^2(z_0/a)^2}\right)\frac{z\,dz}{r}. \tag{68}$$

In the particular case of a static potential it is permissable to take advantage of the fact that $(z_0/a) \ll b$ by taking the limit $z_0 \to 0$. We then obtain

$$\phi^{(2)} = -\left(\frac{\mu}{\hbar^2}\right)^2 \frac{1}{k^3 b^2}\int_0^\infty \frac{d}{dr}(r^3V^2)\,dz\,, \tag{69}$$

which is exactly[11-13] the second-order WKB phase function (= twice the phase shift) for a potential function $V(r)$.

IV. THE SCATTERING AMPLITUDE

Our next task is to calculate the scattering amplitude as a function of the first- and second-order phase functions. Let us first concentrate on the pth-order partial-wave (fixed b) amplitude. It is composed of many terms, all of which are products of n_1 first-order and n_2 second-order phase functions. Obviously, we have $n_1 + 2n_2 = p$. Generally, these terms are of the form

$$\langle\sigma_f e^{-iM_f\varphi}|\Lambda^{(1)}_{k_f-k_{p-1}}|\sigma_{p-1}e^{-iM_{p-1}\varphi}\rangle\phi^{(2)}_{\sigma_{p-1}\sigma_{p-2}\sigma_{p-3}}\cdots\langle\sigma_m e^{-iM_m\varphi}|\Lambda^{(1)}_{k_m-k_{m-1}}|\sigma_{m-1}e^{-iM_{m-1}\varphi}\rangle\cdots\phi^{(2)}_{\sigma_l\sigma_{l-1}\sigma_{l-2}}$$
$$\cdots\langle\sigma_j e^{-iM_j\varphi}|\Lambda^{(1)}_{k_j-k_{j-1}}|\sigma_{j-1}e^{-iM_{j-1}\varphi}\rangle\cdots\phi^{(2)}_{\sigma_2\sigma_1\sigma_i}\,, \tag{70}$$

where $\phi^{(2)}_{\sigma_l\sigma_m\sigma_n}$ is the second-order phase function with fixed states given by Eq. (67), and $\Lambda^{(1)}_{k_1-k_2}$ is the first-order phase function introduced in Paper I (called the modified Glauber phase function)

$$\Lambda^{(1)}_{k_1-k_2} = -\frac{\mu}{\hbar^2 k}\int_{-\infty}^\infty V(\vec{b}+\hat{k}_i z;\vec{\zeta})e^{-i(k_1-k_2)z}\,dz. \tag{71}$$

The fact that each term of the amplitude reduces to a product of one- and two-dimensional integrals follows from a simple approximation which enables us to reduce multiple integrals of $(2n+1)$ dimensions into products of one-dimensional integrals. For further details the reader is referred to Paper I.

The above product, Eq. (70) has to be summed over all intermediate states. To do this, we introduce the average momentum $\bar{k}$ of the scattered particle by

$$\sum_{\sigma_\alpha} H_{\sigma_{\alpha+1}\sigma_\alpha}(\vec{b},z_{\alpha+1})H_{\sigma_\alpha\sigma_{\alpha-1}}(\vec{b},z_\alpha)$$
$$= \langle\sigma_{\alpha+1}e^{-iM_{\alpha+1}\varphi}|V(\vec{b}+\hat{k}_i z_{\alpha+1};\vec{\zeta})V(\vec{b}+\hat{k}_i z_\alpha;\vec{\zeta})|\sigma_{\alpha-1}e^{-iM_{\alpha-1}\varphi}\rangle e^{-i(k_{\alpha+1}-\bar{k})z_{\alpha+1}}e^{-i(\bar{k}-k_{\alpha-1})z_\alpha}. \tag{72}$$

This equation was first introduced in Paper I in the course of the derivation of the modified Glauber amplitude. Summed over all intermediate states, Eq. (70) thus becomes, making use of the closure relation

$$\langle\sigma_f e^{-iM_f\varphi}|\Lambda^{(1)}_{k_f-\bar{k}}\Lambda^{(2)n_2-1}_{0,0}\Lambda^{(1)p-2n_2-1}_0\Lambda^{(2)}_{0,\bar{k}-k_i}|\sigma_i\rangle\,, \tag{73}$$

where the second-order phase function is, according to Eq. (67), defined by

$$\Lambda^{(2)}_{k_2-\bar{k},\bar{k}-k_1} = -\frac{\mu^2}{2(\hbar^2\bar{k})^2}\left[\frac{a}{2\bar{k}b_0 z_0}\int_{-\infty}^\infty \frac{d}{dr}(r^3W_2W_1^*)\tan^{-1}\left(\frac{2b_0rz(z_0/a)}{b_0^2r^2-b^4-z^2(z_0/a)^2}\right)\frac{z\,dz}{r}\right.$$
$$\left. + i\int_{-\infty}^\infty W_1^*\,dz\int_{-z}^z W_2\,dz' - i\int_{-\infty}^\infty W_2\,dz\int_{-z}^z W_1^*\,dz'\right], \tag{74}$$

with the function W_α given by

$$W_\alpha = V(\vec{b}+\hat{k}_i z;\vec{\zeta})e^{-i(k_\alpha-\bar{k})z}. \tag{75}$$

Note that because of the exponential function in W_α, z_0 has to be kept finite in Eq. (74).

In addition to the term of Eq. (73) there are three other terms which contribute to the partial-wave amplitude of order p. They are

$$\langle\sigma_f e^{-iM_f\varphi}\,|\Lambda^{(2)}_{k_f-\bar{k},0}\Lambda^{(2)n_2-1}_{0,0}\Lambda^{(1)p-2n_2-1}_{0}\Lambda^{(1)}_{\bar{k}-k_i}\,|\sigma_i\rangle\,, \tag{76}$$

$$\langle\sigma_f e^{-iM_f\varphi}\,|\Lambda^{(1)}_{k_f-\bar{k}}\Lambda^{(2)n_2}_{0,0}\Lambda^{(1)p-2n_2-2}_{0}\Lambda^{(1)}_{\bar{k}-k_i}\,|\sigma_i\rangle\,, \tag{76$'$}$$

$$\langle\sigma_f e^{-iM_f\varphi}\,|\Lambda^{(2)}_{k_f-\bar{k},0}\Lambda^{(2)n_2-2}_{0,0}\Lambda^{(1)p-2n_2}_{0}\Lambda^{(2)}_{0,\bar{k}-k_i}\,|\sigma_i\rangle\,. \tag{76$''$}$$

Now the pth-order partial-wave amplitude is equal to the sum of these four terms, each multiplied by the appropriate statistical weight factor.

The partial-wave amplitude of order $p \geq 2$ with fixed n_2 thus will be of the form

$$\begin{aligned}T^{(p)}_{n_2}=\Big\langle\sigma_f e^{-iM_f\varphi}\Big|A_{p,n_2}\Big[&\binom{p-n_2-2}{n_2}\Lambda^{(1)}_{k_f-\bar{k}}\Lambda^{(2)n_2}_{0,0}\Lambda^{(1)p-2n_2-2}_{0}\Lambda^{(1)}_{\bar{k}-k_i}\\&+\binom{p-n_2-2}{n_2-1}\Big((\Lambda^{(1)}_{k_f-\bar{k}}\Lambda^{(2)n_2-1}_{0,0}\Lambda^{(1)p-2n_2-1}_{0}\Lambda^{(2)}_{0,\bar{k}-k_i}+\Lambda^{(2)}_{k_f-\bar{k},0}\Lambda^{(2)n_2-1}_{0,0}\Lambda^{(1)p-2n_2-1}_{0}\Lambda^{(1)}_{\bar{k}-k_i}\Big)\\&+\binom{p-n_2-2}{n_2-2}\Lambda^{(2)}_{k_f-\bar{k},0}\Lambda^{(2)n_2-2}_{0,0}\Lambda^{(1)p-2n_2}_{0}\Lambda^{(2)}_{0,\bar{k}-k_i}\Big]\Big|\sigma_i\Big\rangle,\end{aligned} \tag{77}$$

where A_{p,n_2} is an as yet undetermined numerical factor. To find it, let us make use of the pth-order partial-wave amplitude of a spherical symmetric potential in which only the first- and second-order phase functions are kept. We have

$$e^{i(\Lambda^{(1)}+\Lambda^{(2)})}-1=\sum_{p=1}^{\infty}\sum_{n_2=0}^{[p/2]}\frac{(i\Lambda^{(1)})^{p-2n_2}}{(p-2n_2)!}\frac{(i\Lambda^{(2)})^{n_2}}{n_2!}\,. \tag{78}$$

Putting $k_i=k_f=\bar{k}$, the expression inside the matrix element of Eq. (77) becomes a partial-wave, potential scattering amplitude of order p and fixed n_2. Comparison with Eq. (78) thus yields the factor A_{p,n_2}. As

$$\sum_{s=0}^{2}\binom{2}{s}\binom{p-n_2-2}{n_2-s}=\binom{p-n_2}{n_2}\,, \tag{79}$$

we get

$$A_{p,n_2}=\frac{i^{p-n_2}}{(p-n_2)!}\,. \tag{80}$$

Next, let us perform the summation over all orders p, and all possible values of n_2. According to Eq. (77) we are in need of the three series $S^{(j)}_{n_2}$, $j=1,2,3$, for $n_2 \geq j-1$,

$$S^{(j)}_{n_2}=\sum_{p=2n_2-j+3}^{\infty}\frac{1}{(p-n_2)!}\binom{p-n_2-2}{n_2-j+1}x^p=\frac{x^{2n_2-j+3}}{(n_2-j+1)}\sum_{p'=0}^{\infty}\frac{x^{p'}}{p'!\,(p'+n_2-j+2)(p'+n_2-j+3)}\,. \tag{81}$$

Defining the function, for $n \geq 0$

$$F_n(x)=\sum_{p'=0}^{\infty}\frac{x^{p'+n}}{(p'+n+2)!}\,, \tag{82}$$

we get

$$S^{(j)}_{n_2}(x)=\frac{x^{2n_2-j+3}}{(n_2-j+1)!}\frac{d^{n_2-j+1}}{dx^{n_2-j+1}}F_{n_2-j+1}(x)\,. \tag{83}$$

Thus the partial-wave amplitude, summed over all orders $p \geq 2$, with fixed n_2 becomes

$$\begin{aligned}T_{n_2}=\Big\langle\sigma_f e^{-iM_f\varphi}\Big|&i\Lambda^{(1)}_{k_f-\bar{k}}i\Lambda^{(1)}_{\bar{k}-k_i}\frac{1}{n_2!}\Big(i\Lambda^{(2)}_{0,0}\frac{d}{dx}\Big)^{n_2}F_{n_2}+(i\Lambda^{(1)}_{k_f-\bar{k}}i\Lambda^{(2)}_{0,\bar{k}-k_i}+i\Lambda^{(2)}_{k_f-\bar{k},0}i\Lambda^{(1)}_{\bar{k}-k_i})\frac{1}{(n_2-1)!}\Big(i\Lambda^{(2)}_{0,0}\frac{d}{dx}\Big)^{n_2-1}F_{n_2-1}\\&+i\Lambda^{(2)}_{k_f-\bar{k},0}i\Lambda^{(2)}_{0,\bar{k}-k_i}\frac{1}{(n_2-2)!}\Big(i\Lambda^{(2)}_{0,0}\frac{d}{dx}\Big)^{n_2-2}F_{n_2-2}\Big|\sigma_i\Big\rangle,\end{aligned} \tag{84}$$

where the derivatives are evaluated at $x=i\Lambda_0^{(1)}$. Now by Eq. (82), the function F_n is given by

$$F_n(x)=\frac{1}{x^2}\left(e^x-\sum_{l=0}^{n+1}\frac{x^l}{l!}\right), \tag{85}$$

and therefore, the nth derivative is

$$F_n^{(n)}(x)=\frac{d^n}{dx^n}f(x), \tag{86}$$

with

$$f(x)=\frac{1}{x^2}(e^x-x-1). \tag{86'}$$

From this it follows that the expression of Eq. (84) can be summed formally over all values of n_2. We obtain for the partial-wave amplitude

$$T'=\left\langle \sigma_f e^{-iM_f\varphi}\left|(i\Lambda^{(1)}_{k_f-\bar{k}}+i\Lambda^{(2)}_{k_f-\bar{k},0})(i\Lambda^{(1)}_{\bar{k}-k_i}+i\Lambda^{(2)}_{0,\bar{k}-k_i})\exp\left(i\Lambda^{(2)}_{0,0}\frac{d}{dx}\right)f(x)\right|\sigma_i\right\rangle. \tag{87}$$

As

$$\exp\left(i\Lambda^{(2)}_{0,0}\frac{d}{dx}\right)f(x)=f(i\Lambda_0^{(1)}+i\Lambda^{(2)}_{0,0}) \tag{88}$$

T' becomes, making use of the explicit expression of $f(x)$, Eq. (86′)

$$T'=\langle\sigma_f e^{-iM_f\varphi}\,|\,\alpha^{(2)}_{k_f-k_i}[e^{i(\Lambda_0^{(1)}+\Lambda_{0,0}^{(2)})}-i(\Lambda_0^{(1)}+\Lambda^{(2)}_{0,0})-1]\,|\,\sigma_i\rangle \tag{89}$$

with

$$\alpha^{(2)}_{k_2-k_1}=\frac{(\Lambda^{(1)}_{k_2-\bar{k}}+\Lambda^{(2)}_{k_2-\bar{k},0})(\Lambda^{(1)}_{\bar{k}-k_1}+\Lambda^{(2)}_{0,\bar{k}-k_1})}{(\Lambda_0^{(1)}+\Lambda^{(2)}_{0,0})^2}. \tag{89'}$$

As can be seen from Eq. (77) the above expression includes neither the first-order amplitude, nor the second-order amplitude of Eq. (56). As we have seen the latter is directly related to the second-order phase function $\Lambda^{(2)}_{k_f-\bar{k},\bar{k}-k_i}$. These two terms have therefore to be added to the above expression. Hence the complete partial-wave amplitude is of the form

$$T_{\sigma_f\sigma_i}[(l+\tfrac{1}{2})/\bar{k}]=\langle\sigma_f e^{-iM_f\varphi}\,|\,i(\Lambda^{(1)}_{k_f-k_i}+\Lambda^{(2)}_{k_f-\bar{k},\bar{k}-k_i})+\alpha^{(2)}_{k_f-k_i}[e^{i(\Lambda_0^{(1)}+\Lambda_{0,0}^{(2)})}-i(\Lambda_0^{(1)}+\Lambda^{(2)}_{0,0})-1]\,|\,\sigma_i\rangle. \tag{90}$$

This matrix element replaces the matrix element which appears in the expression of the modified Glauber amplitude. Our new amplitude thus becomes (see Appendix of Ref. 5)

$$f_{\sigma_f\sigma_i}(\vec{k}_f,\vec{k}_i)=-i^{M_f+1}\left(\frac{2\pi}{k_fk_i}\right)^{1/2}\sum_l(l+\tfrac{1}{2})^{1/2}Y_l^{-M_l}(\theta_s,\varphi_s)\,T_{\sigma_f\sigma_i}[(l+\tfrac{1}{2})/\bar{k}]. \tag{91}$$

This expression can be converted into an integral over the impact parameter $b=(l+\frac{1}{2})/\bar{k}$, making use of the connection between the Bessel function and spherical harmonics

$$Y_l^{-M}(\theta,\varphi)=\left(\frac{l+\frac{1}{2}}{2\pi}\right)^{1/2}e^{-iM\varphi}J_M(bQ_\perp),$$

valid for large l, and small scattering angles (for more details see Paper I). Hence

$$f_{\sigma_f\sigma_i}(\vec{k}_f,\vec{k}_i)=-i^{M_f+1}(k_fk_i)^{1/2}e^{-iM_f\varphi}\int_0^\infty b\,db\,J_{M_f}(bQ_\perp)\,T_{\sigma_f\sigma_i}(b), \tag{92}$$

$Q_\perp$ being the component of the momentum transfer perpendicular to the direction of the initial beam $\vec{k}_i$.

The above amplitude is a generalization of the modified Glauber amplitude[4,5] in which the second-order phase function has been included consistently to all orders of the interaction. Indeed, putting the second-order phase function equal to zero, the above amplitude reduces to the modified Glauber amplitude which in turn reduces to the conventional Glauber amplitude[14] by putting $k_f=k_i=\bar{k}$. Other amplitudes discussed in the literature are also special cases of our amplitude. For instance, putting $\alpha^{(2)}_{k_f-k_i}=0$ we get the amplitude discussed in Ref. 3, or, expanding the exponential function and keeping only four terms proportional to $\Lambda^{(1)j}$, $j=1,2,3$, and $\Lambda^{(2)}$ in the matrix element reproduces essentially the eikonal-Born-series method.[15]

Like the Glauber theory the present theory is valid for short wave-lengths and small scattering angles. The range of validity is expected to be about $ka>1$, and $(1-\cos\vartheta)<1/ka$, where a is of the order of the dimension of the atom (for more details, see Paper I).

It is of some advantage to replace $Q_\perp$ by Q in Eq. (92). There are two reasons for this.

(i) It is expected that the angular range as well as the accuracy of the amplitude will be increased somewhat, though the above inequalities will still apply. For inelastic scattering, however, a small lower limit $\theta_{\rm min}$ on the angle is introduced because $Q_\perp \to 0$ when $\theta \to 0$, whereas Q remains finite. We find that $\theta^2 > \theta^2_{\rm min}$, where

$$\theta_{\rm min} = \frac{\mu(E_f - E_0)}{k^2}\,, \tag{93}$$

E_0 and E_f being the energies of the ground state and final state of the atom. For $\theta^2 < \theta^2_{\rm min}$, $Q_\perp$ should be used in the expression of the amplitude, Eq. (92). It is quite possible that for elastic and certain inelastic scattering the angular range of validity goes beyond the theoretical limit mentioned above. Only comparison with experimental data will tell how good the theory is at large angles.

(ii) The amplitude becomes symmetric under the interchange of the initial and final states. It is an easy matter to prove that the theorem of reversibility

$$f_{\sigma_f\sigma_i}(\vec{k}_f, \vec{k}_i) = \epsilon_{fi} f_{(-)\sigma_i,(-)\sigma_f}(-\vec{k}_i, -\vec{k}_f) \tag{94}$$

is satisfied. Here $(-)$ means that the signs of the magnetic quantum numbers have been reversed; the phase ϵ_{fi} turns out to be equal ± 1, according to whether $(L_f - M_f)$ is even or odd. In other words, the direct and the time-reversed cross sections are equal.

It should be emphasized that our amplitude satisfies the unitarity theorem. This follows immediately from the proof given in the Appendix of Ref. 5, together with the relations

$$\alpha^{(2)}_{k_f-\bar{k}}(\Lambda^{(1)}_{\bar{k}-k_i} + \Lambda^{(2)}_{0,\bar{k}-k_i}) = (\Lambda^{(1)}_{k_f-\bar{k}} + \Lambda^{(2)}_{k_f-\bar{k},0})\,\alpha^{(2)}_{\bar{k}-k_i}$$

$$= \alpha^{(2)}_{k_f-k_i}(\Lambda^{(1)}_0 + \Lambda^{(2)}_{0,0}) \tag{95}$$

and

$$\alpha^{(2)}_{k_f-\bar{k}}\,\alpha^{(2)}_{\bar{k}-k_i} = \alpha^{(2)}_{k_f-k_i}\,, \tag{95$'$}$$

which are a direct consequence of the definition of $\alpha^{(2)}_{k_2-k_1}$. The proof applies whether one puts $Q_\perp$ or Q in Eq. (92). But in the latter case, Q is also used for $\theta < \theta_{\rm min}$.

There is room for improvement of the present theory. One point is that in introducing the average momentum $\bar{k}$ [see Eq. (72)] the summation was performed over all intermediate states. A more accurate result should be obtained by separating out the elastic intermediate state. This is, in principle, feasible, but the final expression will be less elegant. Another point is that it is not difficult to generalize the derivation of the scattering amplitude by adding phase functions of the third, fourth, etc., order. For instance, the third-order phase function $\Lambda^{(3)}$ can be derived from the third-order amplitude, in the same way as $\Lambda^{(2)}$ is derived from $f^{(2)}$ in Sec. II.

To summarize, we have found an amplitude in the eikonal approximation which is unitary to all orders of the interaction. It is free of *all* the short-comings of the conventional Glauber theory discussed in the Introduction.

Our amplitude is expected to be better than $(f^{(\rm MG)} + {\rm Re}\, f^{(\rm B2)})$, which was examined in Ref. 5 for the special case of elastic e-H scattering.

Application of the present theory to the calculation of differential cross sections and elements of the alignment tensor and orientation vector for electron-atom collisions will be reported in a forthcoming communication.

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APPENDIX A

In this appendix we derive the explicit expression of $C^{(1)}_{l_2L_2}(z)$ for large l_2 defined by Eq. (20). The result is given by Eq. (21). As $l_2 \gg L_2$, we have $l_1 \sim l_2$; we therefore expand the phases $\alpha_{l_1}(k)$ about l_2. As $\partial\alpha_l/\partial l = -\beta$ with $\cos\beta = l/kr$, we have $\alpha_{l_1} = \alpha_{l_2} - (l_1 - l_2)\beta$. Now because of the triangular relation we have $l_1 = l_2 + \mu$ with $\mu = -L_2, \ldots, L_2$, and $\Delta\mu = 2$, the latter because of parity conservation. The function of Eq. (20) thus becomes

$$C^{(1)}_{l_2L_2} = i^{l_2} \sum_{\mu=-L_2}^{L_2} i^{\mu}\, C(l_2L_2l_2+\mu; 00)\, C(l_2L_2l_2+\mu; -M_fM_f-M_\sigma) \cos[\alpha_{l_2}(k_\sigma) - \mu\beta] \cos[\alpha_{l_2}(k_\sigma) + \alpha_{l_2}(k_i) - 2\mu\beta]\,, \tag{A1}$$

which by simple algebra assumes the form

$$C^{(1)}_{l_2L_2}=\tfrac{1}{2}i^{l_2}\sum_\mu i^\mu C(l_2L_2l_2+\mu;00)C(l_2L_2l_2+\mu;-M_fM_f-M_\sigma)$$
$$\times\{\cos\alpha_{l_2}(k_i)\cos\mu\beta+\cos[2\alpha_{l_2}(k_\sigma)+\alpha_{l_2}(k_i)]\cos3\mu\beta$$
$$+\sin\alpha_{l_2}(k_i)\sin\mu\beta+\sin[2\alpha_{l_2}(k_\sigma)+\alpha_{l_2}(k_i)]\sin3\mu\beta\}. \quad \text{(A1}'\text{)}$$

In Appendix C it is shown that for $l_2 \gg L_2$ the Clebsch-Gordan coefficients $C(l_2L_2l_2+\mu;m_2M_2)$ are even (odd) functions of μ if (L_2-M_2) is even (odd). Call $\Delta=L_2-(M_f-M_\sigma)$, then we distinguish between two cases, remembering that $i^\mu C(l_2L_2l_2+\mu;00)$ is always an even function of μ.

i. Δ *even*. The function $C^{(1)}_{l_2L_2}$ is given by the first two terms of Eq. (A1′). In Appendix B it is shown that the two sums over μ are proportional to the spherical harmonics $[Y^{M_f-M_\sigma}_{L_2}(\theta,\varphi)]^*$ and $[Y^{M_f-M_\sigma}_{L_2}(3\theta,\varphi)]^*$, where $\theta=\frac{1}{2}\pi-\beta$. We thus end up with the expression

$$C^{(1)}_{l_2L_2}=i^{l_2+M_f-M_\sigma}\left(\frac{\pi}{2L_2+1}\right)^{1/2}e^{i(M_f-M_\sigma)\varphi}\{\cos\alpha_{l_2}(k_i)[Y^{M_f-M_\sigma}_{L_2}(\theta,\varphi)]^*$$
$$+(-1)^{L_2}\cos[2\alpha_{l_2}(k_\sigma)+\alpha_{l_2}(k_i)][Y^{M_f-M_\sigma}_{L_2}(3\theta,\varphi)]^*\}. \quad \text{(A2)}$$

ii. Δ *odd*. The function $C^{(1)}_{l_2L_2}$ is given by the last two terms of Eq. (A1′). Again the two sums over μ are proportional to $[Y^{(M_f-M_\sigma}_{L_2}(\theta,\varphi)]^*$ and $[Y^{M_f-M_\sigma}_{L_2}(3\theta,\varphi)]^*$, and $C^{(1)}_{l_2L_2}$ is accordingly

$$C^{(1)}_{l_2L_2}=i^{l_2+M_f-M_\sigma+1}\left(\frac{\pi}{2L_2+1}\right)^{1/2}e^{i(M_f-M_\sigma)\varphi}\{\sin\alpha_{l_2}(k_i)[Y^{(M_f-M_\sigma}_{L_2}(\theta,\varphi)]^*$$
$$+(-1)^{L_2}\sin[2\alpha_{l_2}(k_\sigma)+\alpha_{l_2}(k_i)][Y^{M_f-M_\sigma}_{L_2}(3\theta,\varphi)]^*\}. \quad \text{(A3)}$$

The two Eqs. (A2) and (A3) complete the proof of Eq. (21). The derivation of the other three functions $S^{(1)}_{l_2L_2}$, $C^{(2)}_{l_2L_2}$, and $S^{(2)}_{l_2L_2}$ defined by Eqs. (36), (46), and (54), respectively, follows exactly the same lines as above.

APPENDIX B

In this appendix we derive an exact expression for the spherical harmonics in terms of Clebsch-Gordan (CG) coefficients. We start with the coupling rule of the spherical harmonics[16]

$$Y^{m_1}_{l_1}Y^{m_2}_{l_2}=\sum_l\left(\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}\right)^{1/2}C(l_1l_2l;00)C(l_1l_2l;m_1m_2)Y^{m_1+m_2}_l. \quad \text{(B1)}$$

We assume that $l_1\gg l_2$ and $l_1\gg m_1+m_2$. The functions Y^m_l in terms of the associate Legendre polynomials are given by

$$Y^m_l(\theta,\varphi)=\left(\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right)^{1/2}e^{im\varphi}P^m_l(\cos\theta), \quad \text{(B2)}$$

and the asymptotic expansion of P^m_l for $l\gg1$ and $l\gg m$ is[17]

$$P^m_l(\cos\theta)=l^m\left(\frac{2}{\pi l\sin\theta}\right)^{1/2}\cos\left((l+\tfrac{1}{2})\theta-\frac{\pi}{4}-m\frac{\pi}{2}\right). \quad \text{(B3)}$$

Hence Eq. (B1) becomes

$$\cos\left((l_1+\tfrac{1}{2})\theta-\frac{\pi}{4}\right)Y^{m_2}_{l_2}(\theta,\varphi)$$
$$=\left(\frac{2l_2+1}{4\pi}\right)^{1/2}e^{im_2\varphi}\lim_{l_1\to\infty}\sum_{\mu=-l_2}^{l_2}C(l_1l_2l_1+\mu;00)C(l_1l_2l_1+\mu;m_1m_2)$$
$$\times\left[\cos\left((l_1+\tfrac{1}{2})\theta-\frac{\pi}{4}\right)\cos\left(\mu\theta-m_2\frac{\pi}{2}\right)+\sin\left((l_1+\tfrac{1}{2})\theta-\frac{\pi}{4}\right)\sin\left(\mu\theta-m_2\frac{\pi}{2}\right)\right]. \quad \text{(B4)}$$

We show in Appendix C that for $l_1\gg l_2$ the CG coefficients $C(l_1l_2l_1+\mu;m_1m_2)$ are even (odd) functions of μ if $\Delta=l_2-m_2$ is even (odd). Therefore, for even values of m_2 the product of the two CG coefficients in the above expression is an even function of μ, and for odd values of m_2 the product is an odd function of μ. From this it follows that the product of the two CG coefficients multiplied by $\sin(\mu\theta-m_2\pi/2)$ is odd in μ, irrespective of the parity of m_2. The second term in Eq. (B4) is, therefore, identically zero. In conclusion, the spherical harmonics in terms of CG coefficients are given by the expression

$$Y_{l_2}^{m_2}(\theta,\varphi)=\left(\frac{2l_2+1}{4\pi}\right)^{1/2} e^{im_2\varphi}\lim_{l_1\to\infty}\sum_{\mu=-l_2}^{l_2} C(l_1l_2l_1+\mu;\,00)\,C(l_1l_2l_1+\mu;\,0m_2)\cos[\mu\theta-m_2(\tfrac{1}{2}\pi)]\,. \tag{B5}$$

APPENDIX C

In this appendix we derive the explicit expression of the CG coefficients $C(l_1l_2l;\,0m_2)$ for $l_1 \gg l_2$. According to Eq. (B5) the dependence of the spherical harmonics on the angle θ is through $\cos[\mu\theta-m_2(\frac{1}{2}\pi)]$. To find an expansion of this kind with known coefficients we start with the definition[18] of the Gegenbauer functions $C_n^\nu(x)$

$$(1-2\alpha x+\alpha^2)^{-p/2}=\sum_{n=0}^{\infty}C_n^\nu(x)\alpha^n\,. \tag{C1}$$

Put $2x=e^{i\theta}+e^{-i\theta}$, then

$$(1-2\alpha x+\alpha^2)^{-p/2}=(1-\alpha e^{i\theta})^{-p/2}(1-\alpha e^{-i\theta})^{-p/2}=\sum_{\nu=0}^{\infty}\binom{-p/2}{\nu}(-\alpha e^{i\theta})^{\nu}\sum_{\nu'=0}^{\infty}\binom{-p/2}{\nu'}(-\alpha e^{-i\theta})^{\nu'}. \tag{C2}$$

Comparison with the rhs of Eq. (C1) gives

$$C_n^{p/2}(\cos\theta)=(-1)^n\sum_{k=0}^{n}\binom{-p/2}{k}\binom{-p/2}{n-k}e^{-i(n-2k)\theta}. \tag{C3}$$

We now make use of the relation[18] between the Gegenbauer functions and the associate Legendre functions $P_l^m(\cos\theta), m\geq 0$:

$$C_{l-m}^{m+1/2}(\cos\theta)=\frac{2^m m!}{(2m)!}\sin^{-m}\theta\, P_l^m(\cos\theta)\,. \tag{C4}$$

The functions P_l^m are thus given by

$$P_l^m(\cos\theta)=(-1)^{l-m}\frac{(2m)!}{2^m m!}\sin^m\theta\sum_{k=0}^{l-m}\binom{-(m+\frac{1}{2})}{k}\binom{-(m+\frac{1}{2})}{l-m-k}e^{-i(l-m-2k)\theta}. \tag{C5}$$

By some simple algebra this expression becomes

$$P_l^m(\cos\theta)=(-1)^{l-m}\frac{(2m)!}{4^m m!}\sum_{\mu=-l}^{l}C_{(1/2)(l-\mu)}^{(ml)}\cos\left(\mu\theta-m\frac{\pi}{2}\right), \tag{C6}$$

where

$$C_p^{(ml)}=\sum_{n=0}^{l}(-1)^n\binom{m}{n}\binom{-(m+\frac{1}{2})}{p-n}\binom{-(m+\frac{1}{2})}{l-m-p+n}. \tag{C6$'$}$$

For $m=0$, this gives the well known[19] expansion of the Legendre function

$$P_l(\cos\theta)=(-1)^l\sum_{p=0}^{l}\binom{-1/2}{p}\binom{-1/2}{l-p}\cos(l-2p)\theta.$$

Now comparison of Eq. (C6) with Eq. (B4) reveals that the second term of Eq. (B4) must be identically zero, i.e.,

$$\sum_{\mu=-l_2}^{l_2}C(l_1l_2l_1+\mu;\,00)\,C(l_1l_2l_1+\mu;\,0m_2)\sin(\mu\theta-m_2\pi/2)=0\,. \tag{C7}$$

From this we conclude that the product of the two CG coefficients is an even (odd) function of μ if m_2 is even (odd). Furthermore, in conjunction with Eq. (B2) we obtain an explicit expression for the product of the two CG coefficients in *the limit of large* l_1:

$$C(l_1ll_1+\mu;\,00)\,C(l_1ll_1+\mu;\,0m)=(-1)^{l-m}\left(\frac{(l-m)!}{(l+m)!}\right)^{1/2}\frac{(2m)!}{4^m m!}C_{(1/2)(l-\mu)}^{(ml)}\,. \tag{C8}$$

For $m=0$, we find

$$[C(l_1ll_1+\mu;\,00)]^2=(-1)^l\binom{-\frac{1}{2}}{\frac{1}{2}(l-\mu)}\binom{-\frac{1}{2}}{\frac{1}{2}(l+\mu)}\,. \tag{C9}$$

To exhibit the symmetry property of the CG coefficients mentioned above we write $C^{(ml)}_{(1/2)(l-\mu)}$ of Eq. (C6′) in two different ways:

(i) Put $n = n' - \frac{1}{2}(l+\mu)$, then

$$C^{(ml)}_{(1/2)(l-\mu)} = (-1)^{(1/2)(l+\mu)} \sum_{n=m}^{l} (-1)^n \binom{m}{n-\frac{1}{2}(l+\mu)} \binom{-(m+\frac{1}{2})}{l-n} \binom{-(m+\frac{1}{2})}{n-m}. \tag{C10}$$

(ii) Put $n = m + \frac{1}{2}(l-\mu) - n''$, then

$$C^{(ml)}_{(1/2)(l-\mu)} = (-1)^{(1/2)(l-\mu)+m} \sum_{n=m}^{l} (-1)^n \binom{m}{n-\frac{1}{2}(l-\mu)} \binom{-(m+\frac{1}{2})}{l-n} \binom{-(m+\frac{1}{2})}{n-m}. \tag{C10′}$$

Consequently, the product of the two CG coefficients of Eq. (C8) for large l_1 and $m \geq 0$ is of the form

$$C(l_1 l l_1+\mu; 00)C(l_1 l l_1+\mu; 0m) = \frac{(2m)!}{2^{2m+1}m!}\left(\frac{(l-m)!}{(l+m)!}\right)^{1/2}(-1)^{(1/2)(l-\mu)-m}$$

$$\times \sum_{n=m}^{l} (-1)^n \left[\binom{m}{n-\frac{1}{2}(l+\mu)} + (-1)^{l-m}\binom{m}{n-\frac{1}{2}(l-\mu)}\right] \binom{-(m+\frac{1}{2})}{l-n} \binom{-(m+\frac{1}{2})}{n-m}. \tag{C11}$$

For $m<0$, one gets identical results with m replaced by $-m$. This expression shows very clearly that the product of the two CG coefficients is an even (odd) function of μ if m is even (odd). The above analysis does not reveal whether $C(l_1 l l_1+\mu; 0m)$ is even or odd under the transformation $\mu \to -\mu$. A more thorough investigation shows that (for large l_1') it is even (odd) if $(l-m)$ is even (odd). This, of course, is in compliance with the results of Eqs. (C11) and (C9). The analysis presented here is adequate for the purpose of the present paper as we deal here only with products of two CG coefficients.

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