Exact solution of the initial eigenvalue problem for coherent pulse propagation

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The inverse scattering method is used to investigate the resonant propagation of an optical pulse assumed to be initially unchirped and to have a hyperbolic-secant shape. An exact solution is found for the two-component initial eigenvalue problem. The discrete spectrum consists of a finite set of purely imaginary, equidistant values. The number of discrete eigenvalues (number of solitons) is determined by the initial pulse area A. The scattering amplitudes are expressed in terms of Euler gamma functions and hyperbolic functions. In addition to poles in the upper half-plane, the function $\bar{b}/a$ has a pole on the real axis for $A = (2n + 1)\pi$.

I. INTRODUCTION

The one soliton solution of the equations for coherent resonant propagation had been found by McCall and Hahn[1,2] before the optical equations were recognized to be completely integrable. This solution has the form of a hyperbolic secant and describes satisfactorily the main features of self-induced transparency.

With the development of soliton theory, the Bäcklund transformations and the inverse scattering method ISM have been applied to optical equations by Lamb.[3-6] He found the one-, two- and three-soliton solutions and derived a set of conservation laws corresponding to these equations. The use of the two-component eigenvalue problem[7] allowed the inclusion of chirping. The formulas for N-soliton solutions have also been given.[8] Ablowitz, Kaup and Newell[9] solved the general initial value problem for optical propagation, i.e., they showed how to construct the solution from a given initial pulse profile. The complete solution contains solitons and a decaying part of the field referred by the authors as radiation. Their approach can be used to study transient effects as well as the steady-state pulse propagation. Kaup[10] applied the ISM to a detailed analytical and numerical study of optical propagation and compared the results with earlier methods.[2,11]

The first difficulty which is encountered when applying the ISM to a specific propagation problem is how to determine the initial scattering data (sd). In the early papers by Lamb the discrete eigenvalues were not calculated, but introduced as arbitrary parameters in the solution. Later a method of evaluation of soliton parameters from conservation laws has been proposed.[12,13] Some information on the sd can be obtained directly from the structure of the equations.[7,14] The WKB method can also be used[7,10] to find approximate solutions of the eigenvalue problem. For the square profile the eigenvalue equations can be solved exactly,[10] but discrete eigenvalues are determined in an implicit way through a transcendental equation.

The aim of the present paper is to study the initial eigenvalue problem for an entering pulse of hyperbolic secant profile and arbitrarily large amplitude. The scattering data obtained in this paper can be used to investigate particular propagation problems, to find the explicit form of purely soliton solutions, and to determine the evolution of pulse spectral moments.[11,15,16]

In Sec. II the equations of motion are derived in a form which is particularly suitable for treatment by the inverse scattering method. The scattering data for these equations are defined. The initial eigenvalue problem is solved in Sec. III. The solution is modeled on an analogous problem for the Schrödinger equation. Explicit analytic expressions for the discrete spectrum and the scattering amplitudes are found. A detailed analysis of the asymptotic properties of solutions is given in Appendix B.

In Sec. IV the evolution of the scattering data on and off resonance is determined. The scattering data are used to find the precise form of one- and two-soliton solutions. Further optical applications will be the subject of a forthcoming paper.[16]

II. RESONANT PROPAGATION AS AN INVERSE SCATTERING PROBLEM

To fix the notation and make clear the underlying assumptions I give a short derivation of the equations of motion. Assuming that the optical pulses are short and the interaction of atoms with the field is purely coherent the Schrödinger equation can be used to describe the state of atoms. Relaxation effects can be treated as perturbations to the solutions of purely coherent equations.[10]

The Hamiltonian for two-level atoms in the plane-wave optical field $\vec{E}(t,z)$ is[17]

$$H = H_0 - \vec{d}\cdot\vec{E}\,, \tag{2.1}$$

where H_0 is the Hamiltonian for a free two-level system and $\vec{d}$ is the dipole operator. In the Pauli matrix representation

$$H_0 = -\tfrac{1}{2}\hbar\omega\sigma_3, \quad \vec{d} = \vec{P}_r\sigma_1 + \vec{P}_i\sigma_2 , \tag{2.2}$$

and the Schrödinger equation reads

$$i\hbar\partial_t C_1 = -\tfrac{1}{2}\hbar\omega C_1 - \vec{E}\cdot\vec{P}^* C_2 , \tag{2.3a}$$

$$i\hbar\partial_t C_2 = \tfrac{1}{2}\hbar\omega C_2 - \vec{E}\cdot\vec{P} C_1 , \tag{2.3b}$$

where C_1 and C_2 are the amplitudes, respectively, of the ground state (energy equal to $-\frac{1}{2}\hbar\omega$) and the excited state (energy equal to $\frac{1}{2}\hbar\omega$), $\vec{P} = \vec{P}_r + i\vec{P}_i$. Assuming that the pulse is quasimonochromatic and circularly polarized,

$$\vec{E} = \mathrm{Re}(\hat{e}\mathcal{E}e^{-i\chi}) ,$$

where

$$\chi = \omega_0 t - k_0 z + \phi(t,z), \quad \hat{e} = \tfrac{1}{2}(\hat{x} + i\hat{y}) ,$$

one has

$$\vec{E}\cdot\vec{P} = \tfrac{1}{2}\mathcal{P}\mathcal{E}e^{-i\chi} ,$$

where $\mathcal{P} = \hat{e}\cdot\vec{d}_{12}$ and $\vec{d}_{12}$ denotes the dipole transition matrix element in the two-level system. $\mathcal{P}$ is assumed to be real. Equations (2.3) can be given the form[18,19] of the Zakharov-Shabat[7] two-component eigenvalue problem. Defining

$$v_1 = C_1 e^{-i(\chi_0 - \pi)/2} , \tag{2.4a}$$

$$v_2 = C_2 e^{i\chi_0/2} , \tag{2.4b}$$

where $\chi_0 = \omega_0 t - k_0 z$, one finds

$$\partial_t v_1 + i\zeta v_1 = \tfrac{1}{2}\overline{\mathcal{E}} v_2 , \tag{2.5a}$$

$$\partial_t v_2 - i\zeta v_2 = -\tfrac{1}{2}\overline{\mathcal{E}}^* v_1 , \tag{2.5b}$$

where the complex envelope of the pulse $\overline{\mathcal{E}} = \mathcal{P}\mathcal{E}e^{i\phi}/\hbar$, $\zeta = \frac{1}{2}(\omega_0 - \omega)$. Equations (2.5) are identical up to sign with Eqs. (8) in Lamb's paper[5] derived on a different way. Equations (2.5) are more suitable for treatment by the ISM than similar equations derived by Haus[19] where the field $\overline{\mathcal{E}}$ is not real in the absence of phase modulation.

The component of macroscopic polarization which resonates with the field is

$$\hat{e}\cdot\vec{P}_{\mathrm{macro}} = 2in_0\mathcal{P}\langle v_1 v_2^* e^{i\chi_0}\rangle , \tag{2.6}$$

where n_0 is the density of atoms, angular brackets denote averaging over the inhomogeneously broadened atomic line,

$$\langle f\rangle = \int_{-\infty}^{\infty} f(\omega)g(\omega - \overline{\omega})d\omega ,$$

and g is a Gaussian distribution with characteristic width $(T^*)^{-1}$. Introducing the retarded time and dimensionless notation

$$\tau = (t - z/c)\Omega, \quad x = z\Omega/c ,$$

$$\tilde{\mathcal{E}} = \overline{\mathcal{E}}/\Omega, \quad \tilde{\zeta} = \zeta/\Omega, \quad \Omega^2 = 2\pi n_0\mathcal{P}^2\omega_0/\hbar ,$$

one can write the Maxwell equations for slowly varying envelope[2,4] in the form

$$\partial_x\tilde{\mathcal{E}} = \langle 2v_1v_2^*\rangle , \tag{2.7}$$

while the equations for the medium change to

$$\partial_\tau v_1 + i\tilde{\zeta}v_1 = \tfrac{1}{2}\tilde{\mathcal{E}}v_2 , \tag{2.5$'$a}$$

$$\partial_\tau v_2 - i\tilde{\zeta}v_2 = -\tfrac{1}{2}\tilde{\mathcal{E}}^* v_1 . \tag{2.5$'$b}$$

In the sequel we shall write for simplicity $\mathcal{E}$ and ζ instead of $\tilde{\mathcal{E}}$ and $\tilde{\zeta}$. It should be remembered that quantities, such as $\mathcal{E}$, ζ are dimensionless.

Equations for $\lambda = 2v_1v_2^*$ and the population inversion $N = |v_2|^2 - |v_1|^2$ follow from Eqs. (2.5):

$$\partial_\tau\lambda = 2i\zeta\lambda + \mathcal{E}N , \tag{2.8}$$

$$\partial_\tau N = -\tfrac{1}{2}(\mathcal{E}^*\lambda + \mathcal{E}\lambda^*) . \tag{2.9}$$

The general initial value problem consists in solving the system of equations (2.5′) and (2.7) for a given envelope of the entering pulse $\mathcal{E}(\tau, 0)$ and given initial state of the medium $[v_1(-\infty, x) = 1$, $v_2(-\infty, x) = 0$, for the absorbing medium$]$. This problem is exactly solvable by the ISM as shown by Ablowitz, Kaup and Newell.[9]

There are three main steps in the procedure. In the first step, the initial eigenvalue problem is solved, and the asymptotic form of the eigenfunctions is determined, leading to the set of scattering data (sd). The evolution of the sd in x is then studied, and, in the last step, the inversion problem is solved, i.e., the field $\mathcal{E}$ as a function of x and τ is constructed from the sd.

The specific feature of the optical problem is that the eigenvalue equations (2.5) form a part of the evolution equations. Once the form of $\mathcal{E}$ is known, the equations for λ and N reduce to a system of linear equations. The primordial scattering data of the ISM are the reflection coefficients $\overline{b}/a)$, the discrete set of eigenvalues ζ_j, $j = 1, \ldots, N$, for which bound-state solutions exist, and the normalization constants for bound states

$$D_j = [\mathrm{Res}(\overline{b}/a)]_{\zeta=\zeta_j} . \tag{2.10}$$

The scattering amplitudes $\overline{b}$ and a are determined by the asymptotic properties of the solutions.[7,14] Let

$$\Phi = \begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}$$

denote the solution of (2.5) which for $\tau \to -\infty$ approaches $\binom{1}{0}e^{-i\zeta\tau}$ for real ζ, then

$$a(\zeta) = \lim_{\tau\to\infty}(e^{i\zeta\tau}\Phi_1) , \tag{2.11}$$

$$b(\zeta) = \lim_{\tau\to\infty} (e^{-i\zeta\tau}\Phi_2). \tag{2.12}$$

Similarly if $\overline{\Phi}$ is the solution which for $\tau \to -\infty$ approaches $\binom{0}{-1} e^{i\zeta\tau}$, then, for real ζ

$$\overline{a}(\zeta) = \lim_{\tau\to\infty} (-e^{i\zeta\tau}\overline{\Phi}_2), \tag{2.13}$$

$$\overline{b}(\zeta) = \lim_{\tau\to\infty} (e^{i\zeta\tau}\overline{\Phi}_1). \tag{2.14}$$

For real $\mathcal{E}(\tau, 0)$, as it is in our case, one can show[14] that $\overline{a}(\zeta) = a(-\zeta)$ and $\overline{b}(\zeta) = b(-\zeta)$. Moreover in that case the zeros of a in the upper half-plane are either purely imaginary or occur as conjugate pairs ζ_j and $-\zeta_j^*$.

III. THE INITIAL EIGENVALUE PROBLEM

In this section, I consider the two-component eigenvalue problem (2.5) with initial envelope $\mathcal{E}(\tau, 0)$ in the form of a hyperbolic secant,

$$\mathcal{E}(\tau, 0) = \mathcal{E}_0 \operatorname{sech}\tau/\tau_0, \tag{3.1}$$

where $\mathcal{E}_0$ and τ_0 are real. To ensure coherence the pulse duration τ_0 must be small compared to the relaxation time in the medium, $\tau_0 \ll T_2$. The pulse initial area $A = \pi\mathcal{E}_0\tau_0$ and energy flux $\mathcal{T} = \mathcal{E}_0^2\tau_0$ may be arbitrarily large. Real $\mathcal{E}_0$ excludes chirping.

A. Reduction to the hypergeometric equation

Expressing v_2 in terms of v_1 by means of Eq. (2.5′a),

$$v_2 = 2\mathcal{E}^{-1}(\partial_\tau v_1 + i\zeta v_1), \tag{3.2}$$

Eq. (2.5b) can be written in the form of a second-order equation for $\psi = v_1$

$$\partial_\tau^2\psi - \mathcal{E}^{-1}(\partial_\tau\mathcal{E})(\partial_\tau\psi + i\zeta\psi) + (\tfrac{1}{4}\mathcal{E}^2 + \zeta^2)\psi = 0. \tag{3.3}$$

By analogy with the solution of the Schrödinger problem for the hyperbolic secant potential[20,21] one introduces the new variable

$$u = \tfrac{1}{2}(1 + \tanh\tau/\tau_0), \quad 0 < u < 1. \tag{3.4}$$

Hence

$$\mathcal{E}(\tau, 0) = 2u^{1/2}(1-u)^{1/2}$$

and Eq. (3.3) takes the form

$$u(1-u)\partial_u^2\psi - (2u-1)\partial_u\psi + \tfrac{1}{4}\mathcal{E}_0^2\tau_0^2\psi + u^{-1}(u-1)^{-1}[4\zeta^2\tau_0^2 + (2u-1)i\zeta\tau_0]\psi = 0. \tag{3.5}$$

This equation is of the Fuchs class[22] with singular points at $u=0$, 1 and $u=\infty$. Let ρ and $\overline{\rho}$ be the roots of the indicial equations corresponding to (3.5) at the singular points $u=0$ and $u=1$, respectively. The transformation

$$w(u) = (1-u)^{-\overline{\rho}}u^{-\rho}\psi(u) \tag{3.6}$$

leads to the hypergeometric equation[23] for w,

$$u(1-u)\partial_u^2 w + [\gamma - (\alpha+\beta+1)u]\partial_u w - \alpha\beta w = 0, \tag{3.7}$$

where

$$\begin{aligned} &\alpha + \beta = 2(\rho + \overline{\rho}), \\ &\alpha\cdot\beta = 2\rho\overline{\rho} + \tfrac{1}{2}(\rho+\overline{\rho}) - \tfrac{1}{4}\mathcal{E}_0^2\tau_0^2 - \tfrac{1}{2}\zeta^2\tau_0^2, \\ &\gamma = 2\overline{\rho} + \tfrac{1}{2}. \end{aligned} \tag{3.8}$$

The values of ρ and $\overline{\rho}$, as calculated in Appendix A, are

$$\begin{aligned} &\rho_1 = \tfrac{1}{2} + \tfrac{1}{2}i\zeta\tau_0, \quad \rho_2 = -\tfrac{1}{2}i\zeta\tau_0, \\ &\overline{\rho}_1 = \tfrac{1}{2} - \tfrac{1}{2}i\zeta\tau_0, \quad \overline{\rho}_2 = \tfrac{1}{2}i\zeta\tau_0. \end{aligned} \tag{3.9}$$

The asymptotic behavior of ψ for large τ and the convergence of the hypergeometric series w both depend on the choice of the values for ρ and $\overline{\rho}$. Various types of solutions are discussed in Appendix B.

B. Continuous spectrum

In the case of a continuous spectrum, $\zeta = k$ is real and according to Appendix B, one can choose

$$\rho = \tfrac{1}{2}ik\tau_0, \quad \overline{\rho} = -\tfrac{1}{2}ik\tau_0. \tag{3.10}$$

Then, from (3.8)

$$\alpha = \tfrac{1}{2}\mathcal{E}_0\tau_0 = -\beta, \quad \gamma = \tfrac{1}{2} - ik\tau_0. \tag{3.11}$$

Since $\operatorname{Re}(\alpha + \beta - \gamma) = -\frac{1}{2}$ the hypergeometric series is convergent in the interval $0 \le u \le 1$. The solution

$$v_1(\tau) = \psi(u) = u^{(1/2)ik\tau_0}(1-u)^{-(1/2)ik\tau_0} \times F(\tfrac{1}{2}\mathcal{E}_0\tau_0, -\tfrac{1}{2}\mathcal{E}_0\tau_0, \tfrac{1}{2} - ik\tau_0; u) \tag{3.12}$$

satisfies the required "boundary" condition at $\tau \to -\infty$,

$$\psi \to e^{-ik\tau}.$$

It follows from (3.12) that the scattering amplitude is

$$a(k) = \lim_{\tau\to\infty} e^{ik\tau}\psi = F(\tfrac{1}{2}\mathcal{E}_0\tau_0, -\tfrac{1}{2}\mathcal{E}_0\tau_0, \tfrac{1}{2} - ik\tau_0; 1). \tag{3.13}$$

Making use of the properties of the hypergeometric functions (see Appendix C) one finds

$$a(k) = \frac{\Gamma^2(\frac{1}{2} - ik\tau_0)}{\Gamma(\frac{1}{2} - ik\tau_0 - \frac{1}{2}\mathcal{E}_0\tau_0)\Gamma(\frac{1}{2} - ik\tau_0 + \frac{1}{2}\mathcal{E}_0\tau_0)}. \tag{3.14}$$

The second eigenfunction v_2 can be found from (3.2) by differentiation of (3.12). A similar, if somewhat more complicated, computation leads to

$$b(k) = -\frac{\sin\frac{1}{2}\pi\mathcal{E}_0\tau_0}{\cosh\pi k\tau_0}. \tag{3.15}$$

For a real $\mathcal{E}$, the amplitudes $\bar{a}$ and $\bar{b}$ are related to a and b by

$$\bar{a}(k) = a(-k), \quad b(k) = b(-k) .$$

Combining (3.14) and (3.15) one gets the reflection coefficient $\bar{b}/a$. It follows from (3.15) that b vanishes for initial pulse areas $A = 2\pi n$ which are known to describe purely soliton solutions.

In the presence of radiation the nonvanishing $\bar{b}/a$ is the nonlinear analog[14] of the Fourier transform of the field. The simple formula

$$\left|\frac{\bar{b}}{a}\right|^2 = \frac{\sin^2 \frac{1}{2}\pi \mathcal{E}_0 \tau_0}{\cosh^2 \pi k \tau_0 - \sin^2 \frac{1}{2}\pi \mathcal{E}_0 \tau_0} \tag{3.16}$$

is useful in the construction of conserved quantities which will be studied in a subsequent paper.[16]

A basic property of $a(k)$ is that it can be analytically continued to the upper half of the complex ζ plane. The zeros of a which determine the discrete eigenvalues are the poles of the function $\Gamma(\frac{1}{2} - i\zeta\tau_0 - \frac{1}{2}\mathcal{E}_0\tau_0)$. They occur for

$$\zeta_n = i\eta_n = i[\tfrac{1}{2}\mathcal{E}_0\tau_0 - (n + \tfrac{1}{2})]\tau_0^{-1}, \quad \eta_n > 0 . \tag{3.17}$$

Since $\bar{a}(\zeta) = a(-\zeta)$ one has

$$\bar{\zeta}_n = i\bar{\eta}_n = -i[\tfrac{1}{2}\mathcal{E}_0\tau_0 - (n + \tfrac{1}{2})]\tau_0^{-1}, \quad \eta_n < 0 . \tag{3.18}$$

An interesting property of $a(k)$ given by (3.14) is that it has a zero on the real axis at $k = 0$ for pulse areas which are odd multiples of π, i.e., for $\mathcal{E}_0\tau_0 = 2n + 1$. Singularities of $\bar{b}/a$ on the real axis can occur in the two-component eigenvalue problem.[14] The residue of $\bar{b}/a$ at $k = 0$ plays an important role in the analytical description of π pulse solutions.[24]

C. Discrete spectrum

Discrete eigenvalues ζ_j, $j = 1, \ldots, N$, are defined as those values of ζ for which bound-state solutions of Eqs. (2.5) exists. From Table I of Appendix B one sees that bound-state solutions in the upper half-plane are possible for $\rho = -\frac{1}{2}i\zeta\tau_0$ and $\bar{\rho} = \frac{1}{2} - \frac{1}{2}i\zeta\tau_0$, (Im$\zeta > 0$). Similarly, in the lower half plane bound-state solutions may appear for $\rho = \frac{1}{2} + \frac{1}{2}i\zeta\tau_0$ and $\bar{\rho} = \frac{1}{2}i\zeta\tau_0$, Im$\zeta < 0$.

In the first case, one finds from Eqs. (3.8)

$$\alpha = \tfrac{1}{2} - i\zeta\tau_0 - \mathcal{E}_0\tau_0 ,$$

$$\beta = \tfrac{1}{2} - i\zeta\tau_0 + \mathcal{E}_0\tau_0 ,$$

$$\gamma = \tfrac{3}{2} - i\zeta\tau_0 .$$

Since Re$(\alpha + \beta - \gamma) > 0$, a regular solution exists only when the hypergeometric series is a polynomial, i.e., when it breaks off with the term of degree $j = -\alpha$, $j = 0, 1, 2, \ldots$. This leads to

$$\zeta_j = i\eta_j = i[\tfrac{1}{2}\mathcal{E}_0\tau_0 - (j + \tfrac{1}{2})]\tau_0^{-1} . \tag{3.17$'$}$$

In a similar way one finds

$$\bar{\eta}_j = -\eta_j . \tag{3.18$'$}$$

In agreement with the general theory, the eigenvalues (3.17′) and (3.18′) are identical with the zeros of a (ζ) and $\bar{a}(\zeta)$. We see that the discrete spectrum consists of an equidistant set of eigenvalues separated by τ_0^{-1}. The number of eigenvalues N is the maximum integer of $(\frac{1}{2}\mathcal{E}_0\tau_0 + \frac{1}{2})$.

TABLE I. Asymptotic behavior of ψ.

A: Real $\zeta, \zeta = k$

ρ	$\bar{\rho}$	$\lim_{\tau\to-\infty}\psi$	$\lim_{\tau\to\infty}\psi$	Re$(\alpha+\beta-\gamma)$ = Re$(2\bar{\rho}-\frac{1}{2})$	$1-\gamma$	Type of solution
$-\frac{1}{2}ik\tau_0$	$\frac{1}{2}ik\tau_0$	$e^{-ik\tau}$	$-e^{-ik\tau}$	$-\frac{1}{2}$	$\frac{1}{2}+ik\tau_0$	Φ_1
$-\frac{1}{2}ik\tau_0$	$\frac{1}{2}-\frac{1}{2}ik\tau_0$	$e^{-ik\tau}$	$-e^{-\tau/\tau_0}e^{ik\tau}$	$\frac{1}{2}$	$\frac{1}{2}+ik\tau_0$	divergent
$\frac{1}{2}+\frac{1}{2}ik\tau_0$	$\frac{1}{2}ik\tau_0$	$e^{\tau/\tau_0}e^{ik\tau}$	$-e^{-ik\tau}$	$-\frac{1}{2}$	$-\frac{1}{2}-ik\tau_0$	$\bar{\Phi}_1$
$\frac{1}{2}+\frac{1}{2}ik\tau_0$	$\frac{1}{2}-\frac{1}{2}ik\tau_0$	$e^{\tau/\tau_0}e^{ik\tau}$	$-e^{-\tau/\tau_0}e^{ik\tau}$	$\frac{1}{2}$	$-\frac{1}{2}-ik\tau_0$	divergent

B: Complex ζ, Im$\zeta = \eta$

Reρ	Re$\bar{\rho}$	$\lim_{\tau\to-\infty}\psi$	$\lim_{\tau\to\infty}\psi$	Re$(2\bar{\rho}-\frac{1}{2})$	$1-\gamma$	Type of solution
$\frac{1}{2}-\frac{1}{2}\eta\tau_0$	$-\frac{1}{2}\eta\tau_0$	$e^{\tau/\tau_0}e^{-\eta\tau}$	$-e^{\eta\tau}$	$-\frac{1}{2}-\eta\tau_0$	$\eta\tau_0$	$\bar{\Phi}$ $\eta < 0$ discrete eigenvalue
$\frac{1}{2}-\frac{1}{2}\eta\tau_0$	$\frac{1}{2}+\frac{1}{2}\eta\tau_0$	$e^{\tau/\tau_0}e^{-\eta\tau}$	$-e^{-\tau/\tau_0}e^{-\eta\tau}$	$\frac{1}{2}+\eta\tau_0$	$-\frac{1}{2}+\eta\tau_0$	unbounded
$\frac{1}{2}\eta\tau_0$	$\frac{1}{2}+\frac{1}{2}\eta\tau_0$	$e^{\eta\tau}$	$-e^{-\tau/\tau_0-\eta\tau_0}$	$\frac{1}{2}+\eta\tau_0$	$\frac{1}{2}-\eta\tau_0$	Φ $\eta > 0$ discrete eigenvalue
$\frac{1}{2}\eta\tau_0$	$-\frac{1}{2}\eta\tau_0$	$e^{\eta\tau}$	$-e^{\eta\tau}$	$-\frac{1}{2}-\eta\tau_0$	$\frac{1}{2}-\eta\tau_0$	unbounded

For even π pulses ($A=2n\pi$) the number of nonvanishing eigenvalues is $N=n$. For odd π pulses, $A=(2n+1)\pi$, there is one zero eigenvalue and n nonvanishing eigenvalues. The linear dependence of η on the area (3.17) is similar to the numerical results of Kaup[10] for the $\tau e^{-\tau}$ profile and the Gaussian profile.

From the above derivation of the scattering data it is straightforward to see that the ζ_j, $a(k)$, and $b(k)$ are uniquely determined by the initial pulse profile. They cannot be changed by a shift of ζ which in the eigenvalue problem is treated as a parameter. Its relation to the pulse carrier frequency becomes important only when the evolution equation (2.7) is used. In my opinion the eigenvalues cannot be shifted due to off-resonance effects, contrary to the interpretation of Kaup.[10] On the other hand, initial chirping $[\dot{\phi}(\tau,0)\neq 0]$ could change the spectrum and possibly allow for $0-\pi$ solutions[4,25,26] (breathers) which are not possible for real $\mathcal{E}(\tau,0)$ because in that case the eigenvalues are purely imaginary.

To complete our set of scattering data we calculate the coefficients D_j defined by Eq. (2.10). Making use of (3.14), (3.15) and (3.17) one finds (see Appendix C)

$$D_j(0)=-\frac{1}{j!}\,\frac{\Gamma(\mathcal{E}_0\tau_0-j)}{\Gamma^2(\frac{1}{2}\mathcal{E}_0\tau_0)}\,. \tag{3.19}$$

Expressions (3.14), (3.15), (3.17), and (3.19) constitute the complete set of the scattering data for the initial profile (3.1).

IV. EVOLUTION OF THE SCATTERING DATA

The evolution of the initial scattering data has been determined[14] from the asymptotic form of Eqs. (2.5) and (2.7) for large τ. The result can be written

$$\frac{\bar{b}}{a}(k,x)=\frac{\bar{b}}{a}(k,0)\exp\left(-\frac{1}{2}\pi g(k)x-\frac{i}{2}xP\int_{-\infty}^{\infty}\frac{g(\alpha)}{k-\alpha}d\alpha\right) \tag{4.1}$$

and

$$D_j(x)=D_j(0)\exp\left(-\frac{i}{2}\,x\int_{-\infty}^{\infty}\frac{g(\alpha)}{\zeta_j-\alpha}\right)d\alpha\,, \tag{4.2}$$

where P denotes the principal value of the integral, $j=1,\ldots,N$. The eigenvalues ζ_j are constants of the motion. Equations (4.1) and (4.2) are valid at exact resonance. To include the effect of detuning these formulas have to be slightly modified. Let the detuning parameter δ denote the difference between the carrier frequency of the pulse ω_0 and the central frequency of the atomic line $\bar{\omega}$, $\delta=\omega_0-\bar{\omega}$. By definition $\mathrm{Re}\zeta=k=\frac{1}{2}(\omega_0-\omega)$. Out of resonance the right side of Eq. (2.7) becomes

$$\begin{aligned}\langle\lambda\rangle&=\int_{-\infty}^{\infty}g(\omega-\bar{\omega})\lambda[\tfrac{1}{2}(\omega-\omega_0)]d\omega\\&=2\int_{-\infty}^{\infty}g(2\alpha-\delta)\lambda(\alpha)d\alpha\,,\end{aligned} \tag{4.3}$$

where g denotes a symmetric distribution function, e.g., a Gaussian.

Following the method of Ablowitz *et al.*[9] and using (4.3) one finds

$$\frac{\bar{b}}{a}(k,x)=\frac{\bar{b}}{a}(k,0)\exp\left(-\frac{1}{2}g(2k-\delta)x+\frac{i}{2}xP\int_{-\infty}^{\infty}\frac{g(2\alpha-\delta)}{k-\alpha}d\alpha\right) \tag{4.4}$$

and

$$D_j(x)=D_j(0)\exp\left(-\frac{i}{2}\,x\int_{-\infty}^{\infty}\frac{g(2\alpha-\delta)}{\zeta_j-\alpha}d\alpha\right). \tag{4.5}$$

For a given distribution g the above formulas determine the effects of detuning.

As a simple illustration of the method consider the one-soliton case, $N=1$, $\mathcal{E}_0\tau_0=2$, $b=0$. According to (3.17) and (3.19) the scattering data are

$$\eta_0=\tfrac{1}{2}\tau_0^{-1};\qquad D_0(0)=1,\quad D_0(x)=e^{iK_0x-\kappa_0x}\,, \tag{4.6}$$

$$K_0=\tau_0^2\int_{-\infty}^{\infty}\frac{\alpha g(\alpha-\delta)}{1+(\alpha\tau_0)^2}\,d\alpha\,, \tag{4.7}$$

$$\kappa_0=\tau_0\int_{-\infty}^{\infty}\frac{g(\alpha-\delta)}{1+(\alpha\tau_0)^2}\,d\alpha\,. \tag{4.8}$$

Introducing η_0 and D_0 into the general form of the one soliton solution[9] and transforming back to the laboratory coordinates z and t one finds

$$\mathcal{E}(t,z)=(2\hbar/\wp t_0)\,\mathrm{sech}[t_0^{-1}(t+\bar{t}-z/V)]e^{iK_0z/c}\,, \tag{4.9}$$

where $t_0=\tau_0/\Omega$ is the pulse duration in seconds,

$$V^{-1}=c^{-1}\left(1+t_0\Omega\int_{-\infty}^{\infty}\frac{g(\alpha-\delta)}{1+(\alpha t_0)^2}d\alpha\right), \tag{4.10}$$

$$\bar{t}=t_0\ln(t_0\Omega)\,, \tag{4.11}$$

$$K_0=t_0^2\Omega\int_{-\infty}^{\infty}\frac{\alpha g(\alpha-\delta)}{1+(\alpha t_0)^2}\,d\alpha\,. \tag{4.12}$$

The only effect of the resonant medium on an initially hyperbolic secant pulse is to change its velocity (4.10), induce a time delay $\bar{t}$ (4.11), and produce a dispersion effect. The latter effect, due to K_0, is identical with the corresponding one in the linear theory.[27] In the absence of detuning $K_0=0$. For a two-soliton solution ($N=2$, $\mathcal{E}_0\tau_0=4$) the scattering data are

$$\eta_0=\tfrac{3}{2}\tau_0^{-1},\quad \eta_1=\tfrac{1}{2}\tau_0^{-1}\,,\qquad D_0(0)=6,\quad D_1(0)=-2\,. \tag{4.13}$$

At exact resonance the solution is unchirped. Introducing the data (4.13) into the special solution given by Lamb[6] one finds that asymptotically the pulse breaks into two hyperbolic secant pulses whose heights are in the relation 1:3 and their widths in the relation 3:1. Their relative phase shift[4,7] is $\beta=\ln 2$. Out of resonance the 4π pulse is chirped and the products of decay have different carrier frequencies.[5] The solution can be computed in a closed form from the scattering data (4.13).[24]

V. CONCLUDING REMARKS

The main results of the paper are contained in Sec. III where explicit analytic expression for the scattering data are given. From these data various physical solutions can be constructed by the use of Marchenko equations or some other inverse procedure. The physical character of a particular solution depends on the initial area A, the detuning δ and the relation of the pulse width τ_0^{-1} to the width of the atomic line $(T^*)^{-1}$. In the case of even π pulses only discrete eigenvalues contribute to the solution and the inverse problem reduces to solving a system of algebraic equations. A more difficult problem arises when radiation is present. A partial information on the propagating pulse can be obtained from the conservation laws. In this way, in a subsequent paper,[16] I compute the energy attenuation and the shift of the pulse average frequency out of resonance. Finally, it is worth noticing, that the solution for the scattering data (3.14)–(3.19) can be used to study not only optical problems but any problem which deals with the two-component eigenvalue equations, e.g., the sine-Gordon equation.

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APPENDIX A: THE INDICIAL EQUATIONS

Equation (3.5) can be written in the form

$$\partial_u^2\psi + p(u)\partial_u\psi + q(u)\psi = 0\,, \tag{A1}$$

where

$$p(u) = A_1 u^{-1} + A_2(u-1)^{-1}\,, \tag{A2}$$

$$q(u) = B_1 u^{-2} + B_2(u-1)^{-2} + C_1 u^{-1} + C_2(u-1)^{-1}\,, \tag{A3}$$

$$A_1 = A_2 = \tfrac{1}{2}\,,$$

$$B_{1,2} = \tfrac{1}{4}\zeta^2\tau_0^2 \mp \tfrac{1}{4}i\zeta\tau_0\,,$$

$$C_1 = -C_2 = \tfrac{1}{2}\zeta^2\tau_0^2 - \tfrac{1}{4}\mathcal{E}_0^2\tau_0^2\,.$$

The indicial equations[23] read

$$\rho(\rho-1) + A_1\rho + B_1 = 0\,, \tag{A4}$$

$$\bar\rho(\bar\rho-1) + A_2\bar\rho + B_2 = 0\,, \tag{A5}$$

and their roots are

$$\rho_1 = \tfrac{1}{2} + \tfrac{1}{2}i\zeta\tau_0, \quad \rho_2 = -\tfrac{1}{2}i\zeta\tau_0\,,$$

$$\bar\rho_1 = \tfrac{1}{2} - \tfrac{1}{2}i\zeta\tau_0, \quad \bar\rho_2 = \tfrac{1}{2}i\zeta\tau_0\,.$$

APPENDIX B: ASYMPTOTIC PROPERTIES OF SOLUTIONS

The solution of Eq. (3.5) has the form

$$\psi = u^{\rho}(1-u)^{\bar\rho}F(\alpha,\beta,\gamma;u)\,,$$

where F is the hypergeometric series with parameters α, β, and γ determined by ρ and $\bar\rho$ through Eqs. (3.8). The second linearly independent solution of the hypergeometric equation

$$u^{1-\gamma}F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma;u)$$

either vanishes or is divergent at $u=0$ and cannot be used to construct solutions (see Table I). Convergence of the hypergeometric series F depends on the sign of $\mathrm{Re}(\alpha+\beta-\gamma)$.[23] The series is convergent in the interval $0\le u\le 1$ $(-\infty<\tau<\infty)$ when $\mathrm{Re}(\alpha+\beta-\gamma)<0$.

Note that, according to Eq. (3.4) the asymptotic form of u and $1-u$ for large τ is

$$u = \tfrac{1}{2}(1+\tanh\tau/\tau_0)\begin{cases}\sim e^{2\tau/\tau_0}, & \text{for } \tau\to-\infty\\ \sim 1, & \text{for } \tau\to\infty\end{cases}$$

$$1-u = \tfrac{1}{2}(1-\tanh\tau/\tau_0)\begin{cases}\sim 1, & \text{for } \tau\to-\infty\\ \sim e^{-2\tau/\tau_0} & \text{for } \tau\to\infty\,.\end{cases}$$

Asymptotic properties of ψ for different choices of ρ and $\bar\rho$ are listed in Table I. The first part of the table deals with the solutions for the continuous spectrum (real ζ), while in the second part bound states (complex ζ) are considered. In the second case only real parts of ρ and $\bar\rho$ are listed. We see that one should choose $\rho=-\frac{1}{2}ik\tau_0$ and $\bar\rho=\frac{1}{2}ik\tau_0$ to construct the Φ_1 solution, and $\rho=\frac{1}{2}+\frac{1}{2}ik\tau_0$, $\bar\rho=\frac{1}{2}ik\tau_0$ to get $\bar\Phi_1$. The bound-state solutions occur for $\rho=\frac{1}{2}-\frac{1}{2}\eta\tau_0$ and $\bar\rho=\frac{1}{2}+\frac{1}{2}\eta\tau_0$ when ζ lies in the upper half-plane $(\eta>0)$.

APPENDIX C: EVALUATION OF THE AMPLITUDES a AND b

To find $F(\alpha,\beta,\gamma;1)$ one makes use of the following relation for the hypergeometric series[22]

$$F(\alpha,\beta,\gamma;u)=\frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta,\alpha+\beta-\gamma+1;1-u)$$
$$+\frac{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-u)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-u)\,, \tag{C1}$$

where Γ denotes the Euler gamma function. In the limit $u\to 1$ the second term vanishes and one gets

$$F(\alpha,\beta,\gamma;1)=\frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\,. \tag{C2}$$

Inserting α, β, and γ given by (3.11) one finds for $a(k)$ expression (3.14).

The eigenfunction v_2 is determined by v_1 through Eq. (3.2). Making use of (3.4) and (3.12), one finds

$$v_2=(\mathcal{E}_0\tau_0)^{-1}[2\rho u^{\rho-1/2}(1-u)^{\bar\rho+1/2}-2\rho u^{\rho+1/2}(1-u)^{\bar\rho-1/2}$$
$$+ik\tau_0u^{\rho-1/2}(1-u)^{\bar\rho-1/2}]F+2(\mathcal{E}_0\tau_0)^{-1}u^{\rho+1/2}(1-u)^{\bar\rho+1/2}\partial_uF\,, \tag{C3}$$

where $\rho=-\frac{1}{2}ik\tau_0=-\bar\rho$ and F is the solution of the hypergeometric equation with α, β, and γ given by (3.11). It is easy to check that, for $u\to 0$ $(\tau\to-\infty)$, v_2 tends to zero, as required for a Φ-type solution.

To calculate $b(k)$ one considers the limit

$$\lim_{\tau\to\infty}e^{-ik\tau}v_2(\tau)=b(k)\,. \tag{C4}$$

In the limit $\tau\to\infty(u\to 1)$ expressions proportional to F on the right side of (C3) tend to zero. The only contribution to b comes from the derivative of the hypergeometric function[23]

$$\partial_uF(\alpha,\beta,\gamma;u)=+\alpha\beta/\gamma)F(\alpha+1,\beta+1,\gamma+1;u)\,. \tag{C5}$$

Introducing (C5) to (C3) and making use of (C2) one finds

$$b(k)=\frac{\mathcal{E}_0\tau_0}{2ik\tau_0-1}\frac{\Gamma(\frac{3}{2}-ik\tau_0)\Gamma(\frac{1}{2}+ik\tau_0)}{\Gamma(1-\frac{1}{2}\mathcal{E}_0\tau_0)\Gamma(1+\frac{1}{2}\mathcal{E}_0\tau_0)}\,. \tag{C6}$$

By applying the identities[23]

$$\Gamma(1+x)=x\Gamma(x)\,,$$
$$\Gamma(x)\Gamma(1-x)=\pi/\sin\pi x\,, \tag{C7}$$
$$\Gamma(\tfrac{1}{2}-ix)\Gamma(\tfrac{1}{2}+ix)=\pi/\cosh\pi x\,,$$

the expression (C6) can be simplified to

$$b(k)=-\sin\tfrac{1}{2}\pi\mathcal{E}_0\tau_0/\cosh\pi k\tau_0\,. \tag{C8}$$

The amplitude $\bar b(k)$ can also be found directly from the equations. Choosing $\rho=\frac{1}{2}+\frac{1}{2}ik\tau_0$ and $\bar\rho=\frac{1}{2}ik\tau_0$ one can construct the $\bar\Phi_1$ solution of Eqs. (2.5) defined in Sec. I:

$$\bar\Phi_1=Mu^{1/2+(1/2)ik\tau_0}(1-u)^{ik\tau_0/2}F(\tfrac{1}{2}ik\tau_0+\tfrac{1}{2}\mathcal{E}_0\tau_0,\tfrac{1}{2}ik\tau_0-\tfrac{1}{2}\mathcal{E}_0\tau_0,\tfrac{3}{2}+ik\tau_0;u)\,. \tag{C9}$$

Here M is a normalization constant determined from

$$\bar\Phi_2\to -e^{ik\tau},\quad \text{for } \tau\to\infty\,.$$

From Eq. (3.2) and the limiting value of $\bar\Phi_1$ at $\tau\to-\infty$ it follows that

$$M=-\tau_0\mathcal{E}_0(1+2ik\tau_0)^{-1}\,. \tag{C10}$$

Inserting M into (C9) and making use of (C1) one finds for $\bar b$

$$\bar b(k)=\lim_{\tau\to\infty}(e^{ik\tau}\bar\Phi_1)=-\sin\tfrac{1}{2}\pi\mathcal{E}_0\tau_0/\cosh\pi k\tau_0\,, \tag{C8$'$}$$

in agreement with (C8).

Evaluation of $\bar b(\zeta_j)$ requires some care because for ζ approaching ζ_j the denominator of (C8′) tends to zero for $\mathcal{E}_0\tau_0=2n$. Considering this case separately, one finds, for $\mathcal{E}_0\tau_0=2n$,

$$\bar b(i\eta_j)=-\lim_{\epsilon\to 0}\sin(n\pi+\epsilon)/\cos(n+\epsilon-j-\tfrac{1}{2})\pi$$
$$=(-1)^{j+1}\,, \tag{C11}$$

and for $\mathcal{E}_0\tau_0 \neq 2n$,

$$\bar{b}(i\eta_j) = -\sin\tfrac{1}{2}\mathcal{E}_0\tau_0/\cos\pi(j+\tfrac{1}{2}-\tfrac{1}{2}\mathcal{E}_0\tau_0) = (-1)^{j+1}. \tag{C11'}$$

From (3.14) and (3.16) it follows that

$$(\mathrm{Res}\, a^{-1})_{\zeta=\zeta_j} = \Gamma(\mathcal{E}_0\tau_0 - j)\Gamma^{-2}(\tfrac{1}{2}\mathcal{E}_0\tau_0 - j)[\mathrm{Res}\,\Gamma(\zeta)]_{\zeta=-j}, \tag{C12}$$

where

$$[\mathrm{Res}\,\Gamma(\zeta)]_{\zeta=-j} = (-1)^j/j!. \tag{C13}$$

From (C11), (C12), and (C13) the expression for D_j follows.

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