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Comment on the inelastic scattering of relativistic charged particles

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It has long been known that the cross section for the inelastic scattering of relativistic charged particles by atoms rises logarithmically with increasing energy. I rederive this result without introducing the current-current interaction (and hence without quantizing the electromagnetic field). This is done by working in the projectile frame in which the projectile current vanishes. The advantage of using the projectile frame is emphasized. A rough heuristic derivation of the logarithmic divergence is also given with the collision viewed in the projectile frame.

I. INTRODUCTION

The differential and integrated cross section for the inelastic scattering of relativistic charged particles by atoms has been discussed in considerable detail.[1,2] For optically allowed transitions the integrated cross section rises logarithmically with increasing energy. In the laboratory frame, in which the target atom is at rest, the logarithmic rise is attributed to the interaction between the current of the projectile and the current generated by the orbital motion of the electron. The current-current interaction may be viewed as the emission (by, say, the projectile) and the reabsorption (by the electron) of a transverse photon. In the Weizsäcker-Williams method of virtual quanta the projectile is replaced by a pulse of electromagnetic radiation which optically induces a transition in the target atom.[3] It was recently suggested[4] that the current-current interaction must be taken into account to explain the observed size of the cross sections for K-vacancy production by 4.88-GeV protons.

The purpose of this comment is to point out that the relativistic rise in the cross section can be derived without introducing the current-current interaction. This is done by working in the projectile frame in which the projectile current, and hence, the current-current interaction, vanish. In this frame the perturbation is just the instantaneous Coulomb potential and thus it is unnecessary to quantize the electromagnetic field. It is instructive to see how the relativistic rise is obtained. Moreover, because the perturbation is just the unretarded Coulomb potential, the calculation is rather simple, and the details are straightforward to interpret. In particular, the difference between the *non*relativistic and relativistic cross sections is illuminated very clearly in the projectile frame—inspection of Eqs. (15) and (22) below reveals that (in the dipole approximation) the nonrelativistic and relativistic cross sections differ kinematically *only* by the Lorentz contraction of the longitudinal components of the momentum transfer and atomic dipole.

The simplicity gained by working in the projectile frame has also been exploited in the calculation of the first Born cross section for electron capture by relativistic ions.[5] Perhaps the projectile frame can be used to greatest advantage in the calculation of the cross section for magnetic monpoles to scatter inelastically from atoms. (There is considerable interest in magnetic monopole stopping powers.[2]) A vector potential for a *stringless* magnetic monopole *at rest* has been written down by Wu and Yang.[6] This potential is defined over all space as the sum of two terms, with one or the other term vanishing outside a given region. While the form of this potential is not too complicated, the form of the 4-vector potential of a *moving* magnetic monopole (obtained from a Lorentz transformation) is presumably very complicated. Therefore the calculation is expected to be simplest in the projectile frame where the monopole is at rest.

In the next section, a mathematical derivation of the cross section for the inelastic scattering of relativistic charged particles by atoms is given. In Sec. III, a *rough* heuristic derivation is given by treating the problem as one of potential scattering. The relativistic divergence of the cross section is seen to be due to the long-range nature of the effective potential.

II. DERIVATION

For simplicity I will consider the target atom to be hydrogenlike and the projectile to be structureless. Let F and F' denote the laboratory and projectile frames, respectively. In F the target nucleus T is assumed to remain at rest and the projectile P, which is incident with a very high energy and is barely deflected during the collision, is assume to move as a classical particle with constant velocity $\vec{v}=v\hat{v}$. In F' it is P which remains at rest, and T moves as a classical particle with velocity $-\vec{v}$. Let F and F' have the same orientation but let their origins be at T and P, respectively. If $\vec{b}$ is the impact parameter of P relative to T, the relative position vector of P and T is $\vec{b}+\vec{v}t$, and the frame F' is obtained from F by a Lorentz boost along $\vec{v}$ and a translation along $\vec{b}$. Let $x_\mu=(\vec{r},ict)$ and $x'_\mu=(\vec{r}\,',ict')$ be the coordinates of the electron in F and F', respectively. Let $\phi_i(x_\mu)$ and $\phi_f(x_\mu)$ be the initial and final wave functions, and let ϵ_i and ϵ_f be the initial and final energies of the electron in F. In F', the initial and final wave functions are $\phi'_i(x'_\mu)$ and $\phi'_f(x'_\mu)$ where [7] (occasionally dropping the subscript i or f)

$$\phi'(x'_\mu)=S\phi(x_\mu)\,, \tag{1}$$

$$S=\left(\frac{\gamma+1}{2}\right)^{1/2}-\left(\frac{\gamma-1}{2}\right)^{1/2}\hat{v}\cdot\vec{\alpha}\,, \tag{2}$$

where $\gamma=(1-\beta^2)^{-1/2}$, $\vec{\beta}=\vec{v}/c$, and $\vec{\alpha}$ is the Dirac velocity operator. The time dependence of $\phi(x_\mu)$ can be factored out:

$$\phi(x_\mu)=\psi(\vec{r})e^{-i\epsilon t/\hbar}\,, \tag{3}$$

where $\psi(\vec{r})$ is a 4 spinor. With e and $-Z_Pe$ the charges of the electron and P, the interaction between P and the electron in F' is $-Z_Pe^2/|\vec{r}\,'|$. The first Born amplitude for direct excitation in F' is,[7] with $d^4x=c\,d^3r\,dt$,

$$A=iZ_P\alpha\int d^4x'\left(\frac{\phi'_f(x'_\mu)^\dagger\phi'_i(x'_\mu)}{|\vec{r}\,'|}\right), \tag{4}$$

where $\alpha=e^2/\hbar c$. The cross section σ (which is invariant under a Lorentz boost from F to F') is obtained by integrating over all impact parameters:

$$\sigma=\int d^2b\,|A|^2\,. \tag{5}$$

Writing

$$\psi(\vec{r})=(2\pi)^{-3/2}\int d^3k\,\tilde{\psi}(\vec{u})e^{i\vec{u}\cdot\vec{r}}\,, \tag{6}$$

$$\tilde{\psi}'(\vec{u})=S\tilde{\psi}(\vec{u})\,, \tag{7}$$

$$\frac{1}{|\vec{r}\,'|}=\frac{1}{2\pi^2}\int\frac{d^3k}{k^2}\,e^{i\vec{k}\cdot\vec{r}\,'}\,, \tag{8}$$

and using the Lorentz transformation of coordinates, that is,

$$ct=\gamma(ct'+\vec{\beta}\cdot\vec{r}\,')\,, \tag{9a}$$

$$\vec{r}_\parallel=\gamma\vec{r}\,'_\parallel+\gamma\vec{v}t'\,, \tag{9b}$$

$$\vec{r}_\perp=\vec{r}\,'_\perp+\vec{b}\,, \tag{9c}$$

where the subscripts $\parallel$ and $\perp$ indicate components parallel to and perpendicular to $\vec{v}$, respectively, Eq. (4) becomes

$$A=i2Z_P\alpha(2\pi)^{-5}\int d^4x'\int d^3p\int d^3s\int\frac{d^3k}{k^2}\,\tilde{\psi}'_f(\vec{p})^\dagger\tilde{\psi}'_i(\vec{s})\exp[i(-\vec{p}_\perp+\vec{s}_\perp)\cdot\vec{b}]\exp[i(\epsilon_f-\epsilon_i-\hbar\vec{v}\cdot\vec{p}+\hbar\vec{s}\cdot\vec{v})\gamma t'/\hbar]$$
$$\times\exp\{i[\gamma(\epsilon_f-\epsilon_i)\vec{\beta}/(\hbar c)-\vec{p}_\perp+\vec{s}_\perp+\gamma(-\vec{p}_\parallel+\vec{s}_\parallel)+\vec{k}]\cdot\vec{r}\,'\}$$
$$=\frac{iZ_P\alpha}{\pi\gamma\beta}\int\frac{d^2k_\perp}{k^2}\,e^{-i\vec{k}_\perp\cdot\vec{b}}I(\vec{k}_\perp)\,, \tag{10}$$

where in Eq. (10)

$$I(\vec{k}_\perp)=\int d^3s\,\tilde{\psi}'_f(\vec{s}+\vec{q})^\dagger\tilde{\psi}'_i(\vec{s})\,, \tag{11}$$

$$\vec{q}_\perp=\vec{k}_\perp,\quad q_\parallel=\Delta\epsilon/(\hbar v)\,, \tag{12}$$

$$\Delta\epsilon=\epsilon_f-\epsilon_i\,, \tag{13}$$

$$k^2=k_\perp^2+k_\parallel^2\,, \tag{14}$$

$$k_\parallel=\Delta\epsilon/(\gamma\hbar v)=(1/\gamma)q_\parallel\,. \tag{15}$$

Note that the momentum transfer to the target atom is $\hbar\vec{q}$ in F and $\hbar\vec{k}$ in F', with $\vec{q}$ and $\vec{k}$ defined by Eqs. (12) and (15).

Transforming back to coordinate space,

$$I(\vec{k}_\perp)=\int d^3r\,e^{i\vec{q}\cdot\vec{r}}\psi'_f(\vec{r})^\dagger\psi'_i(\vec{r})\,, \tag{16}$$

where $\psi'(\vec{r})=S\psi(\vec{r})$. In bracket notation, Eq. (16) becomes

$$I(\vec{k}_\perp)=\langle f|e^{i\vec{q}\cdot\vec{r}}S^\dagger S|i\rangle\,. \tag{17}$$

Note that $(\hat{v}\cdot\vec{\alpha})^2=1$ and hence

$$S^\dagger S=\gamma-\gamma(\vec{\beta}\cdot\vec{\alpha})\,. \tag{18}$$

It follows that

$$I(\vec{k}_\perp)=\gamma\langle f|e^{i\vec{q}\cdot\vec{r}}|i\rangle-\gamma\langle f|e^{i\vec{q}\cdot\vec{r}}(\vec{\beta}\cdot\vec{\alpha})|i\rangle\,. \tag{19}$$

$I(\vec{k}_\perp)$ will now be reduced to a simple form which is valid in the "long-wavelength limit", $qa\ll 1$, where a is either the initial or final characteristic radius of the target atom, whichever is the smaller. Note that $qa\ll 1$ is equivalent to $k_\perp a\ll 1$. In the long-wavelength limit, $e^{i\vec{q}\cdot\vec{r}}$ can be replaced by $1+i\vec{q}\cdot\vec{r}$ in the first term on the right-hand side of Eq. (19) and by 1 in the second term. In this approximation one has

$$\langle f|e^{i\vec{q}\cdot\vec{r}}|i\rangle \approx i\vec{q}\cdot\langle f|\vec{r}|i\rangle\,, \tag{20a}$$

and, using $\vec{\alpha}=(i/\hbar c)[H,\vec{r}]$, where H is the Hamiltonian of the target atom,

$$\langle f|e^{i\vec{q}\cdot\vec{r}}(\vec{\beta}\cdot\vec{\alpha})|i\rangle \approx (i\Delta\epsilon/\hbar c)\vec{\beta}\cdot\langle f|\vec{r}|i\rangle\,. \tag{20b}$$

Combining Eqs. (12), (15), (19), and (20) yields

$$I(\vec{k}_\perp) \approx i\gamma\vec{k}_\perp\cdot\langle f|\vec{r}_\perp|i\rangle + i\vec{k}_\parallel\cdot\langle f|\vec{r}_\parallel|i\rangle\,. \tag{21}$$

The first term on the right-hand side of Eq. (21) represents the excitation of a transverse dipole (perpendicular to $\vec{v}$); the second term represents the excitation of a longitudinal dipole (parallel to $\vec{v}$). Inserting A from Eq. (10) into Eq. (5) and using Eq. (21) yields

$$\begin{aligned}\sigma &= 4\left(\frac{Z_P\alpha}{\gamma\beta}\right)^2 \int \frac{d^2k_\perp}{k^4}\,|I(\vec{k}_\perp)|^2 \\ &\approx 4\left(\frac{Z_P\alpha}{\beta}\right)^2 \int \frac{d^2k_\perp}{k^4}\left[|\vec{k}_\perp\cdot\langle f|\vec{r}_\perp|i\rangle|^2 + \left|\left(\frac{1}{\gamma}\right)\vec{k}_\parallel\cdot\langle f|\vec{r}_\parallel|i\rangle\right|^2\right],\end{aligned} \tag{22}$$

noting that the interference between the transverse and longitudinal dipole terms in Eq. (21) vanishes upon integration over $\vec{k}_\perp$.[8] The factor $(1/\gamma)$ in the longitudinal dipole term in Eq. (22) accounts for the Lorentz contraction of the longitudinal dipole in F'. Performing the integration gives

$$\sigma \approx 4\pi\left(\frac{Z_P\alpha}{\beta}\right)^2\left\{\left[\ln\left(\frac{\gamma\hbar v}{a\Delta\epsilon}\right)-\tfrac{1}{2}\right]|\langle f|\vec{r}_\perp|i\rangle|^2 + (1-\beta^2)|\langle f|(\vec{r}_\parallel)|i\rangle|^2\right\}, \tag{23}$$

where I have placed an upper limit of $1/a$ rather than ∞ on $k_\perp$ in the integral that gives rise to the logarithmic term since otherwise this integral would diverge. (Recall that the long-wavelength approximation is valid only for $k_\perp a \ll 1$.) Note that the logarithmic term, which diverges as $\gamma \to \infty$, arises from the term in $\langle f|\vec{r}_\perp|i\rangle$ in Eq. (22), that is, from the excitation of the transverse dipole moment. It is, in fact, the excitation of the transverse moment at small momentum transfers (i.e., $\vec{k}_\perp = \vec{q}_\perp \to 0$) that leads to the logarithmic term. Note that as $\vec{q}_\perp \to 0$ the momentum transfer becomes parallel to $\vec{v}$ and in this limit "transverse to $\vec{v}$" means the same as "transverse to $\vec{q}$." Summing and averaging over magnetic quantum numbers, $|\langle f|\vec{r}|i\rangle|^2$ is the same for all directions, and will be indicated as x_{fi}^2, and Eq. (23) becomes

$$\sigma \approx 4\pi(Z_P\alpha/\beta)^2 x_{fi}^2[\ln(\gamma\hbar v/a\Delta\epsilon)^2 - \beta^2]\,. \tag{24}$$

This last result is identical to the result obtained earlier by Bethe,[9] Fano,[1] and others [see Eq. (7) of the first paper of Ref. 1], but the derivation is entirely different.[10] Note from Eq. (23) that the contribution to σ from the transverse dipole is larger than the contribution from the longitudinal dipole by a factor of order $\gamma^2 \ln\gamma$.

III. HEURISTIC DERIVATION

In order to gain more physical insight into the origin of the logarithmic divergence of σ, when the collision is viewed in the projectile frame F', I will simplify (in a *very* heuristic fashion) the problem of the excitation of an atom to that of potential scattering. Thus, working throughout in F', the target atom during the collision is equivalent to an electric dipole oscillating with period $\gamma(\hbar/\Delta\epsilon)$ and moving with velocity $-\vec{v}$, provided multipoles higher than dipoles are neglected. The factor γ in the period of oscillation accounts for time dilation. The collision time is roughly b/v. [It is interesting to note that the collision time in F is $b/(\gamma v)$. The reduction by a factor of $1/\gamma$ accounts for the contraction of the electric field in F generated by the moving point charge P. The electric field in F' generated by the moving atomic dipole is not contracted.] Therefore if $b/v \ll \gamma(\hbar/\Delta\epsilon)$, the oscillation of the atomic dipole is negligible during the collision and the effective interaction between the atom and the projectile is that between a static dipole $e\vec{\mu}$ and a stationary point charge $-Z_P e$. This interaction vanishes when averaged over all directions of $\vec{\mu}$. However, the root mean square of this interaction averaged over all directions of $\vec{\mu}$ does not vanish and is[11]

$$U(r) \equiv (Z_P e^2\mu/3^{1/2}r^2)\,, \tag{25}$$

where $\mu = |\vec{\mu}|$. Guided by intuition I now assume, without proof, that the cross section for excitation of an atom by a relativistic charged particle is the same (to within a factor of order unity) as the cross section for potential scattering where the potential is $U(r)$ in the range $r_{\min} \leqq r \leqq r_{\max}$, where

$$r_{\min} = a\,,\qquad r_{\max} = \gamma\hbar v/\Delta\epsilon\,. \tag{26}$$

For $r < r_{\min}$, multipoles higher than dipoles are important. For $r > r_{\max}$, the interaction time averages to zero over a collision duration of order r/v.

The scattering phase shifts δ_l may be calculated directly from the Klein-Gordon equation using the potential U of Eq. (25). However, as pointed out to be by L. Spruch, the calculation is simplified by noting that the Klein-Gordon equation can be reduced to the form of the Schrödinger equation[12] for the scattering of a particle of momentum $p = \gamma m v$ from an energy-dependent potential

$\gamma U+(U^2/2mc^2)$.

Since $\gamma \gg U(r)/mc^2$ for the γ and r of interest, the term in U^2 can be dropped and the energy-dependent potential is γU. In the Schrödinger framework, the cross section is

$$\sigma=\frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l\,, \tag{27}$$

where $k=p/\hbar$ and where, in the Born approximation (noting $\delta_l \ll 1$),

$$\sin\delta_l \approx \frac{2m}{\hbar p}\int dr[\gamma U(r)]\hat{j}_l^2(kr)\,, \tag{28}$$

with $\hat{j}_l(r)$ denoting the Riccati-Bessel function. The contribution to the integral of Eq. (28) from the region $r \lesssim l/k$ is small, and will be neglected, since in this region $\hat{j}_l^2(kr)$ behaves like r^{2l+2}. In the region $r \gtrsim l/k$ the asymptotic form $\hat{j}_l(kr) \sim \sin(kr-\frac{1}{2}l\pi)$ is valid and $\hat{j}_l^2(kr)$ will be approximated by $\frac{1}{2}$ in this region. Thus taking the lower limit of the integral of Eq. (28) to be l/k,

$$\sin\delta_l \approx \frac{Z_P\alpha\mu k}{3^{1/2}\beta l}\,. \tag{29}$$

Substituting into Eq. (27), and replacing the sum over l by an integral since the major contribution to σ comes from values of $l \gg 1$, gives

$$\sigma \approx \left(\frac{8\pi}{3}\right)\left(\frac{Z_P\alpha}{\beta}\right)^2\mu^2\int\frac{dl}{l}\,. \tag{30}$$

Since the form of U in Eq. (25) is valid only for $r_{\min} \leqq r \leqq r_{\max}$, and since this form was assumed for $r \geqq l/k$ in arriving at Eq. (29), the upper and lower limits of the integral over l in Eq. (30) are $kr_{\max}$ and $kr_{\min}$, respectively. Performing the integration over l yields

$$\sigma \approx (4\pi/3)(Z_P\alpha/\beta)^2\mu^2\ln(\gamma\hbar v/a\Delta\epsilon)^2\,. \tag{31}$$

The right-hand side of this last equation is identical to the logarithmic term in Eq. (24) if μ^2 is set equal to $3x_{fi}^2$. However, I stress that while the result just obtained is correct, this heuristic derivation is merely suggestive since it rests on an assumption that has been given little, if any, justification, namely, that the cross section for excitation of an atom by a relativistic charged particle is the same as that for scattering from the potential $U(r)$, $r_{\min} \leqq r \leqq r_{\max}$.

The connection indicated above between the relativistic rise in σ and the relativistic rise in $r_{\max}$ has been known for a long time. Indeed, this connection was established by Bohr[13] in 1915. Within the framework of classical mechanics, Bohr calculated the energy transferred to an electron, initially free and at rest in F, in a Coulomb collision with a fast charged projectile passing at an impact parameter b. Bohr noted that an electron bound in an atom can be regarded as free only if the atomic period, $1/\nu$ in F, is larger than the collision time, $b/(\gamma v)$ in F. Putting $1/\nu=\hbar/\Delta E$, this condition becomes $b<r_{\max}$. As Bohr noted, for $b>r_{\max}$ the electron behaves as if it is rigidly bound, and therefore energy can be effectively transferred to the electron only for $b<r_{\max}$. He found that the energy transfer increases logarithmically as $r_{\max}$ increases. Later Bloch[14] rederived (in frame F) Bohr's result by treating the electron quantum mechanically, and he demonstrated that Bohr's classical analysis is valid for the distant collisions. The heuristic derivation given here differs from Bloch's work in that here (i) the problem is approximately treated as one of potential scattering, and (ii) the frame F' is used, which results in a simplification of the effective potential. By viewing the problem within the framework of potential scattering, one sees rather directly that the relativistic logarithmic divergence of the cross section is due to the long-range nature of the effective $1/r^2$ potential, and is not an artifact of perturbation theory or the dipole approximation.

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