# Blankenbecler-Goldberger approximation in inelastic $e$-H and $e$-He scattering

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The Blankenbecler-Goldberger approximation to the scattering amplitude is applied to the inelastic $e$-H ($1s \to 2s$) and $e$-He $[(1s)^2 \to (1s)(2s)]$ scattering. The results for the differential cross section indicate a very good agreement with the Glauber approximation, but the Blankenbecler-Goldberger method is more difficult to evaluate than the Glauber approximation.

## I. INTRODUCTION

The application of the Glauber approximation[1] to atomic problems is well documented.[2] In addition we list the following papers individually: Tai *et al.*,[3] Thomas and Gerjuoy,[4] Franco,[5] and Thomas and Chan.[6] In an earlier work[7] one of us had applied an alternate technique, the Blankenbecler-Goldberger (BG) approximation,[8] to the elastic scattering of electrons on H and He. The BG amplitude and the sequence of approximations needed to obtain it are discussed in Ref. 9. The experience[7] gathered from the $e$-H and $e$-He elastic scattering was that the BG amplitude fares very well when compared to the Glauber amplitude.

In the use of the Glauber approximation one does not need to evaluate the eikonal $\chi(b)$. The evaluation of the scattering amplitude employs the most expedient sequence of integrations which bypasses the evaluation of $\chi(b)$ [or the overlap function $\Gamma(b) = 1 - e^{i\chi(b)}$]. In contrast to this the eikonal is an input in the BG approximation and must first be extracted. In Sec. II we follow the technique used earlier for the elastic problem to extract the eikonal for the $e$-H inelastic scattering in which the hydrogen atom is excited to the $2s$ state. As a check on our calculation we substituted this eikonal in the Glauber amplitude and carried out the last remaining $b$ integration to reproduce the known[4] inelastic amplitude. This eikonal is finally substituted in the BG form and the final $b$ integration done to generate the inelastic scattering amplitude from which the differential cross section is calculated. In Sec. III we do a similar calculation for inelastic $e$-He scattering in which one of the electrons is excited from the $1s$ ground state to $2s$ excited state without spin flip, i.e., the final state is $1s2s\,^1S$. The differential cross section is calculated at 100 eV. A brief discussion of results is given in Sec. IV.

The Glauber amplitude for two-body scattering is written as,[1]

$$F(q) = \frac{ik}{2\pi} \int e^{i\vec{q}\cdot\vec{b}}(1 - e^{i\chi(b)})d^2b\,, \tag{1.1}$$

where $k$ is the incident momentum, $\vec{q}$ is the momentum transfer, $b$ is the impact parameter, and $\chi(b)$ is the eikonal which for a potential $V(\vec{b}, z)$ is defined as[1]

$$\chi(b) = \frac{1}{\hbar v} \int_{-\infty}^{\infty} V(\vec{b}, z)dz\,. \tag{1.2}$$

$v$ is the incident velocity. The BG amplitude[8] is, in contrast,

$$F(\vec{q}) = \frac{k}{2\pi} \int e^{i\vec{q}\cdot\vec{b}} \frac{\chi(b)}{1 - (i/2)\chi(b)} d^2b\,. \tag{1.3}$$

To order $\chi^2(b)$ the two amplitudes are identical.

## II. EIKONAL FOR $e$-H INELASTIC SCATTERING

If the target (H or He) is treated as infinitely heavy the reduced mass in the problem is the electron mass. The Glauber amplitude for the scattering of an electron by a neutral atom such that the initial atom, in state $i$, is left finally in state $f$, is

$$F_{fi}(q) = \frac{ik_i}{2\pi} \int u_f^*(\vec{r})u_i(\vec{r})\Gamma(\vec{b}, \vec{r})e^{i\vec{q}\cdot\vec{b}}d^2\vec{b}d^3r\,, \tag{2.1}$$

where $u_i(\vec{r})$ and $u_f(\vec{r})$ are the initial and final wave functions of the atom. $r$ stands for $\{r_1, r_2, \ldots, r_Z\}$ the positions of $Z$ electrons in the atom and $d^3r$ stands for $\prod_{i=1}^{Z} d^2\vec{r}_i$. $\vec{b}$ is the impact parameter, $\vec{q} = \vec{k}_i - \vec{k}_f$ is the three-momentum transfer, and $\vec{k}_i$ and $\vec{k}_f$ are the initial and final momenta of the electron. The relation between the mass and momentum is $\hbar\vec{k} = m\vec{v}$. Let us define a function $\Gamma(\vec{b}, \vec{r})$ in terms of a phase $\chi(\vec{b}, \vec{r})$ as

$$\Gamma(\vec{b}, \vec{r}) = 1 - e^{i\chi(\vec{b}, \vec{r})}\,, \tag{2.2}$$

where $\vec{r}$ again stands for the coordinate set $\{r_1, r_2, \ldots, r_Z\}$ and $\chi(\vec{b}, \vec{r})$ is defined in terms of the potential through

 

$$\chi(\vec{b},\vec{r})=\frac{1}{\hbar v_i}\int_{-\infty}^{+\infty} V(\vec{b},\vec{r};z')dz' . \qquad (2.3)$$

Using the Coulomb potential one obtains[5]

$$\chi(\vec{b},\vec{r})=2\eta \ln(|\vec{b}-\vec{s}|/b) , \qquad (2.4)$$

where $\eta=-e^2/\hbar v_i$ for electron scattering by a neutral atom. The inelastic scattering amplitude can be written as

$$F_{fi}(\vec{q})=\frac{ik_i}{2\pi}\int e^{i\vec{q}\cdot\vec{b}}\Gamma_{\mathrm{H}}^{\mathrm{inel}}(\vec{b})d^2\vec{b} , \qquad (2.5)$$

where

$$\Gamma_{\mathrm{H}}^{\mathrm{inel}}(\vec{b})=\int u_f^*(\vec{r})u_i(\vec{r})\Gamma(\vec{b},\vec{r})d^3r . \qquad (2.6)$$

Note that due to the orthogonality of state $i$ and $f$, if $i\neq f$, the unit factor in the definition of $\Gamma(\vec{b},\vec{r})$ in Eq. (2.2) does not contribute to $\Gamma_{\mathrm{H}}^{\mathrm{inel}}(\vec{b})$.

Thomas and Gerjuoy[4] have shown that the overlap function for the $1s\rightarrow ns$ excitation of hydrogen atom is

$$\Gamma_{\mathrm{H}}^{\mathrm{inel}}(\vec{b})=2\pi A_n\sum_{j=0}^{n-1}\alpha_j(n)(-1)^{j+1}\frac{\partial^{j+1}}{\partial\lambda^{j+1}}\frac{1}{(2\pi)^2}\times\int_0^\infty\frac{e^{-\lambda r}}{r}\left[1-\left(\frac{|\vec{b}-\vec{s}|}{b}\right)^{2i\eta}\right]d^3r , \qquad (2.7)$$

where $\lambda=[(1+1/n)/a_0]$, $a_0$ is the first Bohr radius for H,

$$A_n=\left(\frac{2}{a_0}\right)^3\frac{1}{4}\left(\frac{n!}{n^4(n-1)!}\right)^{1/2}=\left(\frac{2}{a_0}\right)^3\frac{1}{4}n^{-3/2}$$

and

$$\alpha_j(n)=\frac{(1-n)_j}{(2)_j}j!\left(\frac{2}{na_0}\right)^j .$$

$(a)_j$ is the Pochhammer symbol[10] with $(a)_0=1$ for all $a$. We obtain

$$\begin{aligned}\Gamma_{\mathrm{H}}^{\mathrm{inel}}(b)=&-i\frac{2^{11/2}}{3^4}[(1+i\eta)_1F_2(1;1+i\eta,1+i\eta;y^2/4)-(2i\eta)_1F_2(1;i\eta,1+i\eta;y^2/4)\\&+(i\eta-1)_1F_2(1;i\eta-1,i\eta+1;y^2/4)]\\&+2^{9/2}\frac{\Gamma^2(1+i\eta)}{3^4(y/2)^{2i\eta}}[(2i\eta)(1+i\eta)_0F_1(1;y^2/4)-(4i\eta)(y/2)^2_0F_1(2;y^2/4)+(y/2)^4_0F_1(3;y^2/4)] . \qquad (2.8)\end{aligned}$$

This is the final form for the inelastic $1s\rightarrow 2s$ overlap function for hydrogen atom. Having derived the overlap function we can do the $b$ integration indicated in (2.5) to check our result with the known result[4] for $F(q)$ for $1s\rightarrow 2s$ excitation in the Glauber approximation. The final result agrees with that of Thomas and Gerjuoy.[4] This provides us with the check that our eikonal (overlap function) is correctly evaluated. The real and imaginary parts of the overlap function $\Gamma_{\mathrm{H}}^{\mathrm{inel}}(b)$ for $1s\rightarrow 2s$ are plotted in Fig. 1 as a function of $b$.

With the eikonal $\chi(b)$ extracted from (2.8) through $\Gamma(b)=1-e^{i\chi(b)}$ we can evaluate the inelastic amplitude in the BG approximation,

$$F_{\mathrm{H}}^{\mathrm{inel}}(\vec{q})=k\int_0^\infty\frac{\chi(b)}{1-(i/2)\chi(b)}J_0(qb)b\,db . \qquad (2.9)$$

This integral had to be done numerically. The expansion of the hypergeometric functions in powers of $y^2$ in (2.8) was convenient for $b\sim 12a_0$. For integration beyond this value we used the asymptotic form[6] for the modified Lommel functions,

$$\mathcal{L}_{\mu,\nu}(iy)=S_{\mu,\nu}(iy)-2^{\mu-1}\Gamma\left(\frac{1+\mu-\nu}{2}\right)\Gamma\left(\frac{1+\mu+\nu}{2}\right)\frac{2}{\pi}e^{-i\pi\nu/2}\times\cos\left(\frac{\pi}{2}(\mu-\nu)\right)K_\nu(y) , \qquad (2.10)$$

where the asymptotic form of $S_{\mu,\nu}(iy)$ is known.[11] The exponential fall of $K_\nu(y)$ asymptotically allows us to ignore this term for $b\gtrsim 12a_0$.

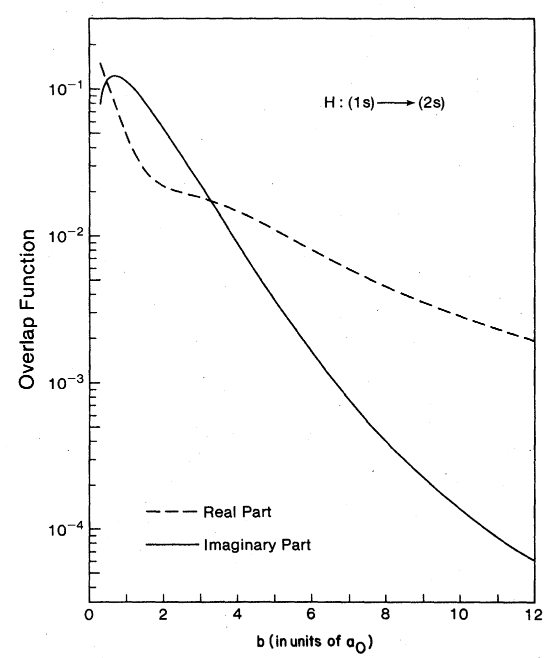


FIG. 1. Real and imaginary parts of $\Gamma_{\mathrm{H}}^{\mathrm{inel}}(b)$ for $1s\rightarrow 2s$ transition in hydrogen.

The differential cross section for inelastic scattering $1s \rightarrow 2s$ is then generated in the usual way. In Fig. 2 we have plotted $d\sigma_{fi}/d\Omega$ (in $\pi a_0^2$) versus $q^2$ (in $a_0^{-2}$) for incident electron energy of 100 eV and $1s \rightarrow 2s$ excitation of hydrogen. We find that the BG approximation works just as well as the Glauber approximation. As the differential cross sections are so very close there will be no discernable difference in the total cross section either.

## III. EIKONAL FOR $e$-He INELASTIC SCATTERING

In order to be able to use the BG form for He excitation we have to extract the corresponding eikonal $\chi(b)$. The analog of Eq. (2.1) for $e$-He inelastic scattering is

$$F_{\mathrm{He}}^{\mathrm{inel}}(\vec{q}) = \frac{ik_i}{2\pi}\int e^{i\vec{q}\cdot\vec{b}}d^2\vec{b}\int u_i(\vec{r}_1,\vec{r}_2)u_f^*(\vec{r}_1,\vec{r}_2)(1-e^{i\chi(\vec{b},\vec{s}_1,\vec{s}_2)})d^3r_1d^3r_2 \equiv \frac{ik_i}{2\pi}\int e^{i\vec{q}\cdot\vec{b}}\Gamma_{\mathrm{He}}^{\mathrm{inel}}(\vec{b})d^2\vec{b}, \tag{3.1}$$

where $\vec{r} = (\vec{s}, z)$ and

$$\Gamma_{\mathrm{He}}^{\mathrm{inel}}(\vec{b}) = -\prod_{j=1}^{2}\int u_f^*(\vec{r}_1,\vec{r}_2)u_i(\vec{r}_1,\vec{r}_2)\left(\frac{|\vec{b}-\vec{s}_j|}{b}\right)^{2i\eta}d^3r_1d^3r_2 . \tag{3.2}$$

For the purposes of comparison of the BG approximation with the Glauber approximation we use the wave functions given by Morse *et al.*[12] for an electron in the $1s$ state:

$$\phi(r) = \frac{1}{\sqrt{\pi}}\left(\frac{\alpha}{a_0}\right)^{3/2}e^{-\alpha r/a_0}; \tag{3.3}$$

for an electron in $2s$ state:

$$\psi(r) = \left(\frac{\mu^5}{3\pi N a_0^3}\right)^{1/2}\left(\frac{r}{a_0}e^{-\mu r/a_0} - \frac{3A}{\mu}e^{-\mu\beta r/a_0}\right), \tag{3.4}$$

where

$$\begin{aligned}
&\alpha = 1.69\,,\\
&\mu = 0.61,\\
&\mu\beta = 1.57\,,\\
&A = \frac{\mu(\alpha+\mu\beta)^3}{(\alpha+\mu)^4}\,,\\
&N = \left(1 - \frac{48A}{(1+\beta)^4} + \frac{3A^2}{\beta^3}\right).
\end{aligned} \tag{3.5}$$

$A$ and $N$ are so chosen that $\phi$ and $\psi$ are orthogonal and both are unit normalized. One obtains in a straightforward way,

$$\begin{aligned}
\Gamma_{\mathrm{He}}^{\mathrm{inel}}(\vec{b}) = \frac{16\pi\alpha^3 b^6}{a_0^3}\left(\frac{\alpha^3\mu^5}{3\pi^2 N a_0^6}\right)^{1/2}&\left(-\frac{2}{y^3} - \frac{(2i\eta)^2(2+2i\eta)}{y(iy)^{2+2i\eta}}\mathfrak{L}_{2i\eta-1,0}(iy) + \frac{i(2i\eta)^2(2i\eta-2)}{(iy)^{2+2i\eta}}\mathfrak{L}_{2i\eta-2,1}(iy)\right)\\
\times\Bigg[\frac{b}{a_0}&\left(\frac{6}{y_1^4} + \frac{(2i\eta)^2(2+2i\eta)(3+2i\eta)}{y_1^2(iy_1)^{2+2i\eta}}\mathfrak{L}_{2i\eta-1,0}(iy_1) - \frac{i(2i\eta)^2(2i\eta-2)(4i\eta+3)}{y_1(iy_1)^{2+2i\eta}}\mathfrak{L}_{2i\eta-2,1}(iy)\right.\\
&\left.\quad - \frac{(2i\eta)^2(2i\eta-2)(2i\eta-4)}{(iy_1)^{2+2i\eta}}\mathfrak{L}_{2i\eta-3,2}(iy_1)\right)\\
&+\frac{3A}{\mu}\left(-\frac{2}{y_2^3} - \frac{(2i\eta)^2(2+2i\eta)}{y_2(iy_2)^{2+2i\eta}}\mathfrak{L}_{2i\eta-1,0}(iy_2) + \frac{i(2i\eta)^2(2i\eta-2)}{(iy_2)^{2+2i\eta}}\mathfrak{L}_{2i\eta-2,1}(iy_2)\right)\Bigg],
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
&y_1 = \lambda_1 b, \quad \lambda_1 = (\alpha+\mu)/a_0\,,\\
&y_2 = \lambda_2 b, \quad \lambda_2 = (\alpha+\mu\beta)/a_0\,,
\end{aligned} \tag{3.7}$$

and

$$y = \lambda b, \quad \lambda = 2\alpha/a_0 .$$

The eikonal $\chi(b)$ defined through $\Gamma(b) = 1 - e^{i\chi(b)}$ is then extracted from (3.6). The remaining procedure is a repeat of what we did for the hydrogen problem. In Fig. 3 we have plotted the real and the imaginary parts of the overlap function for $e$-He inelastic scattering $1s^2 \rightarrow 1s2s$. In Fig. 4 we compare the two cross sections $(d\sigma_{fi}/d\Omega)$ (in $\pi a_0^2$) versus $q^2$ (in $a_0^{-2}$) for the inelastic $1s^2 \rightarrow 1s2s$ $e$-He scattering of a 100-eV electron. We notice that

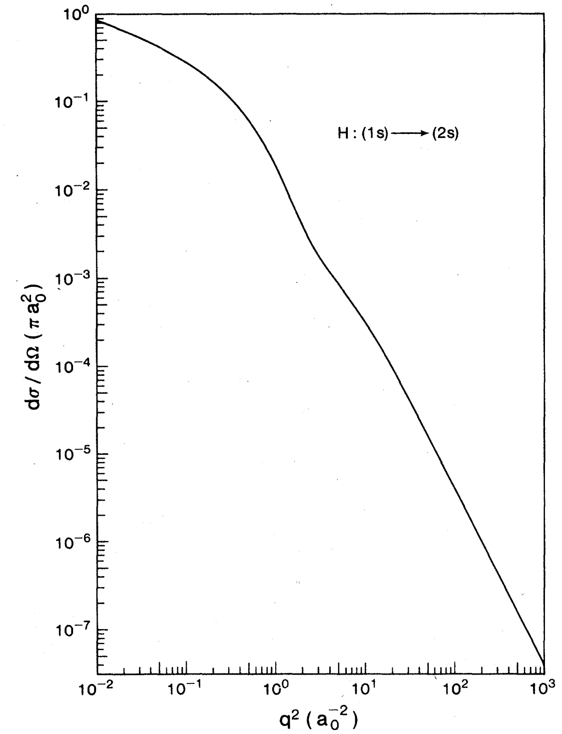


FIG. 2. $d\sigma/d\Omega$ for $1s \rightarrow 2s$ transition in hydrogen. Incident electron energy = 100 eV.

there is a discernable difference in the backward direction. However, as the differential cross section has also dropped by more than an order of magnitude, we do not anticipate any significant difference in the calculated total cross section in the two approximations. In the numerical integration of (2.9) to generate the BG amplitude the power-series expansion for hypergeometric functions up to $b \approx 6.5a_0$ and the asymptotic form for the modified Lommel functions for $b \gtrsim 6.5a_0$.

## IV. CONCLUSION

We have extended a previous calculation[7] which used the BG approximation for the elastic $e$-He and $e$-He scattering to the inelastic $e$-H and $e$-He scattering. We find that as in the elastic case the BG approximation works well for the inelastic scattering also when compared to the Glauber approximation. We suspect that this has to do more with the fact that the electromagnetic coupling is involved than any other subtle effect.

The advantage of using the Glauber form is that the amplitude can be evaluated with some effort in a closed form. The disadvantage of the BG form is that the eikonal has first to be evaluated and the last integral has to be done numerically. The excitation to each state has to be handled separately as against the Glauber form where an analytic result[4] can be obtained for excitation to any state. The BG approximation, however, compares very well with the Glauber approximation for $s \rightarrow s$ excitation.

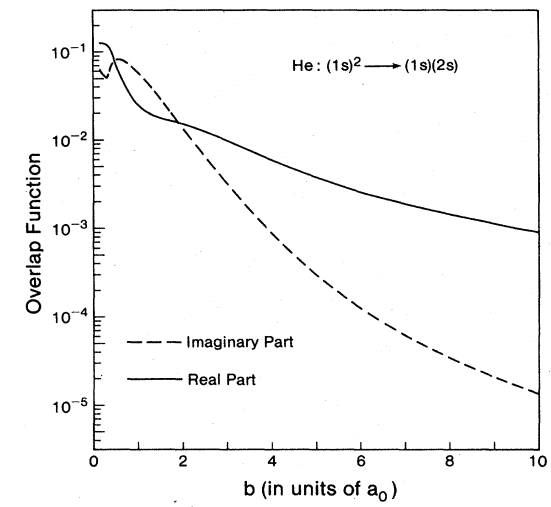


FIG. 3. Real and imaginary parts of $\Gamma_{\mathrm{He}}^{\mathrm{inel}}(b)$ for $1s^2 \rightarrow 1s2s$ transition in helium.

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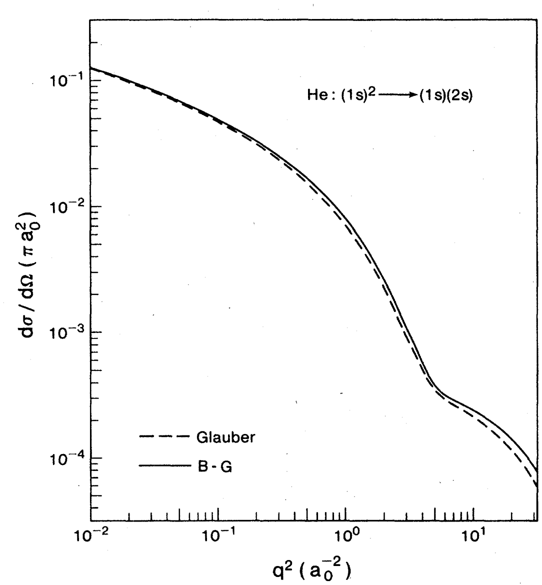


FIG. 4. $d\sigma/d\Omega$ for $1s^2 \rightarrow 1s2s$ transition in helium. Incident electron energy = 100 eV.

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