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Passage-time statistics for the decay of unstable equilibrium states

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The decay of unstable equilibrium states is accompanied by large-scale fluctuations. The statistical properties of such processes can be characterized by using the time at which a representative observable first passes through a fixed threshold value. We present an asymptotic probability distribution for that passage time which is valid when the threshold is set sufficiently far from the initial state. For the simplest example of linear isotropic amplification of an n-component vector we calculate both the exact first-passage time distribution and our asymptotic distribution. We verify that the asymptotic distribution coincides with the exact one in the appropriate limit. We then evaluate our asymptotic distribution for a number of more complicated systems including one in which an n-component vector field in d spatial dimensions departs from an unstable equilibrium state. The resulting expression has a considerable degree of universality. Its form is independent of d and of details of the field dynamics. It is insensitive, in particular, to whether the underlying field considered is conserved or not. Our procedure is applicable to a wide variety of problems in which an order parameter departs spontaneously from an unstable initial value.

I. INTRODUCTION

Under normal circumstances macroscopic observables fluctuate within extremely narrow ranges. In considering the decay of unstable equilibrium states, on the other hand, we must expect large fluctuations. Such states would, once prepared, be infinitely long lived were their decay not initiated by ever-present random forces of microscopic origin.

We can see that large fluctuations must develop in the course of the decay by considering the example of a particle sitting initially at the locally horizontal top of a hill. Owing to microscopic random forces the particle will begin moving randomly. Once it has left the top the particle will experience a systematic force and begin to accelerate downhill. Eventually, the microscopic random force will be dominated completely by the systematic one. After it has traveled a certain distance from the top the particle will tend to follow a deterministic trajectory characteristic of the shape of the hill. The "microscopic" size of the neighborhood of the top within which the particle moves randomly is determined by the strength of the random forces. Each time the particle is released from the top of the hill it will eventually follow a different deterministic trajectory because of the randomness of its early stage motion. In effect, the microscopic random force provides the deterministic trajectory with a random initial condition.

When the particle is observed a macroscopic distance away from the top at a time t after its release, it is likely to have spent an appreciable fraction of that time while still in random motion close to the top; that is true because the particle tends to move most slowly there. Different trajectories will tend, therefore, to arrive at a fixed distance from the top at rather different times. Alternatively, the particle will be found in broadly distributed locations if repeatedly looked for at a fixed time t after its release. Obviously, we can interpret such fluctuations within an ensemble of deterministic trajectories as large-scale manifestations of the microscopic random force which triggers the decay process.

The decay of unstable equilibrium states has been studied in various specific contexts such as the switch on of lasers,[1-4] superfluorescence,[5-8] hydrodynamic instabilities,[9,10] and spinodal decomposition.[11-13] The statistical properties of such processes have thus far been studied experimentally only in the field of quantum optics, however, where the relevant microscopic fluctuations are quantum mechanical in nature.[1-8] We hope that the present paper may serve to stimulate corresponding experiments in other fields.

In our present statistical analysis of the decay of unstable-equilibrium states we shall characterize each trajectory by a first-passage time, i.e., the time at which the variable considered first reaches a certain fixed distance from the point of unstable equilibrium. It may be of interest to note that first-passage-time distributions have already found applications in fields as diverse as anthropology[14] and biology,[15,16] as well as quantum optics[5-8] and electronics.[17] Nonetheless, to a theorist the passage time may at first appear as a somewhat forbidding concept since the classic first-passage-time problem[18,19] has thus far proved solvable only for the very simplest of random processes.

We shall show, however, that an asymptotic approximation to the first-passage-time distribution can be found. The approximation is an excellent one whenever the random forces initiating the decay process are sufficiently weak and as long as the passage is defined for a threshold sufficiently far from the point of unstable equilibrium. Under those conditions the departure of the system from its initial state can be described by using deterministic trajectories and random initial conditions, as we have just seen for the single-particle model (Sec. II). Further simplifications result from the fact that these deterministic trajectories are ordinarily quite unlikely to cross a macroscopic threshold more than once (Sec. III). We find that our asymptotic distribution can often be evaluated in closed form even when there is little hope of evaluating the exact first-passage-time distribution.

We apply our method, in Sec. IV, to the motion of an n-component vector $\vec{S}$, subject to a linear and isotropic amplifying force together with a rapidly changing random force. That system deserves special attention because we can also evaluate its exact first-passage-time distribution (Sec. IX) and verify explicitly that our asymptotic distribution approaches it in the appropriate limit. In Sec. V we investigate saturation effects by including a nonlinear restoring force in the equation of motion for $\vec{S}$. We show that the nonlinearity changes the asymptotic distribution of the passage time in effect only by replacing threshold by a higher one. We then add an inertial term $\sim d^2\vec{S}/dt^2$ to the equation of motion in Sec. VI and again find a change of the threshold and, additionally, a change of the time scale to be the only consequences for the passage-time distribution.

In Sec. VII we study a linear Ginzburg-Landau model for an n-component vector field in d spatial dimensions. Even though the asymptotic passage-time distribution is then defined by a functional integral over the space- and time-dependent noise field, we can construct it in closed form. In the special case $n=2$ our result reduces to one we had previously obtained for the distribution of delay times of superfluorescent pulses.[5] The structure of the result turns out to be remarkably universal. It is independent of the dimensionality of the position space and of the details of the dynamics including the presence or absence of a conservation law for the underlying field.

Finally, in Sec. VIII, we include a cubic saturation term in the Ginzburg-Landau field equation. By using the approximation of Kawasaki, Yalabik, and Gunton[13] for the deterministic trajectories we find the same asymptotic passage-time distribution as for the linear model, but altered once more by a change of the threshold.

As a byproduct of our rigorous treatment of the random n-component vector in Sec. IX we find the exact first-passage-time distribution for the Ornstein-Uhlenbeck process in n dimensions. We discuss that result and its asymptotic simplification in Appendix B.

II. ASYMPTOTIC DYNAMICS

The way in which many systems depart from states of unstable equilibrium can be described in terms of a single time-dependent variable $S(t)$. Typical examples of these variables are the amplitudes of convection rolls in the Bénard and Taylor instabilities and the electric-field strength in a single-mode laser. We shall in the present section treat the dynamics of such processes without specializing the description to any particular one of them.

We let $S(t)$ be a real variable subject to both a force γS proportional to S itself and a random force $f(t)$ which we describe as a Gaussian stochastic process with a white spectrum. The corresponding Langevin equation reads[20]

$$\dot{S}(t)=\gamma S(t)+f(t)\,. \tag{2.1}$$

The noise strength is defined by

$$\langle f(t)f(t')\rangle=\frac{2}{N}\delta(t-t'),\quad \langle f(t)\rangle=0\,. \tag{2.2}$$

Our principal interest is in positive values of the parameter γ, i.e., cases in which $S=0$ would be a point of unstable equilibrium, were the noise force neglected. The linear equation of motion (2.1) has the exact solution[20]

$$S(t)=e^{\gamma t}S(0)+\int_0^t dt'e^{\gamma(t-t')}f(t')\,. \tag{2.3}$$

By considering the moments $\langle S(t)^n\rangle$ obtained by averaging the nth power $S(t)^n$ over the Gaussian noise force we find that the motion of $S(t)$ is appreciably stochastic in nature only if the squared initial value $S(0)^2$ is smaller than or, at most, equal in order of magnitude to the noise strength $2/N$. The noise force is most important, of course, if the initial state is the unstable equilibrium state, $S(0)=0$, since in that case it is only the noise force $f(t)$ which induces change. We shall always be concerned in the present paper with the stochastic limit

$$S(0)^2\lesssim 2/N\,, \tag{2.4}$$

in which the trajectories are most random. For larger values of $S(0)$ the noise force becomes small compared with the systematic force at all times $t>0$ and the trajectories are considerably less random.

The expression (2.3) for the time-dependent variable $S(t)$ admits for large times the asymptotic simplification, for $t>1/\gamma$,

$$S_{as}(t)=e^{\gamma t}\left(S(0)+\int_0^\infty dt'\,e^{-\gamma t'}f(t')\right)=e^{\gamma t}s\,, \quad (2.5)$$

$$s=S(0)+\int_0^\infty dt'e^{-\gamma t'}f(t')\,.$$

This result lends itself to a reinterpretation of $S(t)$ as having a random initial value s but deterministic dynamics for $t>0$.

The discussion just given is easily generalized to include an eventual saturation of the amplification process (2.3). As a simple example, we shall employ the van der Pol equation

$$\dot{S}(t)=\gamma S(t)[1-S(t)^2]+f(t)\,. \quad (2.6)$$

In this case the amplitude $S(t)$ eventually approaches and settles in the neighborhood of the stable equilibrium states $|S|=1$. Since we want these states to be macroscopically distinguishable from the unstable equilibrium state $S=0$ we require the diffusion constant to be small,

$$1/N\ll 1\,. \quad (2.7)$$

The nonlinear damping force $-\gamma S^3$ in Eq. (2.6) begins to compete with the linear amplification force γS, according to Eqs. (2.2), (2.4), and (2.5), at times of the order $(1/2\gamma)\ln N$. Since we study the limit (2.7), this happens long after the time of order $1/2\gamma$ at which the fluctuating force $f(t)$ is overwhelmed by the systematic amplification force. Consequently, the amplitude $S(t)$ keeps on moving deterministically as the nonlinear regime is entered. An approximate solution of the van der Pol Eq. (2.6), which reduces to (2.5) in the linear amplification regime, reads

$$S_{as}(t,s)=e^{\gamma t}s[1+(e^{2\gamma t}-1)s^2]^{-1/2}\,. \quad (2.8)$$

The approximation (2.8) breaks down when the amplitude finally reaches the immediate neighborhood of the stable states $|S|=1$ since there the net systematic force again becomes small and comparable in magnitude with the random force $f(t)$. The solution (2.8) is acceptable as long as we confine our attention to the departure from unstable equilibrium and do not attempt to describe the final approach to stable equilibrium.

It is instructive to reformulate the arguments given up to this point by discussing the time-dependent probability density $P(S,t)$ for the amplitude S. Instead of the Langevin equation (2.1) with the Gaussian white-noise source characterized by Eq. (2.2) we then have to solve the stochastically equivalent Fokker-Planck equation[21]

$$\dot{P}(S,t)=\left(-\frac{\partial}{\partial S}\gamma S+\frac{\partial}{\partial S}\frac{1}{N}\frac{\partial}{\partial S}\right)P(S,t)\,. \quad (2.9)$$

We can take as an initial distribution $P(S,0)$, according to Eq. (2.4), any probability density which is sufficiently localized near $S=0$. We must require then that the initial values of both the squared mean and the mean square of S do not exceed $1/N$ in order of magnitude; if, for instance, the initial state is certain to be the unstable-equilibrium state $S=0$, we have

$$P(S,0)=\delta(S)\,. \quad (2.10)$$

The solution of the Fokker-Planck equation (2.9) obeying the initial condition (2.10) is well known and reads, if S is a simple real variable, then

$$P(S,t)\sim\exp[-S^2N\gamma/2(e^{2\gamma t}-1)]\,. \quad (2.11)$$

As long as γ is positive the random process described by the Fokker-Planck equation (2.9) can be associated with another process, the dynamics of which is purely deterministic.[4] To introduce this process we note that the convolution of $P(S,t)$ with a Gaussian function of width $(N\gamma)^{-1}$,

$$Q(S,t)=\mathfrak{N}\int dS'e^{-(S-S')(N\gamma)(S-S')/2}P(S',t)\,, \quad (2.12)$$

obeys the first-order differential equation

$$\dot{Q}(S,t)=-\frac{\partial}{\partial S}\gamma SQ(S,t)\,. \quad (2.13)$$

The factor $\mathfrak{N}$ in the transformation (2.11) serves to normalize $Q(S,t)$ to unity,

$$\mathfrak{N}^{-1}=\int dS\,e^{-S(N\gamma)S/2}\,. \quad (2.14)$$

The absence of second-order derivatives in Eq. (2.13) implies that the probability Q moves deterministically along the characteristic curves $S\sim\exp(\gamma t)$. The diffusive effects on $P(S,t)$ described by the second-derivative term in Eq. (2.9) are therefore completely represented by the broadening of Q relative to P inherent in the transformation (2.12). The unstable equilibrium state (2.10), in particular, is now represented by the Gaussian distribution

$$Q(S,0)=\mathfrak{N}e^{-SN\gamma S/2}\,. \tag{2.15}$$

The solution of Eq. (2.13) obeying the initial condition (2.15),

$$Q(S,t)\sim\exp(-\tfrac{1}{2}N\gamma e^{-2\gamma t}S^2) \tag{2.16}$$

is also somewhat broader than the corresponding P function (2.11). By comparing the solutions (2.11) and (2.16) we find that the two distributions P and Q become asymptotically equal for large times, as $e^{2\gamma t}\gg 1$. It is thus evident once more that the decay of an unstable state takes place, for large times, along deterministic trajectories and that the initial points of these trajectories may be regarded as having the Gaussian distribution (2.15).

If we include a saturation force as in the nonlinear Langevin equation (2.6), we must consider the corresponding Fokker-Planck equation

$$\dot{P}(S,t)=\left(-\frac{\partial}{\partial S}\gamma S(1-S^2)+\frac{\partial}{\partial S}\frac{1}{N}\frac{\partial}{\partial S}\right)P(S,t)\,. \tag{2.17}$$

The deterministic equation of motion for the associated distribution $Q(S,t)$,

$$\dot{Q}(S,t)=-\frac{\partial}{\partial S}\gamma S(1-S^2)Q(S,t)\,, \tag{2.18}$$

is then arrived at only after dropping higher-order-derivative terms from the combination of Eqs. (2.12) and (2.17). The time development of $Q(S,t)$ is in fact, described by Eq. (2.18) to an excellent degree of accuracy in the limit (2.7). Indeed, it is easy to verify[4] that the terms which are dropped from the exact equation of motion for $Q(S,t)$ do not affect low-order moments of Q by more than corrections of relative magnitude $1/N$ (provided again that the system is not yet close to stable equilibrium, $|S|=1$). Since in the limit (2.7) the saturation effects set in only after the distributions $P(S,t)$ and $Q(S,t)$ have become nearly equal, that equality continues to hold for all times of interest to our present investigation.

In order to demonstrate explicitly the equivalence of the two pictures of the asymptotic dynamics just presented we now evaluate the mean nth power $\langle S(t)^n\rangle$. In the Langevin picture we must, in general, perform two averaging processes. The first of these is a functional average over the random force $f(t)$ with the Gaussian weight

$$W(\{f(t)\})\sim\exp\left(-\tfrac{1}{4}\int_0^\infty dt\,f(t)Nf(t)\right)\,. \tag{2.19}$$

The second average is one over the initial value $S(0)$ with a weight $P(S(0),0)$ describing the initial state of the system. If we recall the definitions (2.5) and (2.8) of the asymptotic trajectories,

$$S_{as}(t,s)=S_{as}\left(t,S(0)+\int_0^\infty d\tau\,e^{-\gamma\tau}f(\tau)\right)\,, \tag{2.20}$$

we may write the mean value in question as

$$\langle S(t)^n\rangle=\int d\{f(t)\}\;W(\{f(t)\})\int dS(0)P(S(0),0)S_{as}(t,s)^n\,, \tag{2.21}$$

where $d\{f(t)\}$ is the differential measure in the space of random functions f.

Since the random force enters the asymptotic trajectory S_{as} only through the integral $\int_0^\infty d\tau \times e^{-\gamma\tau}f(\tau)$, the functional integral over $f(t)$ in Eq. (2.21) can be reduced to a single integral over a new variable s_0 with the weight

$$W(s_0)=\int d\{f(t)\}W(\{f(t)\})\delta\left(s_0-\int_0^\infty d\tau\,e^{-\gamma\tau}f(\tau)\right)\,. \tag{2.22}$$

Because of its linear relation to the random force the variable s_0 likewise has Gaussian statistics and zero mean. In Appendix A we calculate the functional integral in Eq. (2.22) and find

$$W(s_0)=\mathfrak{N}\exp(-\tfrac{1}{2}s_0N\gamma s_0)\,. \tag{2.23}$$

The mean value defined in Eq. (2.21) then takes the form

$$\langle S(t)^n\rangle=\int ds_0W(s_0)\int dS(0)P(S(0),0)S_{as}(t,S(0)+s_0)^n\,. \tag{2.24}$$

By using $S=S(0)+s_0$ as an integration variable we obtain

$$\langle S(t)^n\rangle=\int dS\,Q(S,0)S_{as}(t,S)^n\,, \tag{2.25}$$

where $Q(S,0)$ is related to $P(S,0)$ by the transformation (2.12). An alternative way of presenting this result is reached by letting $S'=S_{as}(t,S)$ and solving for S. Then, since the function $S_{as}(t,S)$ is a characteristic of Eq. (2.18), we find the asymptotic result

$$\langle S(t)^n\rangle=\int dS\,S^nQ(S,t)\,, \tag{2.26}$$

where $Q(S,t)$ satisfies Eq. (2.18). The equivalence of the two pictures presented is thereby fully established.

III. ASYMPTOTIC PASSAGE-TIME DISTRIBUTIONS

We assume here, as in the last section, that the random variable S is initially released from some point $S(0)$ within the microscopic neighborhood (2.4) of the point of unstable equilibrium $S=0$. We shift our interest, however, from the full time development of the trajectory $S(t)$ to the particular time or times at which the trajectory passes through some given "surface," e.g.,

$$S(t)^2 = M^2 , \tag{3.1}$$

where M^2 is a positive number. In principle, the variable $S(t)$ can, since it is driven by the random force $f(t)$, pass through the surface (3.1) any number of times. Because the motion is continuous, passages out of the region enclosed by the surface and into it must alternate. It is therefore necessary, in general, to further specify the definition of the passage time by asking, for example, for the time of the first passage[18,19] through the surface (3.1). The first-passage time depends, of course, on the initial value $S(0)$ and on the Langevin force $f(t)$ and thus is a random variable itself.

The first-passage time is an interesting quantity to study, especially if it is defined with respect to a threshold M^2 well outside the microscopic neighborhood (2.4) of the unstable equilibrium state $S=0$,

$$1/N \ll M^2 . \tag{3.2}$$

This large a threshold is, of course, more easily accessible to laboratory-scale measurements than one near the noise level. We must expect the passage time to display large fluctuations from one trajectory to the next. A simple argument suffices to indicate the reason. Because the variable $S(t)$ experience only a small force in the microscopic neighborhood $S^2 < 1/N$ of the unstable equilibrium, the greater part of its first-passage time tends to elapse before it escapes from that region. Once $S(t)^2$ has attained macroscopic values, however small compared to the threshold M^2, the motion is accelerated rapidly by the systematic force and $S(t)^2$ rushes up to M^2 in a comparatively brief interval of time. The time actually spent in the noise-dominated regime around $S=0$, however, fluctuates greatly since the random force $f(t)$ may as easily accelerate the variable $S(t)$ out of that regime hastily as keep it wandering around inside for more extended periods.

The probability distribution of the first-passage time is determined uniquely by the initial state of the system, i.e., the probability density $P(S,0)$ and the statistical properties of the noise force $f(t)$. It is notoriously difficult to evaluate exactly, though, except in some problems involving a single degree of freedom.[18,19] We shall show, however, that it is often possible to define and evaluate an asymptotic approximation to the first-passage-time distribution which can, for most practical purposes, be used in place of it in studying instability problems.

In order to define the asymptotic passage-time distribution we imagine the threshold M^2 to be set in a domain in which the systematic force on $S(t)$ greatly exceeds the strength $1/\sqrt{N}$ of the random force. The condition (3.2) is thus obeyed as well as analogous ones, requiring that M^2 not be close to any other equilibrium states. We can then be sure that the overwhelming majority of trajectories starting in the neighborhood of $S=0$ will pass through the surface (3.1) once and never return to it. Moreover, the passage will take place while $S(t)$ obeys the asymptotic deterministic dynamics discussed in Sec. II. We can therefore replace the definition (3.1) by

$$S_{as}(t,s)^2 = M^2 . \tag{3.3}$$

For almost all initial values s of S_{as} the Eq. (3.3) should have just one solution $T_1(s, M^2)$.

We now use the identity

$$\delta(S_{as}(t,s)^2 - M^2)\left|\frac{d}{dt}S_{as}(t,s)^2\right| = \sum_p \delta(t - T_p(s, M^2)), \tag{3.4}$$

where the sum on the right is a sum over all successive passage times $T_p, p=1,2,\ldots$, should there be more than one. We also recall that the initial value s of $S_{as}(t,s)$ is a random number with a probability distribution $Q(s,0)$ related to the initial distribution $P(S,0)$ of the observable S by Eq. (2.12). Let us imagine averaging both sides of Eq. (3.4) over the initial value s with the weight $Q(s,0)$. The right-hand side of Eq. (3.4) would then yield a sum of distribution functions. The pth term of that sum would be the distribution function for the time of the pth passage and have a time integral given by the measure in function space of trajectories with at least p passages through the surface (3.3). Inasmuch as our assumptions provide that the terms beyond the first have negligible weight we could use the average mentioned as an asymptotic approximation to the first-passage-time distribution.

In evaluating the average over s in Eq. (3.4) it is convenient to represent the modulus of $(d/dt)S_{as}(t,s)^2$ by using the identity $|x| = x[\Theta(x) - \Theta(-x)]$, where $\Theta(x)$ is the unit step function. We can thereby separate the average in Eq. (3.4) into two sums, one referring to passages with positive values of $(d/dt)S_{as}(t,s)^2$, the other with negative ones. For all trajectories which begin with $s^2<M^2$, that is the overwhelming majority of trajectories according to our assumptions, it is the odd-order passages which have $(d/dt)S_{as}(t,s)^2>0$. Actually, by dropping entirely the contributions of the even-order passages we can secure an even better approximation to the first-passage-time distribution. We shall therefore use, instead of Eq. (3.4), the identity

$$\delta(S_{as}(t,s)^2-M^2)\Theta\left(\frac{d}{dt}S_{as}(t,s)^2\right)\frac{d}{dt}S_{as}(t,s)^2 = \sum_{p\,\mathrm{odd}}\delta(t-T_p) \quad (3.5)$$

and define our asymptotic distribution as the average of the left-hand side of Eq. (3.5),

$$W(t)=\left\langle\delta(S_{as}(t,s)^2-M^2)\Theta\left(\frac{d}{dt}S_{as}(t,s)^2\right)\frac{d}{dt}S_{as}(t,s)^2\right\rangle . \quad (3.6)$$

It is clear that the asymptotic distribution (3.6) is not normalized to unity. For problems with a single degree of freedom the quantity $(d/dt)S_{as}(t,s)^2$ will always increase monotonically in time until the next equilibrium state is reached. In such cases no more than a single passage through the surface (3.3) can take place and the normalization integral cannot exceed unity. It will, in fact, be smaller than unity, since the distribution $Q(s,0)$ will, according to Eqs. (2.4) and (2.12), have a fraction of the order $\exp(-\frac{1}{2}NM^2)$ of initial values s with $s^2>M^2$. Obviously, that fraction is asymptotically negligible in the limit $N\gg1$.

Multiple passages and thus values of the normalization integral larger than unity do become possible, however, for systems with more than one degree of freedom.[7] We shall encounter such effects when dealing with field problems in Sec. VII. Whenever the normalization integral $\int_0^\infty dt \times W(t)$ deviates from unity only insignificantly our asymptotic distribution (3.6) can be expected to be a good approximation to the first-passage-time distribution.

IV. LINEAR ISOTROPIC AMPLIFICATION OF A VECTOR VARIABLE

As the simplest possible application of our asymptotic approximations we here consider an n-component vector S which obeys the linear Langevin equation

$$\frac{\partial\vec{S}}{\partial t}(t)=\vec{S}(t)+\vec{f}(t) . \quad (4.1)$$

We assume the random force vector $\vec{f}(t)$ to have independent components with the autocorrelation function

$$\langle f_i(t)f_j(t')\rangle=\delta_{ij}(2/N)\delta(t-t') . \quad (4.2)$$

We shall now evaluate the asymptotic distribution (3.5) of the times at which the vector $\vec{S}(t)$ passes through the hypersphere $\vec{S}^2(t)=M^2$. As we shall see presently the condition (3.2) on the value of M^2 must be strengthened to $NM^2\gg n$.

The asymptotic trajectory (2.5) of the vector $\vec{S}$ now reads

$$\vec{S}_{as}(t,\vec{s})=\vec{s}e^t \quad (4.3)$$

with the effective initial value

$$\vec{s}=\vec{s}(0)+\int_0^\infty dt\, e^{-t}\vec{f}(t) . \quad (4.4)$$

If the observable $\vec{S}$ starts out precisely at the point of unstable equilibrium, $\vec{S}(0)=0$, the effective initial vector $\vec{s}$ has the distribution (2.15), i.e.,

$$Q(\vec{s},0)=[\tfrac{1}{2}\Omega_n(2/N\gamma)^{n/2}\Gamma(n/2)]^{-1}\exp(-N\gamma\vec{s}^2/2) , \quad (4.5)$$

where $\gamma=1$ and $\Omega_n=2\pi^{n/2}/\Gamma(n/2)$ is the surface of the unit sphere in n dimensions. The distribution (3.6) then takes the form of an n-fold integral which is most easily evaluated in spherical coordinates. The $n-1$ angular integrals cancel the factor $1/\Omega_n$ form the normalization factor of $Q(\vec{s},0)$. The remaining integral over the modulus $s\equiv|\vec{s}|$,

$$W(t)=\frac{2}{\Gamma(n/2)}(N/2)^{n/2}\int_0^\infty ds\, s^{n-1}\exp(-\tfrac{1}{2}Ns^2) \times\delta(S_{as}(t,s)^2-M^2) \times\frac{d}{dt}[S_{as}(t,s)^2] , \quad (4.6)$$

yields the result

$$W(t)=\frac{2}{\Gamma(n/2)}(\tfrac{1}{2}NM^2e^{-2t})^{n/2}\exp(-\tfrac{1}{2}NM^2e^{-2t}) , \quad (4.7)$$

the maximum of which occurs when $NM^2\exp(-2t)=n$. A plot of $W(t)$ is shown in Fig. 1.

It will be useful for the futher discussion of $W(t)$ to evaluate its moments

$$\overline{t^\nu}=\int_0^\infty dt\, t^\nu W(t) , \quad (4.8)$$

or, equivalently, the characteristic function

$$\tilde{W}(\lambda)=\int_0^\infty dt\, e^{-\lambda t}W(t) . \quad (4.9)$$

The latter function has the moments (4.8) as its

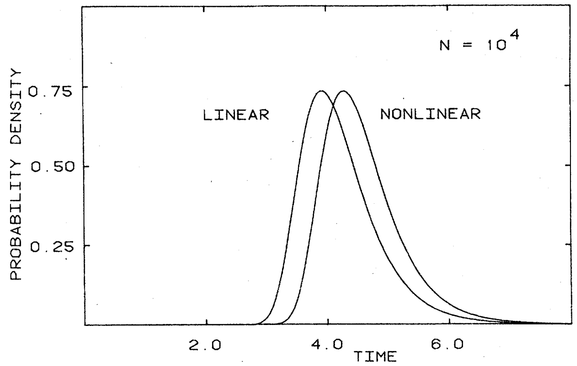

FIG. 1. The asymptotic passage-time distribution according to Eqs. (4.7) and (5.4) for $M^2=\frac{1}{2}$, $N=10^4$.

Taylor coefficients according to

$$\tilde{W}(\lambda)=\sum_{\nu=0}^{\infty}\frac{1}{\nu!}(-\lambda)^{\nu}\,\overline{t^{\nu}}\ . \tag{4.10}$$

Since the asymptotic distribution (4.7) is meaningful only for large values of the parameter NM^2 we can, inevaluating the integral (4.9), tolerate an error of the order $\exp(-NM^2)$ and replace the lower integration limit by $-\infty$. The characteristic function can then be expressed by means of the gamma function as

$$\tilde{W}(\lambda)=(\tfrac{1}{2}NM^2)^{-\lambda/2}\,\Gamma\left(\frac{\lambda+n}{2}\right)\Big/\Gamma\left(\frac{n}{2}\right)\ . \tag{4.11}$$

Although the result $W(0)=1$ seems to imply that the normalization integral of $W(t)$ is unity, it is actually somewhat smaller because of the approximation we have made in extending the integral (4.9) to $-\infty$. The error of that approximation is evidently of order

$$\frac{2}{\Gamma(n/2)}\,(\tfrac{1}{2}NM^2)^{n/2}\exp(-\tfrac{1}{2}NM^2) \tag{4.12}$$

and is thus negligible as long as $NM^2\gg n$. It follows that the asymptotic passage-time distribution (4.7) is an excellent approximation to the exact first-passage-time distribution. We shall see, in fact, in Sec. IX that Eq. (4.11) can also be obtained from the latter in the limit $NM^2\gg 1$.

By differentiating the expression (4.11) we find the mean passage time[22]

$$\bar{t}=\tfrac{1}{2}\left[\ln\left(\frac{NM^2}{2}\right)-\psi\left(\frac{n}{2}\right)\right]\ , \tag{4.13}$$

and its relative variance,

$$(\overline{t^2}-\bar{t}^2)/\bar{t}^2=\psi'\left(\frac{n}{2}\right)\Big/\left[\ln\left(\frac{NM^2}{2}\right)-\psi\left(\frac{n}{2}\right)\right]^2\ , \tag{4.14}$$

where ψ and ψ' denote the digamma and the trigamma functions,[23] respectively. We should emphasize that the relative variance depends only logarithmically on the small parameter $1/N$. It therefore tends to remain of order unity even when N takes on enormous values. This result offers quantitative support to our earlier argument that the passage time fluctuates strongly.

V. SATURATION EFFECTS

Since the greater part of the passage time elapses while the variable S is still close to the unstable equilibrium point $S=0$ where nonlinearities in the systematic force are quite negligible, we may surmise that such nonlinearities have little quantitative influence on the passage-time statistics. In order to check that expectation we here generalize the Langevin Equation (4.1) to include a saturation term,

$$\frac{\partial\vec{S}}{\partial t}(t)=\vec{S}(t)[1-\vec{S}(t)^2]+\vec{f}(t)\ . \tag{5.1}$$

According to the argument given in Sec. II we must then replace the asymptotic trajectory (4.3) by

$$\vec{S}_{as}(t,\vec{s})=\vec{s}e^{t}[1+\vec{s}^{\,2}(e^{2t}-1)]^{-1/2}\,. \tag{5.2}$$

If we again assume the observable $\vec{S}$ to start our at $\vec{S}(0)=0$ the distribution (4.5) of effective initial vectors remains unchanged. The asymptotic passage-time distribution (4.6) then reads

$$W(t)=\frac{2}{\Gamma(n/2)}\left(\frac{1-M^2}{1-M^2+M^2e^{-2t}}\right)\left(\frac{1}{2}\,\frac{NM^2e^{-2t}}{1-M^2+M^2e^{-2t}}\right)^{n/2}\times\exp\left(-\frac{1}{2}\,\frac{NM^2e^{-2t}}{(1-M^2+M^2e^{-2t})}\right)\ . \tag{5.3}$$

If we set the value of M^2 so that neither it nor $1-M^2$ is close to zero, the distribution (5.3), as well as the one given in Eq. (4.7), has practically all of its weight concentrated at times at which Ne^{-2t}/n is of order unity. We can then, in Eq. (5.3), neglect M^2e^{-2t} against $1-M^2$ without thereby affecting the moments of $W(t)$ by more than corrections of order $1/N$. The resulting distribution,

$$W(t)=\frac{2}{\Gamma(n/2)}\left(\frac{1}{2}\,\frac{NM^2}{1-M^2}\,e^{-2t}\right)^{n/2}\exp\left(-\frac{1}{2}\,\frac{NM^2}{1-M^2}e^{-2t}\right)\ , \tag{5.4}$$

and its moments differ from the corresponding expressions obtained for the linear amplification problems, Eqs. (4.7), (4.12), and (4.13) only by the replacement of M^2 by $M^2/(1-M^2)$. The corresponding shift of the passage times to slightly larger values is an intuitively obvious saturation effect since the nonlinear trajectories for $S_{as}(t)^2$

rise less steeply with time than their linear counterparts. We show in Fig. 1 plots of $W(t)$ for both the linear and the nonlinear theory.

VI. INERTIA EFFECTS

We here generalize the analysis of Sec. IV by including an inertial term in the equation of motion of the vector variable,

$$m\frac{d^2\vec{S}}{dt^2}+\frac{d\vec{S}}{dt}-\vec{S}=\vec{f}(t)\ . \tag{6.1}$$

We shall discuss the decay of the unstable-equilibrium state initiated by the random force $\vec{f}(t)$ alone, i.e., we pose the initial condition[24]

$$\vec{S}(0)=0,\ \ \frac{d\vec{S}}{dt}(0)=0\ . \tag{6.2}$$

The qualitatively new feature of the problem so defined is that the two eigenvalues of the homogeneous part of Eq. (6.1),

$$\gamma_\pm=[\pm(1+4m)^{1/2}-1]/2m\ , \tag{6.3}$$

differ in sign. The state $\vec{S}=0$ is thus an unstable-equilibrium state for one eigenvector of the system but a stable one for the other.

The solution of the equation of motion (6.1) obeying the initial condition (6.2) reads

$$\vec{S}(t)=\frac{1}{(1+4m)^{1/2}}\int_0^t dt'(e^{\gamma_+(t-t')}-e^{\gamma_-(t-t')})\vec{f}(t')\ . \tag{6.4}$$

In order to find the relative weight of the two modes contributing in Eq. (6.4) we must consider, e.g., the expectation value

$$\langle S_i(t)S_j(t)\rangle=\frac{1}{1+4m}\,\frac{2}{N}\,\delta_{ij}\int_0^t dt'\,(e^{\gamma_+t'}-e^{\gamma_-t'})^2\ . \tag{6.5}$$

Obviously, for large times, when $e^{\gamma_+t}\gg 1$, all expectation values and thus the random vector $\vec{S}(t)$ itself, too, are dominated by the growing mode. As an asymptotic approximation to the exact solution (6.4) we therefore have

$$\vec{S}_{as}(t,\vec{s})=\vec{s}\,e^{\gamma_+t} \tag{6.6}$$

with the effective initial vector

$$\vec{s}=\frac{1}{(1+4m)^{1/2}}\int_0^\infty dt\,e^{-\gamma_+t}\vec{f}(t)\ . \tag{6.7}$$

The derivation of the probability distribution (4.5) for $\vec{s}$ may now be repeated. The scale change by the factor $1/(1+4m)^{1/2}$ in Eq. (6.7) than leads to the distribution $(1+4m)^{n/2}\,Q[\vec{s}(1+4m)^{1/2},0]$ with $Q(\vec{s},0)$ as in Eq. (4.5).

The evaluation of the asymptotic passage-time distribution (3.6) proceeds as in Sec. IV and yields

$$W(t)=\frac{2\gamma_+}{\Gamma(n/2)}\,[\tfrac{1}{2}N\gamma_+M^2(1+4m)e^{-2\gamma_+t}]^{n/2}\times\exp[-\tfrac{1}{2}N\gamma_+M^2(1+4m)e^{-2\gamma_+t}]\ . \tag{6.8}$$

This result differs from Eq. (4.7) only by a change of the time scale by a factor γ_+ and a change of the effective threshold, M^2 to $M^2(1+4m)$. With these scale changes we obtain the moments of the distribution (6.7) from the previous results (4.13) and (4.14). The mean and the squared relative variance of the passage time are

$$\bar{t}=\frac{1}{\gamma_+}\{\ln[\tfrac{1}{2}N\gamma_+M^2(1+4m)]-\psi(n/2)\} \tag{6.9}$$

and

$$(\overline{t^2}-\bar{t}^2)/\bar{t}^2=\psi'(n/2)/\{\ln[\tfrac{1}{2}N\gamma_+M^2(1+4m)]-\psi(n/2)\}^2\ , \tag{6.10}$$

respectively. Since the time-scale factor γ_+ cancels from the relative variance the latter quantity depends much less sensitively on the mass parameter m than the mean passage time does.

VII. LINEAR GINZBURG-LANDAU MODEL

Macroscopic systems must often be described by means of position-dependent fields rather than a discrete set of coordinates. As a simple example[12,13] we consider an n-component vector field $\vec{S}(\vec{x},t)$ defined in d spatial dimensions. Such fields are generally encountered in the Ginzburg-Landau theory of order-parameter relaxation. The field $\vec{S}$ could, for example, be the magnetization of a ferromagnet below the Curie temperature. An initial state of zero magnetization is then unstable and relaxes towards a state of finite magnetization. In the linear approximation the relaxation is described by equations such as[25]

$$\frac{\partial}{\partial t}\vec{S}(\vec{x},t)=(-\nabla^2)^a[\vec{S}(\vec{x},t)+\nabla^2\vec{S}(\vec{x},t)]+\vec{f}(\vec{x},t) \tag{7.1}$$

with $a=0$ or 1. The decay of the unstable-equilibrium state $\vec{S}=0$, is for these equations, initiated by the microscopic random force $\vec{f}$. We assume the latter to be Gaussian in nature with zero mean and an autocorrelation function

$$\langle f_i(\vec{x},t)f_j(\vec{x}',t')\rangle=\frac{2}{N}\,\delta_{ij}(-\nabla^2)^a\delta^d(\vec{x}-\vec{x}')\delta(t-t')\ . \tag{7.2}$$

In the case $a=0$ the field equation (7.1) generalizes Eq. (4.1) only by the inclusion of spatial diffusion through the term $-\nabla^2\vec{S}$. By letting $a=1$, on the other hand, we impose the conservation law[12]

$$\frac{d}{dt}\int d^dx\,\vec{S}(\vec{x},t)=0 \quad \text{for} \quad a=1 \ . \tag{7.3}$$

The equation of motion (7.1) is most easily solved after spatial Fourier transformation since the Fourier components $\vec{S}(\vec{k},t)$ move independently of one another. The homogeneous equation for $\vec{S}(\vec{k},t)$ is satisfied by exponential functions $\exp(\gamma_k t)$ with

$$\gamma_k=(\vec{k}^2)^a(1-\vec{k}^2) \ . \tag{7.4}$$

The parameter γ_k is a growth rate for $\vec{k}^2<1$ (except for the case $a=1$, where $k=0$ labels the conserved mode) and a damping rate for larger wave numbers. For each growing mode there is a time domain $\gamma_k t\gg 1$, in which the exact solution

$$\vec{S}(\vec{k},t)=\vec{S}(\vec{k},0)e^{\gamma_k t}+\int_0^t dt'\,e^{\gamma_k(t-t')}\vec{f}(\vec{k},t') \tag{7.5}$$

of Eq. (7.1) can be accurately replaced by the asymptotic approximation

$$\vec{S}_{as}(\vec{k},t)=\vec{s}_{\vec{k}}\,e^{\gamma_k t} \ , \tag{7.6}$$

where the effective initial vector is given by

$$\vec{s}_{\vec{k}}=\vec{S}(\vec{k},0)+\int_0^\infty dt\,e^{-\gamma_k t}\vec{f}(\vec{k},t) \ . \tag{7.7}$$

If the initial state of the system is the unstable-equilibrium state, $\vec{S}(\vec{x},0)=0$, Eq. (2.15) or (4.5) shows that the probability density for the random vectors $\vec{S}_k$ with $|\vec{k}|<1$ is given by

$$Q(\{\vec{s}\})=\prod_k\left\{\left[\tfrac{1}{2}\Omega_n\left(\frac{1}{N(1-k^2)}\right)^{n/2}\Gamma\left(\frac{n}{2}\right)\right]^{-1}(\Delta k)^{d/2}\times\exp[-\tfrac{1}{2}N(1-k^2)|\vec{s}_{\vec{k}}|^2(\Delta k)^d]\right\}. \tag{7.8}$$

If, more generally, the initial state is specified by a probability density $P(\{S_{\vec{k}}\},0)$, we obtain the distribution for $\vec{s}_{\vec{k}}$ as the convolution of $P(\{\vec{S}_{\vec{k}}\},0)$ with the density given by Eq. (7.8) by analogy with Eq. (2.12).

There are now various ways in which passage times may be defined. We may, for instance, think of systems where scattering experiments are carried out to observe the growth of single Fourier components $\vec{S}(\vec{k},t)$. It would then be natural to define a passage time for the Fourier component under observation, i.e., by $|\vec{S}(\vec{k},t)|^2=M^2\gg n/N$. Because of the independence of different Fourier components the asymptotic distribution of that passage time is given by Eq. (4.7) (with γ_k^{-1} as the unit of time).

We may also imagine the local growth of the field $\vec{S}(\vec{x},t)$ in the neighborhood of some point $\vec{x}$ to be observed experimentally. An appropriate passage time would then be determined by $\vec{S}(\vec{x},t)^2=M^2\gg n/N$. In fact, if we insist on such a strictly local definition, the presence of decaying as well as growing Fourier components in $\vec{S}(\vec{x},t)$ precludes the applicability of our asymptotic approximation (7.6) for the trajectories. However, $\vec{S}(\vec{x},t)$ will be dominated, for sufficiently large times, by its growing Fourier components. As we shall see, it is possible to approximate the passage-time distribution as well as $\vec{S}(\vec{x},t)$ by omitting the Fourier components for which the asymptotic approximation breaks down. What we shall do, in other words, is to use a cutoff sum over Fourier components instead of the strictly local field $\vec{S}(\vec{x},t)$. In the case of a nonconserved field $a=0$, an ultraviolet cutoff Λ,

$$|\vec{k}|\leqslant\Lambda<1 \quad \text{for} \quad a=0 \ , \tag{7.9}$$

suffices to secure the validity of Eq. (7.6) for times $t>1/\gamma_\Lambda$, while an infrared cutoff λ must be included as well if the field is conserved

$$0<\lambda\leqslant|\vec{k}|\leqslant\Lambda<1 \quad \text{for} \quad a=1. \tag{7.10}$$

With the above arguments in mind we understand all wave-vector integrals encountered below to be cut off in the sense of (7.9) or (7.10). We take, in particular, as an asymptotic approximation to the local field $\vec{S}(\vec{x},t)$ the cutoff Fourier integral

$$\vec{S}_{as}(\vec{x},t,\{s\})=\int_{\text{cutoff}}\frac{d^dk}{(2\pi)^d}\,\vec{S}_{as}(\vec{k},t,\vec{s}_{\vec{k}})e^{-i\vec{k}\vec{x}} \ , \tag{7.11}$$

and define our passage time by

$$\vec{S}_{as}(\vec{x},t,\{s\})^2=M^2\gg n/N \ . \tag{7.12}$$

For our asymptotic approximation to be self-consistent the probability distribution of the passage time just defined must concentrate almost all of its weight on times large compared to all characteristic times $1/\gamma_k$ associated with the Fourier components we have retained. Moreover, the moments of the passage-time distribution must not depend strongly on the cutoff wave number(s). As always, the normalization integral of the asymptotic distribution must be close to unity.

In order to find the asymptotic passage-time distribution we have to perform a functional average over all configurations of the effective initial vector field $\vec{s}(\vec{x})$ in

$$W(t)=\left\langle\delta(|\vec{S}_{as}|^2-M^2)\,\frac{d|\vec{S}_{as}|^2}{dt}\,\Theta\left(\frac{d|\vec{S}_{as}|^2}{dt}\right)\right\rangle . \tag{7.13}$$

For the sake of simplicity we shall suppose that the field $\vec{S}$ is certain to vanish initially so that the average in Eq. (7.13) is one with the function $Q(\vec{s}_k,0)$ given by Eq. (7.8) as the weight for $\vec{s}_k$.

The unit step function in Eq. (7.13) now plays an essential role. In fact, we can easily verify that not all trajectories of $\vec{S}_{as}(\vec{x},t,\{\vec{s}\})^2$ always increase with time. To do that we express the linear relation of $\vec{S}_{as}(\vec{x},t,\{\vec{s}\})$ with its effective initial value by means of an appropriate integral kernel $G(\vec{x},t)$ as

$$\vec{S}_{as}(\vec{x},t,\{\vec{s}\})=\int d^dx'\,G(\vec{x}-\vec{x}',t)\vec{s}(\vec{x}') . \tag{7.14}$$

The time derivative of S_{as}^2 is then a quadratic form in $\vec{s}(\vec{x})$,

$$\frac{d}{dt}[\vec{S}_{as}(\vec{x},t,\{\vec{s}\})]^2=\int d^dx'\int d^dx''K(\vec{x}',\vec{x}'')\vec{s}(\vec{x}')\vec{s}(\vec{x}'') , \tag{7.15}$$

with the kernel

$$K(\vec{x}',\vec{x}'')=\frac{d}{dt}[G(\vec{x}-\vec{x}',t)G(\vec{x}-\vec{x}'',t)] . \tag{7.16}$$

We can find eigenvectors $V(\vec{x})$ of the kernel $K(\vec{x}',\vec{x}'')$,

$$\int d^dx''K(\vec{x}',\vec{x}'')V(\vec{x}'')=\lambda V(\vec{x}') , \tag{7.17}$$

by superposing G and its time derivative

$$V(\vec{x}')=\alpha G(\vec{x}-\vec{x}',t)+\beta\dot{G}(\vec{x}-\vec{x}',t) . \tag{7.18}$$

Inserting the ansatz (7.18) into (7.17) reduces the eigenvalue problem for the kernel K to that of a two-by-two matrix. An elementary calculation then shows that one eigenvalue of K is always negative and hence that $(d/dt)S_{as}^2$ is not positive definite.

The further evaluation of the functional integral slightly generalizes a previous calculation of the delay-time statistics of superfluorescent pulses.[7] We represent the delta function and the step function in Eq. (7.13) by Fourier integrals

$$\delta(\xi)=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,e^{-i\omega\xi} ,$$
$$\theta(\xi)=\lim_{\epsilon\to 0+}\int_{-\infty}^{+\infty}\frac{d\Omega}{2\pi}\,\frac{e^{-i\Omega\xi}}{-i\Omega+\epsilon} , \tag{7.19}$$

and write our passage-time distribution in the form

$$W(t)=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d\Omega}{2\pi}\,\frac{e^{-i\omega M^2}}{(-i\Omega+\epsilon)}\,\frac{\partial}{\partial(-i\Omega)}D(\omega,\Omega), \tag{7.20}$$

where

$$D(\omega,\Omega)=\left\langle\exp\left(+i\omega|\vec{S}_{as}(\vec{x},t,\{\vec{s}\})|^2-i\Omega\frac{d}{dt}|\vec{S}_{as}(\vec{x},t,\{\vec{s}\})|^2\right)\right\rangle \tag{7.21}$$

is a Gaussian functional integral. We can express the latter, after using Eq. (7.8), as the nth power of a similar integral over one vector component of the field $\vec{s}$,

$$D(\omega,\Omega)=\left[\mathfrak{N}\int d\{s(\vec{x})\}\exp\left(-\int d^dx'\int d^dx''M(\omega,\Omega,\vec{x}',\vec{x}'')s(\vec{x}')s(\vec{x}'')\right)\right]^n . \tag{7.22}$$

Here, the normalization constant $\mathfrak{N}$ is determined by the requirement $D(0,0)=1$ and the kernel M is given by

$$M(\omega,\Omega,\vec{x}',\vec{x}'')=\frac{N}{2}(1+\nabla'^2)\delta^d(\vec{x}'-\vec{x}'')+i\omega G(\vec{x}-\vec{x}',t)G(\vec{x}-\vec{x}'',t)+i\Omega\frac{d}{dt}[G(\vec{x}-\vec{x}',t)G(\vec{x}-\vec{x}'',t)] . \tag{7.23}$$

We defer the further evaluation of $D(\omega,\Omega)$ to Appendix A. The result can be expressed in terms of the average

$$\frac{1}{n}\langle|\vec{S}_{as}|^2\rangle=\frac{1}{N}\int\frac{d^dk}{(2\pi)^d}\,\frac{e^{2\gamma_k t}}{1-k^2}\equiv\Sigma(t) , \tag{7.24}$$

its time derivative $\dot{\Sigma}(t)$, and the mean-squared time derivative of the field,

$$\frac{1}{n}\left\langle\left|\frac{d\vec{S}_{as}}{dt}\right|^2\right\rangle=\frac{1}{N}\int\frac{d^dk}{(2\pi)^d}\,\frac{\gamma_k^2e^{2\gamma_k t}}{1-k^2} . \tag{7.25}$$

We use the convenient abbreviation

$$\Delta(t)^2=\frac{1}{n^2}\left(\langle|\vec{S}_{as}|^2\rangle\left\langle\left|\frac{d\vec{S}_{as}}{dt}\right|^2\right\rangle-\left\langle\vec{S}_{as}\cdot\frac{d\vec{S}_{as}}{dt}\right\rangle^2\right) \tag{7.26}$$

and write the result derived in Appendix A as

$$D(\omega,\Omega)=[1+i\omega\Sigma(t)+i\Omega\dot{\Sigma}(t)+\Omega^2\Delta(t)^2]^{-n/2} . \tag{7.27}$$

The remaining two integrals in Eq. (7.20) are elementary and yield our final result for the asymptotic passage-time distribution,

$$W(t)=\frac{1}{\Gamma(n/2)}\left(\frac{M^2}{\Sigma(t)}\right)^{n/2-1}\exp\left(-\frac{M^2}{\Sigma(t)}\right)\times\left\{\frac{M^2\dot{\Sigma}(t)}{2\Sigma(t)^2}\left[1+\mathrm{erf}\left(\frac{\dot{\Sigma}(t)^2M^2}{4\Sigma(t)\Delta(t)^2}\right)^{1/2}\right]+\left(\frac{M^2\Delta(t)^2}{\pi\Sigma(t)^3}\right)^{1/2}\exp\left(-\frac{\dot{\Sigma}(t)^2M^2}{4\Sigma(t)\Delta(t)^2}\right)\right\}. \tag{7.28}$$

We may point out that the asymptotic passage-time distribution takes the same form whatever the dimensionality d of the position space is. It is not even bound to the special form of the deterministic (homogeneous) part of the field Eq. (7.1). In particular, it holds whether or not the field $\vec{S}$ is conserved, i.e., for $a=0$ and 1. The case $n=2$ has found an application in superfluorescence,[7] a problem with field equations quite different in appearance from (7.1). We shall show in the next section that the validity of Eq. (7.28) may even extend to some nonlinear field equations. It is indispensable, though, that the distribution of the effective initial field be Gaussian in nature. While the form (7.28) for $W(t)$ is, as we have noted, rather universal, the expectation values $\langle|\vec{S}_{as}(t)|^2\rangle$ and $\langle|d\vec{S}_{as}(t)/dt|^2\rangle$ do indeed depend on the dimensionality as well as on details of the field dynamics. We must still check whether the expectation values $\Sigma(t)$ and $\Delta(t)^2$ are sufficiently insensitive to the wave-vector cutoff for the result (7.28) to be an acceptable approximation to the first-passage-time distribution of the Ginzburg-Landau model. We shall here present such a consistency check for the case $a=0$.

It is obvious from Eq. (7.24) and (7.25) that, in contrast to the function $\Sigma(t)$, neither the time derivative $\dot{\Sigma}(t)$ nor the mean-square derivative (7.25) diverges as the cutoff Λ approaches unity. We split the potentially dangerous function $\Sigma(t)$ into a singular part

$$\Sigma(0)=\frac{2}{N}\frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^\Lambda dk\,\frac{k^{d-1}}{1-k^2}, \tag{7.29}$$

and a regular part

$$\int_0^t dt'\dot{\Sigma}(t')=\frac{4}{N}\frac{1}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^t dt'\int_0^\Lambda dk\,k^{d-1}e^{2\gamma_k t'}, \tag{7.30}$$

which can be expressed in terms of hypergeometric functions. Since the singularity in $\Sigma(0)$ is only a logarithmic one, we can expect that $\Sigma(t)$ and thus $W(t)$ suffer no appreciable cutoff dependence except when Λ is quite close to unity.

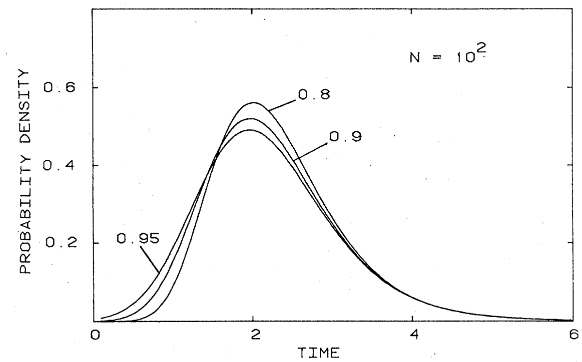

FIG. 2. The asymptotic passage-time distribution according to Eq. (7.28) for $n=d=2$, $M^2=1$, $N=10^2$, and various values of the cutoff, $\Lambda=0.8$, 0.9, and 0.95.

This expectation is fully borne out when $\Sigma(t)$ and $W(t)$ are evaluated numerically for a sufficiently large value of N. In Fig. 2 we show the distribution $W(t)$ obtained for $n=d=2$, $N=10^2$, $M^2=1$, and $\Lambda=0.8$, 0.9, and 0.95. Even though the moments of these three distributions do not differ by more than a few percent, the cutoff dependence of the distribution $W(t)$ itself is quite noticeable. If, however, N is increased to 10^4, we obtain a distribution $W(t)$ which is shown in Fig. 3. To within the accuracy of the plot there is no cutoff dependence in the range $0.7\lesssim\Lambda\lesssim 0.999$.

We can draw similar conclusions from Fig. 4 where we plot, as a rough measure of the mean passage time, the solution τ of the equation

$$\Sigma(\tau)=M^2/n=\tfrac{1}{2}. \tag{7.31}$$

The lower curve pertains to $N=10^2$ and shows a significant dependence on Λ, whereas the upper curve, corresponding to $N=10^4$, describes a constant to within 0.5% for $0.7\lesssim\Lambda\lesssim 0.999$.

As the strength $1/N$ of the microscopic noise is further decreased the cutoff independence of

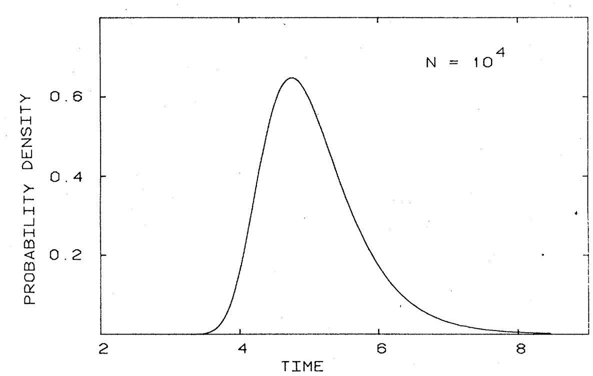

FIG. 3. Same as Fig. 2 but for $N=10^4$; there is no visible cutoff dependence for $0.7\leqslant\Lambda\leqslant 0.999$.

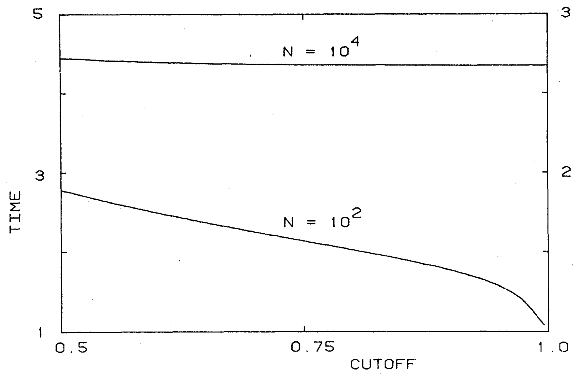

FIG. 4. Cutoff dependence of the approximate average passage time defined in Eq. (7.31). The drop of the curve for $N=10^4$ which occurs near $\Lambda=1$ is not resolved in the plot. The left and right vertical scales refer to the the cases $N=10^4$ and 10^2, respectively.

τ [as well as that of $W(t)$] must extend to ever smaller values of Λ. The reason is simply that the preponderance of the fastest growing modes $\vec{S}_{\vec{k}}(t)$ over the slower ones will be more pronounced when the gap between the threshold M^2 and typical effective initial values of $|\vec{S}_k|$ becomes larger.

Obviously, then, for sufficiently large N we can choose a Λ from within the range of cutoff independence such that even the most slowly growing mode retained reaches its asymptotic behavior (7.6) for times near the typical passage time τ. This argument demonstrates the self-consistency of our asymptotic result (7.28).

VIII. NONLINEAR GINZBURG-LANDAU MODEL

When we add a saturation term to the field equation (7.1) we can, in general, no longer find nontrivial exact solutions in closed form. We could then try to find the passage-time statistics by numerical means. To that end we would have to calculate a large number of deterministic solutions $\vec{S}_{as}(\vec{x},t,\{\vec{s}\})$ the effective initial vectors of which adequately represent the distribution (7.8).[5] Alternatively, we can use closed-form approximations to $\vec{S}_{as}$. To illustrate that method let us consider the nonlinear Ginzburg-Landau equation

$$\frac{d\vec{S}}{dt}=\vec{S}(1-\alpha\vec{S}^2)+\nabla^2\vec{S}+\vec{f} \tag{8.1}$$

with $\alpha>0$. The noise force $\vec{f}$ is assumed to have the same properties as the one we used in Sec. VII. Suitable approximate solutions for this equation have been found by Kawasaki, Yalabik, and Gunton[13] for the case $n=1$. We follow these authors by introducing an associated field $S_0(\vec{x},t)$ by means of the transformation

$$S(\vec{x},t)=S_0(\vec{x},t)[1+\alpha S_0(\vec{x},t)^2]^{-1/2} \tag{8.2}$$

The deterministic field equation for S_0,

$$\dot{S}_0=S_0+\nabla^2 S_0+3\alpha S_0(\nabla S_0)^2/(1+\alpha S_0^2)\,, \tag{8.3}$$

appears to be somewhat more complicated than Eq. (8.1). However, the nonlinear term in Eq. (8.3), in contrast to the one in Eq. (8.1), contains the squared gradient of the field. We expect the nonlinearity in Eq. (8.3) to be less important than the original saturation term in Eq. (8.1). By the time saturation effects become appreciable the diffusion term in the field equation has tended to smooth out spatial inhomogeneities in the field S, so that $S_0(\nabla S_0)^2$ will indeed tend to be considerably smaller than S_0^3. In fact, we see by comparison with Eq. (5.2) that for a strictly homogeneous field the ansatz (8.2) represents the correct long-time behavior of $S(\vec{x},t)$ if $S_0(\vec{x},t)$ is evaluated from Eq. (8.3) with $\alpha=0$. Since the nonlinearity is negligible for small times in any case we should obtain a reasonable approximation for $S(\vec{x},t)$ if we set $\alpha=0$ in Eq. (8.3) and thus reduce it to a linear equation; the saturation of the field $S(\vec{x},t)$ is then accounted for only by the nonlinear relation (8.2) between S_0 and S.

We could not expect the approximation just described to be meaningful for all Fourier components $S(\vec{k},t)$ of the field $S(\vec{x},t)$. The behavior of the components with $|\vec{k}|\simeq 1$, for example, will be especially distorted by neglecting the third term on the right-hand side of Eq. (8.3). These short-wavelength Fourier components will have no appreciable influence, however, on the field $S(\vec{x},t)$, provided the initial configuration $S(\vec{x},0)$ is sufficiently smooth.

The qualitative argument just presented makes no reference to a specific number of components n of the field $\vec{S}$. We shall, in fact, use the ansatz (8.2) for any n, to determine the asymptotic time dependence from the linear field equation

$$\frac{d\vec{S}_0}{dt}(\vec{x},t,\{\vec{s}\})=(1+\nabla^2)\vec{S}_0(\vec{x},t,\{\vec{s}\})\,, \tag{8.4}$$

and take the effective initial field to be distributed according to Eq. (7.8). Starting from the general expression (3.5) we can easily reduce the asymptotic passage-time distribution to the one found in Sec. VII. We first observe that $d\vec{S}^2/dt$ and $d\vec{S}_0^2/dt$ have, according to the transformation (8.2), the same sign. We then verify that

$$\delta(\vec{S}^2-M^2)\,\frac{d\vec{S}^2}{dt}=\delta(\vec{S}_0^2-M_0^2)\,\frac{d\vec{S}_0^2}{dt} \tag{8.5}$$

with

$$M_0^2 = M^2/(1-\alpha M^2) \tag{8.6}$$

and conclude that

$$W(t) = \left\langle \delta(\vec{S}_{0,as}^2 - M_0^2) \frac{d\vec{S}_{0,as}^2}{dt} \Theta\left(\frac{d}{dt}\vec{S}_{0,as}^2\right)\right\rangle , \tag{8.7}$$

where $S_{0,as}$ is the asymptotic solution of Eq. (8.4). The only difference between Eqs. (8.7) and (7.13) is the replacement of the threshold M^2 by M_0^2. By that replacement we then obtain the final result for the passage-time distribution $W(t)$ from Eq. (7.28). Since $M_0^2 > M^2$ the passage times tend to be larger than in the linear problem. The nonlinearity does not otherwise have any qualitative influence on the statistics of the passage time.

IX. SOME RIGOROUS RESULTS FOR THE FIRST-PASSAGE-TIME DISTRIBUTIONS

In two classic papers Siegert and Darling[18,19] solved the first-passage-time problem for a single random variable the motion of which is describable by a Fokker-Planck equation. We shall here employ their method to treat the isotropic first-passage-time problem for a random n-component vector $\vec{S}$.

We denote by $P(\vec{S},t\,|\,\vec{S}_0)$ the conditional probability density for finding the random vector at time t within a differential volume element $d^n S$ at $\vec{S}$ provided it had the value $\vec{S}_0$ initially. The time dependence of the probability density, we assume, is given by the Fokker-Planck equation

$$\dot{P} = \sum_{i=i}^{n} \left(-\mu \frac{\partial}{\partial S_i} S_i + \frac{1}{N}\frac{\partial^2}{\partial {S_i}^2}\right) P . \tag{9.1}$$

For $\mu = +1$ the model so specified has the instability already considered in Sec. IV. For $\mu = 0$, Eq. (9.1) describes the Einstein-Wiener diffusion process and for $\mu = -1$ the Ornstein-Uhlenbeck dissipation process.

We now let the variable $\vec{S}$ have at $t=0$, an isotropic distribution with the fixed modulus r, that is the probability density $\delta(|\vec{S}| - r)/r^{n-1}\Omega_n$. We seek the distribution $f_R(r,t)$ of its first passages through the sphere $|\vec{S}| = R$. Eventually, we shall be interested in the first-passage-time distribution obtained by setting $r=0$ and letting R be the threshold M. Because of the isotropy of the problem states we can relate the distribution $f_R(r,t)$ to the reduced conditional probability $P(\rho,t\,|\,r)\,\rho^{n-1}d\rho$ that the vector is found in a spherical shell with radius ρ and thickness $d\rho$ provided it was equally likely to be found anywhere in similar shell of radius r initially. A well-known relation between the two quantities f_R and P, the "renewal principle,"

$$P(\rho,t\,|\,r) = \int_0^t dt' f_R(r,t') P(\rho, t-t'\,|\,R) , \tag{9.2}$$

with $r<R<\rho$, expresses $P(\rho,t\,|\,r)$ as a superposition of contributions, each of which refers to a previous passage of $|\vec{S}|$ through an intermediate shell of radius R. If we rewrite the identity (9.2) as one for the temporal Laplace transforms

$$f_R(r,\lambda) = \int_0^\infty dt\, e^{-\lambda t} f_R(r,t) , \tag{9.3}$$

and similarly for P, we find

$$P(\rho,\lambda\,|\,r) = f_R(r,\lambda) P(\rho,\lambda\,|\,R) . \tag{9.4}$$

Since, on the right-hand side in Eq. (9.4), $f_R(r,\lambda)$ is independent of ρ, while $P(\rho,\lambda\,|\,R)$ is independent of r, it follows that the conditional probability P can be written as a product of suitable functions of these variables

$$P(\rho,\lambda\,|\,r) = u(r) v(\rho) . \tag{9.5}$$

By inserting the representation (9.5) in Eq. (9.4) we conclude

$$f_R(r,\lambda) = u(r)/u(R), \quad r<R . \tag{9.6}$$

The identity (9.6) permits us to calculate the first-passage-time distribution by solving the adjoint of the Fokker-Planck equation (9.1)

$$\dot{P}(\vec{S},t\,|\,\vec{S}_0) = \sum_{i=i}^{n} \left(+\mu S_{0i}\frac{\partial}{\partial S_{0i}} + \frac{1}{N}\frac{\partial^2}{\partial S_{0i}^2}\right) P(\vec{S},t\,|\,\vec{S}_0) \tag{9.7}$$

for the reduced distribution $P(\rho,t\,|\,r)$. After a temporal Laplace transformation and setting $|\vec{S}_0| = r$, $|\vec{S}| = \rho$ the adjoint Fokker-Planck equation reduces to

$$\left[\frac{1}{N}\frac{d^2}{dr^2} + \left(\mu r + \frac{n-1}{Nr}\right)\frac{d}{dr} - \lambda\right] P(\rho,\lambda\,|\,r) = 0 . \tag{9.8}$$

We can use any solution $u(r)$ of this ordinary differential equation which is regular at $r=0$ to find $f_R(r,\lambda)$ from Eq. (9.6).

By means of the transformation $z = -\mu N r^2/2$ we easily reduce Eq. (9.8) to the differential equation for the confluent hypergeometric function. As a solution regular at the origin we can take Kummer's function[23]

$$u(r) = M\left(-\frac{\lambda}{2\mu}, \frac{n}{2}, -\mu\tfrac{1}{2}Nr^2\right). \tag{9.9}$$

Since $M(-\lambda/\mu 2, n/2, 0) = 1$, the Laplace transform of the first-passage-time distribution reads

$$f_M(0,\lambda) = \left[M\left(-\frac{\lambda}{2\mu}, \frac{n}{2}, -\mu\tfrac{1}{2}NM^2\right)\right]^{-1} . \tag{9.10}$$

This result holds for any value of the threshold

M^2 and the diffusion constant $1/N$. In order to compare with the asymptotic result obtained in Sec. IV we set $\mu=+1$ and use the asymptotic form of the Kummer function for large argument,[23] i.e., for $NM^2\to\infty$,

$$M\left(-\frac{\lambda}{2},\frac{n}{2},-\frac{NM^2}{2}\right)\to\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{\lambda+n}{2}\right)}(\tfrac{1}{2}NM^2)^{\lambda/2}\,. \quad (9.11)$$

Obviously, the result (4.11) is thereby recovered.

For the sake of illustration we shall discuss the dependence of the first moment of the exact distribution on the parameter NM^2. That moment is obtained from Eq. (9.10) as

$$\begin{aligned}\bar{t}&=-\frac{\partial}{\partial\lambda}f_M(0,\lambda)\bigg|_{\lambda=0}\\&=-\tfrac{1}{2}\sum_{\nu=1}^{\infty}\frac{\Gamma(n/2)}{\nu\Gamma(\nu+n/2)}(-\tfrac{1}{2}NM^2)^{\nu}\\&=\frac{NM^2}{2n}\,{}_2F_2\left(1,1;1+\frac{n}{2},2;-\tfrac{1}{2}NM^2\right),\end{aligned} \quad (9.12)$$

where ${}_2F_2$ is the generalized hypergeometric function.[26] For $n=2$, the result (9.12) can also be expressed by means of the exponential integral $E_1(x)$ (Ref. 23) and Euler's constant γ as

$$\bar{t}=\tfrac{1}{2}[E_1(\tfrac{1}{2}NM^2)+\ln(\tfrac{1}{2}NM^2)+\gamma]\,. \quad (9.13)$$

This expression differs from the asymptotic result (4.12) only by the first term. Since $1/x<e^xE_1(x)\le 1/(x+1)$, the asymptotic approximation already becomes accurate for rather small values of the parameter $\frac{1}{2}NM^2$. Since our main concern in the present paper is the decay of unstable equilibrium states we defer discussion of the rigorous result (9.10) for $\mu=-1$ to Appendix B.

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APPENDIX A: EVALUATION OF TWO FUNCTIONAL INTEGRALS

We first evaluate the functional integral in Eq. (2.22). To save space we assume the observable s to be a single real variable. If we represent the delta function by a Fourier integral the functional integral takes the simple form

$$W(s)\sim\int\frac{d\omega}{2\pi}e^{i\omega s}\int d\{f(t)\}\exp\left(-\frac{N}{4}\int_0^\infty dt f(t)^2-i\omega\int_0^\infty dt\,e^{-\gamma t}f(t)\right). \quad (A1)$$

We take the time variable to be discrete, whereupon the functional integral in (A1) becomes a multiple Gaussian integral

$$W(s)\sim\int d\omega\,e^{i\omega s}\prod_i\left[\int_{-\infty}^{+\infty}df_i\exp\left(-\frac{N}{4}\Delta t f_i^2-i\omega e^{-\gamma t_i}\Delta t f_i\right)\right], \quad (A2)$$

which is easily evaluated and yields

$$\begin{aligned}W(s)&\sim\int d\omega\,e^{i\omega s}\prod_i[\exp(-\Delta t\omega^2e^{-2\gamma t_i}/N)]\\&\sim\int d\omega\,e^{i\omega s}\exp\left(-\omega^2\int_0^\infty dt\,e^{-2\gamma t}/N\right)\\&\sim\int d\omega\,e^{i\omega s}\exp(-\omega^2/2\gamma N)\\&\sim\exp(-\tfrac{1}{2}\gamma N s^2)\,,\end{aligned} \quad (A3)$$

which is the result given in Eq. (2.23).

Next, we consider the quantity $D(\omega,\Omega)$ defined by the functional average in Eq. (7.22). Since we are treating a spatially homogeneous system, we can simplify that average by writing the kernel (7.23) as

$$\begin{aligned}M(\omega,\Omega,\vec{x}',\vec{x}'')=&\frac{N}{2}(1+\nabla'^2)\delta^d(\vec{x}'-\vec{x}'')\\&+i\omega G(\vec{x}',t)G(\vec{x}'',t)\\&+i\Omega\frac{d}{dt}[G(\vec{x}',t)G(\vec{x}'',t)]\,.\end{aligned} \quad (A4)$$

By Fourier transforming with respect to the spatial variables and letting the wave numbers take discrete values we transform the right-hand side of Eq. (7.22) into a multiple Gaussian integral

$$D(\omega,\Omega)=\left[\mathfrak{N}\int\left(\prod_i dS_i\right)\exp\left(-\sum_{ij}M_{ij}(\omega,\Omega)S_iS_j\right)\right]^n$$
$$=\mathfrak{N}^n\{\det[M(\omega,\Omega)]\}^{-n/2}$$
$$=\mathfrak{N}^n\exp\left(-\frac{n}{2}\operatorname{tr}\ln[M(\omega,\Omega)]\right), \qquad (A5)$$

with

$$M_{ij}(\omega,\Omega)=(\Delta k)^{2d}\left(\delta_{ij}(1-\vec{k}_i^2)N/2(\Delta k)^d+i\omega G_iG_j+i\Omega\frac{d}{dt}(G_iG_j)\right). \qquad (A6)$$

The normalization constant $\mathfrak{N}$ can be eliminated with the help of the condition $D(0,0)=1$. We then obtain

$$D(\omega,\Omega)=\exp\left(-\frac{n}{2}\operatorname{tr}\ln[M(\omega,\Omega)M(0,0)^{-1}]\right)$$
$$\equiv\exp\left(-\frac{n}{2}\operatorname{tr}\ln[m(\omega,\Omega)]\right) \qquad (A7)$$

with

$$m_{ij}(\omega,\Omega)=\delta_{ij}+(\Delta k)^d(i\omega 2/N(1-\vec{k}_i^2))(G_iG_j)+(\Delta k)^d(i\Omega 2/N(1-\vec{k}_i^2))\frac{d}{dt}(G_iG_j). \qquad (A8)$$

Since the matrix $(m_{ij}-\delta_{ij})$ is built up by the direct products of the vectors G_i and $\dot{G}_i$ its powers and thus its logarithm must have that structure too. By expanding $\ln m$ in powers of $m-1$ we then easily find the result (7.27).

APPENDIX B: EXACT FIRST-PASSAGE-TIME DISTRIBUTION FOR THE ORNSTEIN-UHLENBECK PROCESS

For $\mu=-1$ we find the mean first-passage time from Eq. (9.10) to be

$$\bar{t}=\frac{1}{n}(\tfrac{1}{2}NM^2)\,{}_2F_2\left(1,1;1+\frac{n}{2},2;\tfrac{1}{2}NM^2\right)$$
$$=\tfrac{1}{2}\sum_{\nu=1}^{\infty}\frac{\Gamma(n/2)}{\nu\Gamma(\nu+n/2)}(\tfrac{1}{2}NM^2)^{\nu}. \qquad (B1)$$

It is interesting to consider the limit $\frac{1}{2}NM^2\to 0$ of this expression,

$$\bar{t}=\frac{1}{2n}NM^2 \quad \text{for} \quad \tfrac{1}{2}NM^2\ll 1, \qquad (B2)$$

which is quite reminiscent of Einstein's expression for the diffusion constant in Brownian motion. Obviously, in the limit considered, the variable $\vec{S}$ diffuses to its first encounter with the threshold $\vec{S}^2=M^2$ without taking notice of the linear force. In fact, Seshadri and Lindenberg[27] have previously obtained the result (B2) for free Brownian motion.

In contrast to the mean first-passage time (9.12) the one we consider here grows not logarithmically but exponentially with the parameter $\frac{1}{2}NM^2$. That behavior is due to the potential barrier $\frac{1}{2}M^2$ which the variable $\vec{S}$ has to climb while propelled by random forces of strength $1/\sqrt{N}$. The dependence of $\bar{t}$ on $(\frac{1}{2}NM^2)$ is especially transparent for $n=2$, where Eq. (B1) simplifies to

$$\bar{t}=\tfrac{1}{2}[\mathrm{Ei}(\tfrac{1}{2}NM^2)-\gamma-\ln(\tfrac{1}{2}NM^2)]$$
$$=\tfrac{1}{2}\sum_{\nu=1}^{\infty}\frac{1}{\nu\nu!}(\tfrac{1}{2}NM^2)^{\nu}. \qquad (B3)$$

For large $(\frac{1}{2}NM^2)$ the moment $\bar{t}$ is then dominated by the asymptotic approximation to the exponential integral $\mathrm{Ei}(x)$,[23]

$$\bar{t}=e^{NM^2/2}/NM^2 \quad \text{for} \quad \tfrac{1}{2}NM^2\to\infty. \qquad (B4)$$

We obtain a general asymptotic approximation for arbitrary n most easily by using the asymptotic form of the Kummer function for large argument[23] in Eq. (9.10),

$$f_M(0,\lambda)=\left[\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-\lambda}{2}\right)}e^{i\pi\lambda/2}\left(\frac{NM^2}{2}\right)^{-\lambda/2}+\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{\lambda}{2}\right)}e^{NM^2/2}\left(\frac{NM^2}{2}\right)^{(\lambda-n)/2}\right]^{-1}. \qquad (B5)$$

We should point out that the asymptotic form (B5) of $f_M(0,\lambda)$ is not valid for $\lambda\to\infty$. In fact, since the right-hand side in Eq. (B5) does not vanish for $\lambda\to\infty$, the approximate $f_M(0,\lambda)$ is not even a Laplace transform of any well behaved function of the time. The expression (B5) is useful though as a generating functional for the mean values $\overline{t^m}$ with finite m since the latter are determined by the behavior of $f_M(0,\lambda)$ near $\lambda=0$.

The first term in large squares bracket in Eq. (B5) is negligible compared with the second one except when λ is close to a negative integer or zero. The main role of the first term is to shift the poles of $f_M(0,\lambda)$ slightly away from

$$\lambda=0,-2,-4,\ldots \qquad (B6)$$

and to modify the corresponding residues as well. These shifts are especially important near $\lambda=0$ since the behavior of $f_M(0,\lambda)$ in that region determines the moments $\overline{t^m}$. The mean passage time which follows from Eq. (B5) is

$$\bar{t}=\frac{1}{2}\left(\frac{2}{NM^2}\right)^{n/2}\Gamma\left(\frac{n}{2}\right)e^{NM^2/2}. \qquad (B7)$$

This asymptotic result reduces, of course, to the one given in Eq. (B4) for $n=2$. It also includes a result previously obtained by Lindenberg and Seshadri[28] for $n=1$. As already noted above, the mean passage time $\bar{t}$ includes the exponential $\exp(\frac{1}{2}NM^2)$ which we must expect on the basis of qualitative statistical arguments. The specific nature of the passage-time problem is reflected in the factor which precedes the exponential. That factor shows an interesting dependence on the number of components n of the random vector $\vec{S}$.

The poles of the asymptotic expression in Eq. (9.10) near the points (B6) in the complex λ plane give the long-time behavior of the first-passage-time distribution. The leading term is

$$f_M(0,t)\simeq -\lambda_0 e^{\lambda_0 t}=\frac{1}{\bar{t}}e^{-t/\bar{t}}\,, \tag{B8}$$

where λ_0 is related to the average (B7) by

$$\lambda_0=-1/\bar{t}\,. \tag{B9}$$

The first correction to Eq. (B8) is characterized by a decay constant,

$$\lambda_1=-2-2(\tfrac{1}{2}NM^2)^{-(n+2)/2}\left[\Gamma\left(\frac{n+1}{2}\right)^{-1}\right]e^{-NM^2/2}\,, \tag{B10}$$

which is very large compared to the leading one. The result of (B8) and (B9) has also been obtained previously.[29,30]

The exponential form (B8) of the first-passage-time distribution would be rigorously correct at all times for a delta-correlated process, for which the values taken on by the variable $S(t)$ at different times are independent. It can be derived as an asymptotic approximation, however, by a well-known heuristic argument.[29]

Both the mean and the root mean square of the first passage time must be expected, on general statistical grounds, to be of the order $\exp(\frac{1}{2}NM^2)$. Such times are enormously long compared to the unit of time characterizing both the decay of the correlation function $\langle S(t)S(t')\rangle$ and the loss of memory of any initial conditions. Before the variable S finally arrives at the reference displacement M it has presumably many times reached maximum values such that $1/N\ll S^2<M^2$. On each such occasion the process has lost memory of prior excursions to maxima of this magnitude. In other words, on a time scale of order $\exp(\frac{1}{2}NM^2)$ the random process $S(t)$ indeed appears approximately to be a delta-correlated process.

The argument just presented makes no explicit use of the Gaussian nature of the stochastic process described by the Fokker-Planck equation (9.1). It holds, in fact, for any Markovian motion in a binding potential $V(S)$ with $V\to+\infty$ as $S^2\to\infty$.[29,30]

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$$S(t)=\left(\exp\int_0^t dt'\gamma(t')\right)\times\left[S(0)+\int_0^\infty dt'\left(\exp-\int_0^{t'}dt''\gamma(t'')\right)f(t')\right]$$

for $\int_0^t dt'\gamma(t') > 1$.

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