Properties of the momentum-translation approximation

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It is shown that the momentum-translation approximation (MTA) for the treatment of the interaction of bound states with a plane-wave field can be developed in at least five different ways. This multiplicity of approaches is used to exhibit some of the basic features of the approximation, as well as to contrast it with a gauge transformation, with which it is sometimes confused. Validity conditions are $eaa_0\omega/E \ll 1$, $\omega a_0 \ll 1$ (where a is the amplitude of the vector potential in the Coulomb gauge for a plane wave of circular frequency ω, E is a characteristic bound-state energy, and a_0 is a characteristic size of the bound system), and an absence of intermediate near resonances in a transition. The momentum-translation technique predicts replica states very directly, but level shifts are found only by a method equivalent to perturbation theory. A general formalism is developed for transitions caused by linearly or circularly polarized applied plane-wave fields, in which a second field may or may not be involved. In first order, the MTA always reduces exactly to first-order perturbation theory, but when treated in their entireties, it is shown that a perturbation series and a momentum-translation series are rearrangements of each other. All the criticisms which have been directed at the MTA are reviewed and evaluated. The principal limitations of the MTA are found to be the difficulty of treating problems with intermediate near resonances, and the absence of a systematic way to improve on the basic approximation. The strengths of the MTA are its simple analytical form, its good accuracy when there are no intermediate near resonances, and the fact that high-order multiphoton processes are treated as easily as first-order processes.

I. INTRODUCTION

The momentum-translation approximation (MTA) was introduced[1-3] a decade ago as an extremely simple calculational tool for the approximate treatment of interactions of a bound system with an applied electromagnetic field. It has the major advantages that it gives analytical forms of very simple type; the same general expressions hold true for a wide variety of bound systems in interaction with a plane-wave field; and the approach is applicable to all multiphoton orders, with no extra complications as the order increases.

The method has fallen into disuse in recent years because of a number of criticisms[4-9] leveled against it. Although the answers to most of the objections were contained, explicitly or implicitly, in earlier papers,[1-3,10] no response specifically directed to the criticisms has yet been published.

It is the purpose of this paper to review the bases for the MTA, and to outline the areas of usefulness and the limitations inherent in the method. In the course of this evaluation each of the objections which has been raised is discussed and resolved.

In Sec. II, the MTA wave function for a bound charged particle in an electromagnetic field is generated by three totally different methods: From perturbation theory, by a path-integral technique, and by a unitary transformation approach. In the perturbation method, it is shown in first-order and in second-order time-dependent perturbation theory that the assumption that the energy of a photon of the applied field is much less than level spacing in the bound-system spectrum, leads directly to the MTA. Although the inference that this result holds true for arbitrary order is not developed, this approach clearly shows one aspect of the nature of the MTA. The path-integral method[11] shows in one simple step that the MTA follows from the assumption that the wavelength of the radiation is so much in excess of the size of the bound system that details of the path are irrelevant. The development of the MTA by a unitary transformation is the technique used originally.[1-3] This method is the most useful of the three alternatives, but it is reserved for last because the unitary transformation employed is of the same form as the gauge transformation from Coulomb gauge to electric-field gauge.[12-14] Although no gauge transformation is implied in the MTA, this correspondence in transformation functions has caused confusion in the literature.[4,9,15,16] Presentation of two alternate methods to generate the MTA apart from the unitary transformation technique should serve to avoid such misunderstandings. In addition, quite independent calculational methods devised by Babiker[17,18] and by Campos and Krüger[19] yield the MTA as a special case. Therefore, at least five different ways to arrive at the MTA are known.

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In Sec. III, it is shown that the MTA wave function automatically predicts replica states. The simple MTA does not contain a level shift, but a calculation of the level shift in a momentum-translation framework is shown to be identical to a perturbation calculation of this effect.

Section IV is devoted to a development of techniques for calculating transition probabilities. This exposition is undertaken both because the most familiar procedures are based on perturbation theory, and also because there is some controversy about the way in which the MTA is to be applied to two-field problems.[5] The procedure employed is a simple S-matrix approach, valid for time-dependent interaction Hamiltonians, but done entirely in the Schrödinger picture. This is more direct than the Dyson interaction-picture approach, and it has the advantage of being directly applicable also to time-independent (except for adiabatic switching) interaction Hamiltonians. It is shown that transitions can be viewed as taking place between an unperturbed state and one of the replica states of an interacting state. Both single- and two-field problems are considered.

Section V is concerned with the formal relationship between the momentum-translation method, including all supplementary terms, and the standard perturbation series, also including all terms. As is expected on general principle, one series is simply a rearrangement of the other, since, when all terms are considered, the result in both instances is exact. The rearrangement properties were pointed out in part by Tewari[20] and then by Haque.[21] A more complete treatment is given by Rachman and Laplanche.[6] Only the leading, or MTA, term in the momentum-translation method has any physical meaning or predictive power. Adding supplementary terms to the MTA produces a hybrid of momentum translation and perturbation theory which has no usefulness either physically or numerically.

Problems in which a bound charged particle interacts only with a single electromagnetic field are treated in Sec. VI. When transitions occur between adjacent states, results obtained with the simple MTA range from very accurate if there are no nearby states[23,24] to accuracy within an order of magnitude or better if there are nearby states.[10,25-27] For first-order transitions, MTA in the low-intensity limit always gives exactly the same result as perturbation theory. For higher orders, agreement to within a factor of 2 or so is usual. However, the ease of obtaining results in MTA is not lost as order increases, and a high-order result accurate to within less than an order of magnitude and obtained with minor calculational effort, can be very useful indeed. The situation is quite different if transition between nonadjacent levels is considered, and intervening levels can be nearly resonant at a lower order than the overall transition. It has been found both in bound-bound[10,27] and bound-free[6,28-33] transitions with intermediate near resonances possible, that the simple MTA approach can be seriously in error.

Section VII is concerned with transitions in which a bound charged particle interacts both with an applied electromagnetic field, and with a second field as well. For instance, Raman scattering problems, which can be multiphoton in the incident field, come under this heading. If most of the energy in a transition involves the incident field, then the single-field remarks about MTA apply here as well. However, in cases where most of the transition energy is associated with the second field, then MTA can become very accurate. In such cases, there is no limitation to factor of 2 or 10 accuracy. Furthermore, the accessible intensity domain grows as the fraction of transition energy in the applied field shrinks. As always with the MTA, there is no extra calculational difficulty associated with very high-order processes.

In Sec. VIII, all of the objections raised about the MTA are listed and evaluated. For most of the objections, this is just a matter of summarizing material presented earlier in the paper. Finally, Sec. VIII gives a brief survey of the strengths and limitations of the MTA.

II. BASES OF THE MTA

Generation of the momentum-translation approximation is interpreted here to mean development of the MTA wave function. The way in which this wave function is used to calculate transition probabilities is considered in later sections. The starting point is the single-particle Schrödinger equation containing the effects of both a binding potential and an external field, expressed in Coulomb gauge, i.e.,

$$i\partial_t \Psi(\vec{r},t) = [(2m)^{-1}(-i\vec{\nabla} - e\vec{A})^2 + V(\vec{r})]\Psi(\vec{r},t)\,. \quad (1)$$

Units with $\hbar = c = 1$ are employed.

A. Induction from perturbation theory

In Eq. (1), an unperturbed Hamiltonian operator

$$H_0 = p^2/2m + V(\vec{r}) \quad (2)$$

and a perturbation Hamiltonian operator

$$H'(t) = -e\vec{A}(t)\cdot\vec{p}/m + e^2\vec{A}^2(t)/2m \quad (3)$$

can be identified. The first-order perturbation solution of Eq. (1) is

$$\Psi^{(1)}(t)=\Phi(t)-i\sum_n \Phi_n(t)\int_{-\infty}^{t} dt_1(\Phi_n, H'^{(1)}\Phi)_{t_1}, \tag{4}$$

where the set $\{\Phi_n\}$ is a complete set of noninteracting states [i.e., solutions of Eq. (1) without H'], dependence on $\vec{r}$ is not explicitly indicated, the subscript t_1 on the scalar product in the integrand means that all quantities in the scalar product depend on t_1, and $H'^{(1)}$ as it appears in the scalar product contains only the first-order term in Eq. (3). With the standard operator theorem

$$i\vec{p}/m=[\vec{r}, H_0], \tag{5}$$

the first-order interaction is

$$H'^{(1)}=ie\vec{A}\cdot[\vec{r}, H_0]. \tag{6}$$

Now consider the interaction term to arise from a linearly polarized plane wave in long-wavelength approximation

$$\vec{A}=\vec{a}\cos\omega t, \tag{7}$$

and write the stationary states $\Phi_n(\vec{r},t)$ as

$$\Phi_n(\vec{r},t)=\phi_n(\vec{r})\exp(-iE_n t). \tag{8}$$

The integral which appears in Eq. (4) is then

$$\int_{-\infty}^{t} dt_1(\Phi_n, H'^{(1)}\Phi)_{t_1}=-(E_n-E)\left(\phi_n, \frac{e\vec{a}\cdot\vec{r}}{2}\phi\right)\times\left(\frac{e^{i(E_n-E+\omega)t}}{(E_n-E+\omega)}+\frac{e^{i(E_n-E-\omega)t}}{(E_n-E-\omega)}\right), \tag{9}$$

in which it is implicit that the vector potential expressed in Eq. (7) is adiabatically switched at $|t|\to\infty$. If it is now supposed that

$$\omega \ll |E_n-E| \tag{10}$$

for any n, then ω can be neglected in the denominators in the second set of large parentheses in Eq. (9), leading to a common denominator which is canceled by the prefactor in Eq. (9). Equation (9) can then be rewritten as

$$\int_{-\infty}^{t} dt_1(\Phi_n, H'^{(1)}\Phi)_{t_1}\approx -(\Phi_n, e\vec{A}\cdot\vec{r}\Phi)_t. \tag{11}$$

When Eq. (11) is substituted into Eq. (4), the summation can be accomplished and the result is

$$\Psi^{(1)}(t)\approx(1+ie\vec{A}\cdot\vec{r})\Phi(t). \tag{12}$$

The second-order perturbation solution of Eq. (1) is

$$\Psi^{(2)}(t)=\Phi(t)-i\sum_n \Phi_n(t)\int_{-\infty}^{t} dt_1(\Phi_n, H'\Phi)_{t_1}+(-i)^2\sum_n\sum_m \Phi_n(t)\int_{-\infty}^{t} dt_1(\Phi_n, H'^{(1)}\Phi_m)_{t_1}\times\int_{-\infty}^{t_1} dt_2(\Phi_m, H'^{(1)}\Phi)_{t_2}, \tag{13}$$

where, in the term with a single sum, H' is the full expression given in Eq. (3), whereas the double sum term contains only the first-order $H'^{(1)}$ of Eq. (6). The terms in Eq. (13) will be considered in reverse order. The t_2 integral in the last term is just like Eq. (9), except that the result is a function of t_1. After the t_1 integration is done, and Eq. (10) employed, the last term in Eq. (13) is

$$-\sum_n\sum_m \Phi_n(\Phi_n, e\vec{A}\cdot\vec{r}\Phi_m)(\Phi_m, e\vec{A}\cdot\vec{r}\Phi)\frac{(E_n-E_m)}{(E_n-E)}. \tag{14}$$

When the single-integral term in Eq. (13) is considered in two parts corresponding to the two terms in H', the linear part of H' is just like Eq. (11), and the quadratic part of H' gives

$$-\sum_n \Phi_n\left(\Phi_n, \frac{e^2\vec{A}^2}{2m}\Phi\right)\frac{1}{(E_n-E)}. \tag{15}$$

Equations (14) and (15) can be combined if $\vec{A}^2$ in Eq. (15) is first rewritten with the help of the commutator

$$-\vec{A}^2=[(\vec{A}\cdot\vec{r}), (\vec{A}\cdot i\vec{p})], \tag{16}$$

and then the p operator is replaced by Eq. (5). The result of these manipulations is that Eq. (15) becomes

$$\sum_n\sum_m \Phi_n(\Phi_n, e\vec{A}\cdot\vec{r}\Phi_m)(\Phi_m, e\vec{A}\cdot\vec{r}\Phi)\frac{1}{2}\left(\frac{E-2E_m+E_n}{E_n-E}\right). \tag{17}$$

Equations (14) and (17) combine directly, both sums can be done, and the outcome is $-\frac{1}{2}(eA\cdot r)^2\Phi$. The overall result for Eq. (13) is, therefore,

$$\Psi^{(2)}(t)\approx[1+ie\vec{A}\cdot\vec{r}+\tfrac{1}{2}(ie\vec{A}\cdot\vec{r})^2]\Phi(t). \tag{18}$$

Equations (12) and (18) can be recognized as successive approximations to the momentum-translation wave function

$$\Psi(t)=e^{ie\vec{A}\cdot\vec{r}}\Phi(t). \tag{19}$$

The essential input is the condition expressed in Eq. (10). Any attempt to extend Eq. (18) to higher orders involves a large amount of labor. It is not worthwhile, since more direct paths to Eq. (19) exist.

B. Path-integral method

Some years ago, a number of attempts to develop a manifestly gauge-invariant quantum electrodynamics made use of a solution of the form[34-40]

$$\Psi(x)=\exp\left(-ie\int_{\text{path}}^{x} A\cdot dx'\right)\Phi(x), \tag{20}$$

where the context is relativistic. That is, A^μ is

a four-vector potential, $A \cdot dx$ is a four-vector product with the meaning $A \cdot dx = A^0 dt - \vec{A} \cdot d\vec{r}$, and x written as an argument of the wave functions means $\vec{r}, t$. In Eq. (20), $\Phi(x)$ is a solution of the quantum-mechanical wave equation in the absence of A^μ, and $\Psi(x)$ is the solution with A^μ. Because of the path dependence of the path integral, one finds that a sum over all paths must be considered if $\Phi(x)$ is to be truly a noninteracting solution. If only a single path is considered, then $\Phi(x)$ itself must be path dependent. With some guidance from the very different appearing Feynman path-integral formalism,[41,42] approximate forms of Eq. (20) were developed.[11] The form appropriate to the present set of circumstances, where A^μ has only a space part, and the problem is nonrelativistic, is

$$\Psi(\vec{r},t) \approx \exp\left(ie \int_{V\ \text{path}}^{\vec{r}} \vec{A} \cdot d\vec{r}'\right) \Phi(\vec{r},t), \tag{21}$$

where the designation "V path" refers to paths determined by the $V(r)$ potential which establishes the noninteracting solution $\Phi(r,t)$. With the assumption that A is essentially constant over the region to which the particle path is constrained by the binding potential V, then Eq. (21) converts directly to the MTA wave function of Eq. (19).

The path-integral formalism developed in Ref. 11 leads to the validity conditions

$$\omega a_0 \ll 1, \tag{22}$$

$$eaa_0(\omega/E) \ll 1, \tag{23}$$

where a_0 is a characteristic size of the bound system (e.g., a Bohr radius), and E is a characteristic energy of the bound system. Equation (22) is simply the long-wavelength-approximation condition, and Eq. (23) is an extension of Eq. (10).

C. Unitary transformation method

If the unitary transformation

$$U = \exp(-ie\vec{A} \cdot \vec{r}) \tag{24}$$

is applied to the wave function Ψ of Eq. (1),

$$\bar{\Psi} = U\Psi, \tag{25}$$

then $\bar{\Psi}$ satisfies the equation

$$i\partial_t \bar{\Psi}(\vec{r},t) = [(2m)^{-1}(-i\vec{\nabla})^2 + V(\vec{r}) - e\vec{r} \cdot \vec{E}(t)]\bar{\Psi}(\vec{r},t), \tag{26}$$

where $\vec{E}(t)$ is the long-wavelength-approximation electric field associated with the vector potential $\vec{A}(t)$. If the term $e\vec{r} \cdot \vec{E}$ is much smaller in magnitude than any characteristic energy or energy difference in the problem,

$$|e\vec{r} \cdot \vec{E}| \ll E, \tag{27}$$

then this term can be dropped, Eq. (26) becomes the noninteracting equation with solution $\Phi(\vec{r},t)$, and so Eq. (25) implies

$$\Psi \approx U^{-1}\Phi. \tag{28}$$

When combined with Eq. (24), Eq. (28) is just the same as Eq. (19) for the MTA.

In Eq. (27), if $|\vec{E}|$ is replaced by ωa, and $|\vec{r}|$ is replaced by a_0, the result is precisely the condition stated in Eq. (23). The long-wavelength approximation, Eq. (22), has also been used.

In principle, an exact solution follows if an exact expression for $\bar{\Psi}$ is used in Eq. (25), instead of adopting the approximation Φ, as in Eq. (28). The standard formal solution for $\bar{\Psi}$ gives the expression[1]

$$\Psi = \exp(ie\vec{A} \cdot \vec{r})(\bar{\Psi}, \bar{\Psi})^{-1/2} \times \left[\Phi + \sum_n \Phi_n\left(-i \int_{-\infty}^{t} dt' (\Phi_n, -e\vec{r} \cdot \vec{E}\bar{\Psi})_{t'}\right)\right], \tag{29}$$

where the quantity in the square brackets is $\bar{\Psi}$, and can be expanded by iteration. The terms in the sum over n were referred to in Ref. 1 as "correction" terms, but it is more suitable to speak of them as "supplementary" terms, since (unless the summation is carried out in its entirety) successive terms from this sum cannot be used to improve the MTA. This point is discussed more fully in Sec. VI.

Some important comments must be made here. Equation (24) is exactly the transformation used to achieve a gauge transformation of the potentials from Coulomb gauge to electric-field (Göppert-Mayer) gauge.[12-14] It is not used in that sense here. Equation (24) is used to accomplish a unitary transformation of the wave function *within* Coulomb gauge. That status is assured by employing H' as stated in Eq. (3) as the transition-inducing Hamiltonian in all transition matrix elements. Equation (3) is, of course, the Coulomb-gauge interaction Hamiltonian. Equation (28) is an approximate solution of Eq. (1) in Coulomb gauge in exactly the same sense that Eq. (18) [and its logical extension, Eq. (19)] or Eq. (21) is an approximate solution of Eq. (1) in Coulomb gauge. No gauge transformation is implied in any of these cases.

Another point worth noting is that if $\vec{A}$ is used in its full (non-long-wavelength approximation) form when the transformation (25) is applied, and long-wavelength approximation is reinstituted after the transformation, then a vector potential term will appear in addition to the scalar potential term in Eq. (26).[14] The correct equation is

$$i\partial_t \bar{\Psi}(\vec{r},t) = \{(2m)^{-1}[-i\vec{\nabla} + \omega^{-1}\vec{k}\vec{r} \cdot \vec{E}(t)]^2 + V(\vec{r}) - e\vec{r} \cdot \vec{E}(t)\}\bar{\Psi}(\vec{r},t). \tag{30}$$

However, it is easily shown[14] that the vector potential in Eq. (30) is unimportant compared to the scalar term when long-wavelength approximation is justified. These remarks are based on the status of Eq. (1) as an equation in relative coordinates between the bound charged particle and the system to which it is bound. That is, the m and e parameters in Eq. (1) are really reduced mass and reduced charge.[14]

D. Other approaches

The MTA emerges as a special case of more general (and hence more complicated) methods of solution in electrodynamics. Babiker's method[17,18] is to introduce a unitary transformation akin to Eq. (24), in which the generating function is not $e\vec{A}\circ\vec{r}$, but is instead

$$S=\int d^3r\vec{P}\cdot\vec{A}\,,$$

where $\vec{P}$ is the polarization operator. Babiker's work is relativistic and it reduces to the MTA in the nonrelativistic limit when $\vec{P}$ is limited to electric-dipole contributions.

Another interesting approach is that of Campos and Krüger.[19] They develop a time-evolution operator method based on a Magnus expansion[43] for solution of the equation of motion for the time-evolution operator. The leading term in this technique is just the momentum-translation result.

E. Gauge considerations

As emphasized in Sec. IIC above, the MTA does not involve a gauge transformation. The fact that it has been asserted to be a gauge transformation by Cohen-Tannoudji *et al.*[4] has led some authors[9,15] to refer to the MTA as a "gauge-transformation method." It is quite clear from the variety of ways given above to infer the MTA, that this is a *non sequitur*. In fact, the momentum-translation approach is a gauge-specific technique appropriate to Coulomb gauge. The minimal electromagnetic coupling substitution $\vec{p}\to\vec{p}-e\vec{A}$ to incorporate electromagnetic fields into the equations of motion is what suggests[1] the unitary transformation (28) as an approximate way to translate away the electromagnetic part. This point of view is lost in gauges of the Göppert-Mayer type.

III. REPLICA STATES AND LEVEL SHIFTS

The primary utility of the MTA is for the calculation of transition probabilities. It confers no special advantages for the calculation of level shifts, since any attempt to do this within the momentum-translation approach simply reduces to a standard perturbation theory or "quasienergy state"[44-47] calculation. However, a simple result for the so-called replica states (or satellite states, or sidebands) emerges immediately.

The MTA wave function, Eq. (19), obviously does not possess any well-defined energy eigenvalue. Nevertheless, the energy eigenvalue associated with the unperturbed state $\Phi(t)$ is retained as a convenient label, even though it is no longer a true eigenvalue. With a simple periodic vector potential as given in Eq. (7), the MTA wave function can be rewritten as

$$\Psi(\vec{r},t)=\phi(r)\sum_{n=-\infty}^{\infty}i^nJ_n(e\vec{a}\cdot\vec{r})e^{-iEt+in\omega t}\,. \tag{31}$$

The MTA wave function is thus a superposition of a principal substate ($n=0$) with the unperturbed energy, plus a sequence of replica states of energy $E\pm|n|\omega t$ ($n=1,2,\ldots$) distributed symmetrically about the principal substate.

To treat level shifts in the most elementary fashion within the MTA, consider the expectation value of the total Hamiltonian

$$E+\Delta E=(\Psi,(H_0+H')\Psi)$$
$$\approx(\Phi,e^{-ie\vec{A}\cdot\vec{r}}(H_0+H')e^{ie\vec{A}\cdot\vec{r}}\Phi)\,. \tag{32}$$

The commutator theorem[1,3] in the long-wavelength approximation gives the immediate result

$$e^{-ie\vec{A}\cdot\vec{r}}(H_0+H')e^{ie\vec{A}\cdot\vec{r}}=H_0\,, \tag{33}$$

so that ΔE as expressed in Eq. (32) vanishes. This vanishing of the level shift in the MTA is also evident from the symmetry under $n\to-n$ of Eq. (31).

The level shift is now treated in more detail in terms of the quasienergy formalism.[47] If Eq. (29) is written as

$$\Psi=\exp(ie\vec{A}\circ\vec{r})\overline{\Psi}\,,$$

where $\overline{\Psi}$ is the solution of Eq. (26), and $\overline{\Psi}$ is written in the quasienergy form

$$\overline{\Psi}=e^{-i\mathcal{E}t}u(\vec{r},t)\,,$$

then the quasienergy form for Ψ is

$$\Psi=e^{-i\mathcal{E}t}v(\vec{r},t)\,,$$

with

$$v(\vec{r},t)=\exp(ie\vec{A}\cdot\vec{r})\,u(\vec{r},t)\,.$$

The quasienergy $\mathcal{E}$ is given by

$$\mathcal{E}=\frac{\int_{-\pi/\omega}^{\pi/\omega}dt(v,(H_0+H'-i\partial_t)v)}{\int_{-\pi/\omega}^{\pi/\omega}dt(v,v)} \tag{34}$$

If $v(r,t)$ is normalized, this gives

$$\mathcal{E}=\frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega}dt\,(u,(H_0+H_I-i\partial_t)u)\,, \tag{35}$$

where Eq. (33) has been used, and where $H_I = -e\vec{r}\cdot\vec{E}$ is the interaction Hamiltonian of Eq. (26). That is, the quasienergy sought by Eq. (34) for the wave function in the momentum-translation formalism (if one wishes to go beyond the simple MTA) is identical to the quasienergy of Eq. (35), which can be treated by perturbation theory.[47]

In summary, the MTA directly gives a spectrum of replica states associated with any given unperturbed state, but it predicts no level shift. If a level shift is sought within the momentum-translation approach by going beyond the MTA, the procedure for finding the level shift is identical to that employed in perturbation theory.

IV. TRANSITION PROBABILITY FORMALISM

A. Single-field case

The formalism for calculating transition probabilities and cross sections is established in general terms. The technique used is for time-dependent interaction Hamiltonians H', done in the Schrödinger picture, and with the presumption that H' is turned off at asymptotic times $|t|\to\infty$. The basic equations are

$$(1i\partial_t - H_0 - H')\Psi(t) = 0 \tag{36}$$

for the interacting state Ψ,

$$(1i\partial_t - H_0)\Phi(t) = 0 \tag{37}$$

for the noninteracting state Φ, and

$$(1i\partial_t - H_0)G(t, t_0) = 1\delta(t - t_0) \tag{38}$$

for the noninteracting Green's operator. In Eqs. (36)–(38), Ψ and Φ are vectors in the Hilbert space, H_0, H', G, and 1 are Hilbert space operators, and time is a parameter external to the Hilbert space. In the following, the unit operator 1 will no longer be stated explicitly. Equation (38) has the retarded solution

$$G^{(+)}(t, t_0) = -i\Theta(t - t_0)\sum_j |\Phi_j(t)\rangle\langle\Phi_j(t_0)| \tag{39}$$

and the advanced solution

$$G^{(-)}(t, t_0) = G^{(+)\dagger}(t_0, t) = i\Theta(t_0 - t)\sum_j |\Phi_j(t)\rangle\langle\Phi_j(t_0)| , \tag{40}$$

where it is convenient to introduce Dirac bra and ket notation for Eqs. (39) and (40) only. The j sum encompasses a complete set of states $\{\Phi_j\}$.

The S matrix is the probability amplitude for an interacting initial state to be asymptotically in a particular noninteracting state. This is stated as

$$S_{fi} = \lim_{t\to\infty} (\Phi_f, \Psi_i^{(+)})_t , \tag{41}$$

where the subscript t on the inner product indicates the time parameter appropriate for both states in the product. A form equivalent to Eq. (41), more suitable for some problems, is

$$S_{fi} = \lim_{t\to-\infty} (\Psi_f^{(-)}, \Phi_i)_t . \tag{42}$$

The in states $\Psi^{(+)}$ and out states $\Psi^{(-)}$ follow from the solutions of Eq. (36) with retarded and advanced Green's operators, Eqs. (39) and (40). These solutions are

$$\Psi^{(\pm)}(t) = \Phi(t) + \int dt_1 G^{(\pm)}(t, t_1)H'(t_1)\Psi^{(\pm)}(t_1) . \tag{43}$$

Equation (43), when used in Eq. (41), gives

$$S_{fi} = \delta_{fi} + \lim_{t\to\infty}\int dt_1(\Phi_f(t),\ G^{(+)}(t, t_1)H'(t_1)\Psi_i^{(+)}(t_1)) . \tag{44}$$

In Eq. (44), the adjoint of $G^{(+)}(t, t_1)$ can be taken to operate on $\Phi_f(t)$ and, from Eq. (40), this adjoint is the advanced operator with reversed time dependence. The action of the Green's operators on noninteracting states is

$$G^{(+)}(t, t_0)\Phi(t_0) = -i\Theta(t - t_0)\Phi(t) , \tag{45}$$

$$G^{(-)}(t, t_0)\Phi(t_0) = i\Theta(t_0 - t)\Phi(t) . \tag{46}$$

From Eq. (46), the action of $G^{(-)}(t_1, t)$ on $\Phi_f(t)$ produces $\Phi_f(t_1)$, and a theta function $\Theta(t - t_1)$. In the limit $t\to\infty$, this theta function is just unity, and so Eq. (44) becomes

$$(S - 1)_{fi} = -i\int dt_1(\Phi_f, H'\Psi_i^{(+)})_{t_1} . \tag{47}$$

An alternative form which follows from Eq. (42) is

$$(S - 1)_{fi} = -i\int dt_1(\Psi_f^{(-)}, H'\Phi_i)_{t_1} . \tag{48}$$

Equations (47) and (48) are quite general. If the vectors and operators in Eq. (47) are rendered in configuration representation, and the interacting state is given by the MTA expression Eq. (19), then (47) is

$$(S - 1)_{fi} = -i\int dt_1(\Phi_f, H'e^{ie\vec{A}\cdot\vec{r}}\Phi_i)_{t_1} . \tag{49}$$

The commutator theorem[1,3] in the long-wavelength approximation is

$$H'e^{ie\vec{A}\cdot\vec{r}} = [e^{ie\vec{A}\cdot\vec{r}}, H_0] , \tag{50}$$

which puts Eq. (49) in the form

$$(S - 1)_{fi} = -i(E_i - E_f)\int dt_1(\Phi_f, e^{ie\vec{A}\cdot\vec{r}}\Phi_i)_{t_1} . \tag{51}$$

1. Linear polarization

For a linearly polarized monochromatic field, the vector potential can be written

$$\vec{A} = \vec{a}\cos\omega t = a\vec{\epsilon}\cos\omega t, \quad \vec{\epsilon}^{\,2} = 1 \tag{52}$$

in long-wavelength approximation. The exponential factor in Eq. (51) is then

$$e^{ie\vec{A}\cdot\vec{r}} = \sum_{n=-\infty}^{\infty} i^n J_n(e\vec{a}\cdot\vec{r})e^{in\omega t}, \tag{53}$$

as in Eq. (31), so that the S matrix is

$$(S-1)_{fi} = -2\pi i(E_i - E_f) \times \sum_n i^n(\phi_f, J_n(e\vec{a}\cdot\vec{r})\,\phi_i)\,\delta(E_f - E_i \pm n\omega). \tag{54}$$

In Eq. (54), the stationary-state character of the noninteracting solutions is used with the terminology of Eq. (8). The delta function appearing in Eq. (54) contains $\pm n\omega$, even though the straightforward result would have $+n\omega$ only. The reason for this extension is that, since n ranges from $-\infty$ to $+\infty$, the substitution $n \to -n$ can be made. The only consequence of this is to change the sign of $n\omega$ in the delta function. It is convenient to think of n as positive, and the sign ambiguity in the delta function permits the application of Eq. (54) to either emission or absorption processes while viewing n as positive in either case.

Equation (54) can be used to define a T matrix

$$T_{fi}^{(n)} = (E_i - E_f)i^n(\phi_f, J_n(e\vec{a}\cdot\vec{r})\,\phi_i), \tag{55}$$

so that

$$(S-1)_{fi} = -2\pi i \sum_n T_{fi}^{(n)}\,\delta(E_f - E_i \pm n\omega) \tag{56}$$

in conformity with the usual connection between S and T matrices.

The elementary transition probability per unit time is found from the S matrix by

$$w = \lim_{t\to\infty} \frac{1}{t}\,|(S-1)_{fi}|^2. \tag{57}$$

The delta function in Eq. (56) causes the separate terms to contribute incoherently when the square is taken, so Eq. (57) leads to

$$w = 2\pi \sum_n |T_{fi}^{(n)}|^2\,\delta(E_f - E_i \pm n\omega). \tag{58}$$

The total transition probability W is found by integrating w over available final states. The cross section follows from W by dividing it by the incident flux.

As an example of the last two steps just stated, suppose the problem to be treated involves photoionization or photodetachment, so the initially bound particle is a free particle in the final state. The the total transition probability per unit time is

$$W = \int \frac{Vd^3p}{(2\pi)^3}\,w, \tag{59}$$

where p is the momentum of the free particle, and V is the normalization volume. With the conversion

$$d^3p = p^2dp\,d\Omega = (2m^3E_f)^{1/2}\,dE_f\,d\Omega, \tag{60}$$

Eqs. (58) and (59) yield

$$W = \frac{(2m^3)^{1/2}V}{(2\pi)^2}\sum_n \int E_f^{1/2}\,dE_f\,d\Omega\,|T_{fi}^{(n)}|^2\,\delta(E_f - E_i \pm n\omega). \tag{61}$$

Use of the delta function gives the differential transition probability

$$\frac{dW}{d\Omega} = \frac{(2m^3)^{1/2}V}{(2\pi)^2}\sum_n (E_i \mp n\omega)^{1/2}\,|T_{fi}^{(n)}|^2, \tag{62}$$

and the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{8\pi}{a^2\omega}\,\frac{dW}{d\Omega}, \tag{63}$$

where it is understood that wherever E_f appears in the T matrix, it is to be replaced by $E_i \mp n\omega$.

As a footnote to the above derivation, it should be observed that the fully interacting state Ψ, as it appears in the S matrix of Eqs. (47) and (48), is treated as in Eq. (31). This means that the transition can be viewed as taking place between one of the replica states associated with the interacting state Ψ_i (or Ψ_f) and the unperturbed state Φ_f (or Φ_i).

2. Circular polarization

For a circularly polarized monochromatic field in dipole approximation, the vector potential can be written

$$\vec{A} = \tfrac{1}{2}a(\vec{\epsilon}e^{i\omega t} + \vec{\epsilon}^{\,*}e^{-i\omega t}), \quad \vec{\epsilon}^{\,2} = \vec{\epsilon}^{\,*2} = 0, \quad \vec{\epsilon}\cdot\vec{\epsilon}^{\,*} = 1. \tag{64}$$

It is convenient to select coordinate axes with the z axis in the direction of propagation of the field. Then the polarization vector is

$$\vec{\epsilon} = 2^{-1/2}(\hat{x} \mp i\hat{y}), \tag{65}$$

where $\hat{x}$ and $\hat{y}$ are unit vectors along the x and y axes, the upper sign refers to right circular polarization, and the lower sign to left circular polarization. The argument of the exponential that appears in Eq. (51) is then

$$e\vec{A}\cdot\vec{r} = \beta(x\cos\omega t \pm y\sin\omega t), \quad \beta \equiv 2^{-1/2}ea. \tag{66}$$

The exponential function is

$$e^{ie\vec{A}\cdot\vec{r}} = e^{i\beta x\cos\omega t} e^{\pm i\beta y\sin\omega t} . \quad (67)$$

The first factor on the right in Eq. (67) is of the form given in Eqs. (52) and (53), and the second factor can be represented by

$$e^{\pm i\beta y\sin\omega t} = \sum_{l=-\infty}^{\infty} J_l(\beta y) e^{\pm il\omega t} . \quad (68)$$

Equation (67) is then the double sum

$$e^{ie\vec{A}\cdot\vec{r}} = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} i^k J_k(\beta x) J_l(\beta y) e^{i(k\pm l)\omega t} . \quad (69)$$

If the k,l sums are transformed to k,n sums where $n = k \pm l$, then Eq. (69) is

$$e^{ie\vec{A}\cdot\vec{r}} = \sum_n e^{in\omega t} \sum_k i^k J_k(\beta x) J_{\mp(k-n)}(\beta y) . \quad (70)$$

The sum over k in Eq. (70) can be done with the help of Graf's addition theorem.[48] For left-handed polarization, the result is

$$\sum_k i^k J_k(\beta x) J_{k-n}(\beta y) = (-i)^n \left(\frac{x+iy}{(x^2+y^2)^{1/2}}\right)^n \times J_{-n}(\beta(x^2+y^2)^{1/2}) . \quad (71)$$

It is convenient to introduce cylindrical coordinates ρ, φ, z,[49] with

$$\rho = (x^2+y^2)^{1/2}, \quad \varphi = \arctan(y/x) .$$

Equation (71) then simplifies to

$$\sum_k i^k J_k(\beta x) J_{k-n}(\beta y) = (-i)^n e^{in\varphi} J_{-n}(\beta\rho) . \quad (72)$$

The corresponding result for the right-handed case is

$$\sum_k i^k J_k(\beta x) J_{-k+n}(\beta y) = (-i)^n e^{-in\varphi} J_{-n}(\beta\rho) . \quad (73)$$

When Eqs. (72) and (73) are substituted into Eq. (70), the outcome can be expressed as

$$e^{ie\vec{A}\cdot\vec{r}} = \sum_n (\mp i e^{\mp i\varphi})^n J_n(\beta\rho) e^{in\omega t} , \quad (74)$$

or as

$$e^{ie\vec{A}\cdot\vec{r}} = \sum_n (\mp i e^{\pm i\varphi})^n J_n(\beta\rho) e^{-in\omega t} , \quad (75)$$

where the upper sign refers to right, and the lower to left circular polarization. Equation (74) is the analog of Eq. (53).

When Eq. (74) or (75) is used in the S matrix, the matrix element which appears in Eq. (54) for linear polarization is altered to

$$(\phi_f, e^{\pm in\varphi} J_n(2^{-1/2} ea\rho)\phi_i)$$

for circular polarization. The $\exp(\pm in\varphi)$ factor makes clear immediately a fundamental property of the circular-polarization case. If, for example, ϕ_i represents an S state, then ϕ_f must possess a total angular momentum of at least n units, and only that portion of the ϕ_f state with a z projection of n can contribute. The angular momentum constraints inherent in the linear polarization matrix element $(\phi_f, J_n(e\vec{a}\cdot\vec{r})\phi_i)$ are much less rigid.

The remainder of the development of transition probability expressions given for linear polarization carries through in the same way for circular polarization. That is, Eqs. (56)–(63) are true for the circular case as well. The only difference is that Eq. (55) must be replaced by

$$T_{fi}^{(n)} = (E_i - E_f)(-i)^n (\phi_f, e^{\mp in\varphi} J_n(2^{-1/2} ea\rho)\phi_i) \quad (76)$$

for right circular polarization, and

$$T_{fi}^{(n)} = (E_i - E_f) i^n (\phi_f, e^{\pm in\varphi} J_n(2^{-1/2} ea\rho)\phi_i) \quad (77)$$

for left circular polarization, where the ambiguous signs in Eqs. (76) and (77) are correlated with the ambiguous sign in the delta function in Eq. (56) or (58).

B. Two-fields case

A two-fields situation will occur when energy conservation in a transition cannot be accomplished with any integer number of photons from the applied electromagnetic field. For example, if an atom in a metastable state is irradiated by a field for which no multiphoton order can lead exactly to the ground state, stimulated emission can still occur with the emission of a Raman photon (the "second field") of energy appropriate for overall energy conservation. Both fields are switched off asymptotically, transitions can occur only through the simultaneous action of both fields, and so the formalism developed in Sec. IV A above can be applied directly if H' in Eq. (36) refers to the interaction Hamiltonian of both fields. All steps through Eq. (51) remain true with two fields.

Attention will be directed first to the linear polarization case, for which the two-fields formalism was given in Ref. 3. The vector potential in Eq. (52) is to be supplemented by a second term representing the second field, whose intensity is taken to be much smaller than that of the applied field. The second field is thus treated always in first order. Equation (52) is to be replaced by

$$\vec{A} = a\vec{\epsilon}\cos\omega t + \tilde{a}\vec{\epsilon}'\cos(\tilde{\omega}t + \alpha) , \quad (78)$$

where α is a phase displacement between the two fields. Instead of employing Eq. (53) for the second field, it is treated by the expansion

$$\exp[ie\tilde{a}\vec{r}\cdot\vec{\epsilon}'\cos(\tilde{\omega}t+\alpha)]\approx 1+\tfrac{1}{2}ie\tilde{a}\vec{r}\cdot\vec{\epsilon}'\,e^{i(\tilde{\omega}t+\alpha)}+\tfrac{1}{2}ie\tilde{a}\vec{r}\cdot\vec{\epsilon}'\,e^{-i(\tilde{\omega}t+\alpha)}. \quad (79)$$

The first term in Eq. (79) leads to the same result as if no second field were present, and by hypothesis, this cannot satisfy energy conservation. Only the last two terms need to be retained. The analog of Eq. (54) that arises is

$$(S-1)_{fi}=-2\pi i(E_i-E_f)\times e^{\pm i\alpha}(e\tilde{a})\sum_n i^{n+1}(\phi_f,\vec{r}\cdot\vec{\epsilon}'J_n(e\vec{a}\cdot\vec{r})\phi_i)\times\delta(E_f-E_i\pm n\omega\pm\tilde{\omega}), \quad (80)$$

where $\pm n\omega$ and $\pm\tilde{\omega}$ are not correlated, so that four different possibilities are represented in the delta function. The implication of Eq. (80) is that Eqs. (55) and (56) are to be replaced by

$$T_{fi}^{(n)}=(E_i-E_f)i^{n+1}e^{\pm i\alpha}(e\tilde{a})(\phi_f,\vec{r}\cdot\vec{\epsilon}'J_n(e\vec{a}\cdot\vec{r})\phi_i) \quad (81)$$

and

$$(S-1)_{fi}=-2\pi i\sum_n T_{fi}^{(n)}\delta(E_f-E_i\pm n\omega\pm\tilde{\omega}). \quad (82)$$

The delta function in Eq. (58) must be generalized by the appearance of $\pm\tilde{\omega}$ in the argument. With that modification in mind, the remaining equations in Sec. IV A can be employed.

With circular polarization in the two-fields problem, Eq. (78) is replaced by

$$\vec{A}=\tfrac{1}{2}a(\vec{\epsilon}e^{i\omega t}+\vec{\epsilon}^{*}e^{-i\omega t})+\tfrac{1}{2}\tilde{a}(\vec{\epsilon}'e^{i(\tilde{\omega}t+\alpha)}+\vec{\epsilon}^{*\prime}e^{-i(\tilde{\omega}t+\alpha)}). \quad (83)$$

The portion of $\exp(ie\vec{A}\cdot\vec{r})$ arising from the second field in Eq. (83) is approximated by

$$\exp[ie\tfrac{1}{2}\tilde{a}(r\cdot\vec{\epsilon}'e^{i(\tilde{\omega}t+\alpha)}+\vec{r}\cdot\vec{\epsilon}^{*\prime}e^{-i(\tilde{\omega}t+\alpha)})]\approx 1+\tfrac{1}{2}\tilde{\beta}\rho\cos(\tilde{\omega}t+\alpha\mp\varphi) \quad (84)$$

in place of Eq. (79), where cylindrical coordinates ρ,φ have been introduced in the right-hand side of the equation. The parameter $\tilde{\beta}$ is defined to be $\tilde{\beta}=2^{-1/2}e\tilde{a}$ in analogy to the definition for β in Eq. (66). As in the linear polarization case, the unit term on the right-hand side of Eq. (84) corresponds to the single-field case, and will not contribute here. Equation (80) for linear polarization is now replaced by

$$(S-1)_{fi}=-2\pi i(E_i-E_f)e^{i\alpha\frac{1}{2}}2^{-1/2}e\tilde{a}i\sum_n(\mp i)^n(\phi_f,\rho e^{i(\mp n\mp 1)\varphi}J_n(2^{-1/2}ea\rho)\phi_i)\delta(E_f-E_i+n\omega+\tilde{\omega})$$
$$-2\pi i(E_i-E_f)e^{-i\alpha\frac{1}{2}}2^{-1/2}e\tilde{a}i\sum_n(\mp i)^n(\phi_f,\rho e^{i(\mp n\pm 1)\varphi}J_n(2^{-1/2}ea\rho)\phi_i)\delta(E_f-E_i+n\omega-\tilde{\omega}) \quad (85)$$

for circular polarization. Equation (85) is written only for the case where the δ function contains $+n\omega$, which is the natural form for emission of n photons of frequency ω. One can also write the corresponding S-matrix expression for n-photon absorption in the same fashion as done earlier. The ambiguous signs in Eq. (85) in the factors like $\exp[i(\mp n\mp 1)\varphi]$ are independent. For example, this factor, from the first of the two terms in Eq. (85), represents emission of n right-handed ω photons and one right-handed $\tilde{\omega}$ photon if both upper signs are used. If both lower signs are used, the emission is of left-handed photons in both fields. The other two possibilities are mixed cases. The second term in Eq. (85) is for emission of n photons of ω type and absorption of one photon of $\tilde{\omega}$ type, with either handedness of the circular polarization depending on the choice of either the upper or lower sign. The same remarks apply to the form of Eq. (85) altered to represent absorption of n photons of ω type.

C. Asymptotic-field case

A variation of the two-fields case can occur in which the applied field exists as an "asymptotic" field. What is meant by this is that the bound system is subjected to an applied field even at asymptotic times, so that observations on the initial and final states are made in space and time regions that include the field. This is the only type of two-fields case considered by Decoster.[5]

The formalism of Sec. IV A can be applied directly to this case by reinterpreting the H_0 of Eqs. (36)–(38) to include the applied field, and restricting H' in Eq. (36) to refer only to the effects of a second perturbing field. There is no difficulty in this, since the formalism permits H_0 to have explicit time dependence as well as H'. However, to denote the fact that Φ now includes effects of a time-dependent applied field, it will be relabeled as Υ. The interaction Hamiltonian operator in this asymptotic-field case is denoted H_a', and is

$$H_a'=-m^{-1}e\vec{\mathfrak{A}}\cdot\vec{p}+m^{-1}e^2\vec{\mathfrak{A}}\cdot\vec{A}, \quad (86)$$

since $\vec{\mathfrak{A}}$ is to be retained only to first order. The asymptotically present field is $\vec{A}$, and the perturbing field is $\vec{\mathfrak{A}}$. The second term in Eq. (86) gives the cross coupling between the perturbing

field and the asymptotic field. The S matrix in Eq. (47) is now

$$(S-1)_{fi}=i\int dt_1(\Upsilon_f, H'_a\Psi_i^{(+)})_{t_1}. \tag{87}$$

With H'_a to be treated as a weak field, the interacting state in Eq. (87) can be replaced by the noninteracting state so $\Psi_i^{(+)}\approx\Upsilon_i$. Furthermore, the noninteracting states will be represented by the MTA, so

$$\Upsilon\approx\exp(ie\vec{A}\cdot\vec{r})\Phi\,, \tag{88}$$

with Φ having its earlier meaning of the state with no applied fields. By implication, the Hilbert space vectors and operators in Eq. (87) are now given in configuration representation in the Schrödinger picture. The inner product in Eq. (87) takes the form

$$(\Upsilon_f, H'_a\Psi_i^{(+)})\approx(e^{ie\vec{A}\cdot\vec{r}}\Phi_f, H'_a e^{ie\vec{A}\cdot\vec{r}}\Phi_i)\,. \tag{89}$$

However, the action of the momentum-translation factors on H'_a is to transform it to

$$e^{-ie\vec{A}\cdot\vec{r}}H'_a e^{ie\vec{A}\cdot\vec{r}}=-m^{-1}e\vec{\mathcal{A}}\cdot\vec{p}\,, \tag{90}$$

thus eliminating the cross-coupling term in Eq. (86). The result of Eqs. (89) and (90) is that

$$(\Upsilon_f, H'_a\Psi_i^{(+)})\approx(\Phi_f, -m^{-1}e\vec{\mathcal{A}}\cdot\vec{p}\Phi_i)\,, \tag{91}$$

just as if no asymptotic field were present at all. That is, the MTA predicts no effects of the asymptotic field.

The above conclusion is altered if initial and final states differ in the charge state of the bound particle. The MTA factors for initial and final states are then $\exp(ie_i\vec{A}\cdot\vec{r})$ and $\exp(ie_f\vec{A}\cdot\vec{r})$, with $e_i\neq e_f$, and so Eqs. (90) and (91) no longer hold. For example, if an initially bound electron is detached in the final state, it is no longer describable by the MTA, which is applicable only to bound electrons. The final state is not represented by Eq. (88), but will be given instead by

$$\Upsilon_f\approx\Phi_{rf}\Psi_{ef}\,, \tag{92}$$

where Φ_r is the state of the residual system without the detached electron, and Ψ_e is the state of the free electron. The MTA treatment of the initial state then leads to nontrivial consequences of the asymptotic field.

V. CONNECTIONS BETWEEN THE MTA AND PERTURBATION THEORY

A. First-order S matrix

It is known[1] that the MTA is exactly the same as perturbation theory to first order in the field. The result is very easy to demonstrate. If Eq. (49) for the MTA is to be restricted to first order, then H' is replaced by $H'^{(1)}$ [the first-order term in Eq. (3)], and the exponential factor is replaced by unity. The first-order form of Eq. (49) is thus

$$(S-1)_{fi}^{(1)}=-i\int dt_1(\phi_f, H'^{(1)}\Phi_i)_{t_1}\,, \tag{93}$$

which is exactly the first order S matrix for perturbation theory in Coulomb gauge.

The form of the MTA S matrix in Eq. (51) leads very easily to the electric-field gauge perturbation theory when the first-order limit is considered. To obtain a first-order contribution in Eq. (51), the exponential factor must be expanded to two terms

$$e^{ie\vec{A}\cdot\vec{r}}\approx 1+ie\vec{A}\cdot\vec{r}\,,$$

so that

$$(S-1)_{fi}^{(1)}=(E_i-E_f)\int dt_1(\Phi_f, e\vec{A}\cdot\vec{r}\Phi_i)_{t_1}. \tag{94}$$

An integration by parts in Eq. (94) yields

$$(S-1)_{fi}^{(1)}=-i\int dt_1(\Phi_f, H_I\Phi_i)_{t_1}\,, \tag{95}$$

where

$$H_I\equiv-e\vec{E}\cdot\vec{r}\,. \tag{96}$$

Equation (95) can be viewed as the electric-field gauge result in first-order perturbation theory, but a refinement can be inserted. The work done in this paper is all in Coulomb gauge (except for peripheral discussion about electric-field gauge), and so the set of noninteracting states $\{\Phi_j\}$ refers to Coulomb gauge. If expressed in electric-field gauge, the set is $\{e^{ie\vec{A}\cdot\vec{r}}\Phi_j\}$. This can be incorporated trivially in Eq. (95) by writing it as

$$(S-1)_{fi}^{(1)}=-i\int dt_1(e^{ie\vec{A}\cdot\vec{r}}\Phi_f, H_I e^{ie\vec{A}\cdot\vec{r}}\Phi_i)\,, \tag{97}$$

which follows immediately from

$$[H_I, e^{ie\vec{A}\cdot\vec{r}}]=0\,. \tag{98}$$

Equation (97) is the first equation in this paper in which $\exp(ie\vec{A}\cdot\vec{r})$ is employed as a gauge transformation.

B. Complete S matrix

The complete S matrix in the momentum-translation approach is found from Eq. (47) with the interacting state represented by Eq. (29). Explicitly, this is

$$\begin{aligned}(S-1)_{fi}=&-i\int_{-\infty}^{\infty}dt(\Phi_f, H'e^{ie\vec{A}\cdot\vec{r}}\Phi_i)_t\\&-i\int_{-\infty}^{\infty}dt\Big(\Phi_f, H'e^{ie\vec{A}\cdot\vec{r}}\sum_n\Phi_n\Big)_t\\&\times\Big(-i\int_{-\infty}^{t}dt_1(\Phi_n, H_I\Psi_i)_{t_1}\Big)\,,\end{aligned} \tag{99}$$

in the notation of Eq. (96). The commutator theorem of Eq. (50) puts Eq. (99) in the form

$$(S-1)_{fi} = -i(E_i - E_f)\int_{-\infty}^{\infty} dt(\Phi_f, e^{ie\vec{A}\cdot\vec{r}}\Phi_i)_t - i\sum_n (E_n - E_f)\int_{-\infty}^{\infty} dt(\Phi_f, e^{ie\vec{A}\cdot\vec{r}}\Phi_n) \times\left(-i\int_{-\infty}^{t} dt_1(\Phi_n, H_I\bar{\Psi}_i)_{t_1}\right). \quad (100)$$

Integration by parts in the first term in Eq. (100) yields

$$-i(E_i - E_f)\int_{-\infty}^{\infty} dt(\Phi_f, e^{ie\vec{A}\cdot\vec{r}}\Phi_i)_t = (\Phi_f, e^{ie\vec{A}\cdot\vec{r}}\Phi_i)\Big|_{-\infty}^{\infty} - i\int_{-\infty}^{\infty} dt(\Phi_f, H_I e^{ie\vec{A}\cdot\vec{r}}\Phi_i), \quad (101)$$

in view of the fact that

$$i\partial_t e^{ie\vec{A}\cdot\vec{r}} = -H_I e^{ie\vec{A}\cdot\vec{r}}. \quad (102)$$

Because the field is switched off at asymptotic times, both limits in the first term on the right in Eq. (101) give δ_{fi}, and so they cancel each other, leaving only the second term. Integration by parts in the t integral of the second term in Eq. (100) is more complicated because of the t in the upper limit of the t_1 integral. The result is

$$-i\sum (E_n - E_f)\int_{-\infty}^{\infty} dt(\Phi_f, e^{ie\vec{A}\cdot\vec{r}}\Phi_n)\left(-i\int_{-\infty}^{t} dt_1(\Phi_n, H_I\bar{\Psi}_i)_{t_1}\right) = -i\int_{-\infty}^{\infty} dt(\Phi_f, H_I\bar{\Psi}_i)_t - i\int_{-\infty}^{\infty} dt\left(\Phi_f, H_I e^{ie\vec{A}\cdot\vec{r}}\sum_n \Phi_n\right)\left(-i\int_{-\infty}^{t} dt_1(\Phi_n, H_I\bar{\Psi}_i)_{t_1}\right) + i\int_{-\infty}^{\infty} dt(\Phi_f, e^{ie\vec{A}\cdot\vec{r}}H_I\bar{\Psi}_i)_t. \quad (103)$$

When Eqs. (101) and (103) are added, Eq. (101) combines with the second term in Eq. (103) to cancel the third term in Eq. (103). As a result, Eq. (100) becomes

$$(S-1)_{fi} = -i\int_{-\infty}^{\infty} dt(\Phi_f, H_I\bar{\Psi}_i)_t. \quad (104)$$

As pointed out following Eq. (95), the states Φ_f and $\bar{\Psi}_i$ have been defined in a Coulomb-gauge context. Equation (104) is entirely in electric-field gauge if it is written as

$$(S-1)_{fi} = -i\int dt(e^{ie\vec{A}\cdot\vec{r}}\Phi_f, H_I e^{ie\vec{A}\cdot\vec{r}}\bar{\Psi}_i)_t. \quad (105)$$

This last step is purely formal. In practice one would simply use Eq. (104) as the complete expression from which a perturbation series can be developed.

The fact that the momentum-translation expression, Eq. (99), is fully equivalent to Eq. (104) or (105) means that a momentum-translation development with all terms included is equal to a perturbation series with all terms included. This is as it should be, since the complete sum in both cases is exact. Nevertheless, in practical application, the two approaches will be very different. The MTA is used in the lowest-order form shown in Eq. (51) regardless of the multiphoton order of the process. By contrast, a multiphoton process of order n requires that a perturbation expansion of Eq. (104) be carried to nth order to exhibit the leading term. Despite these major differences, the fact that the full momentum-translation series is a rearrangement of the full perturbation series has led to some confusion. It certainly does not mean that momentum translation is "just" perturbation theory when corrections are applied.[22] This is much like saying that perturbation theory is just MTA when corrections are applied, since a partial sum of perturbation-theory terms leads to the MTA. *Any* approximation method, when all terms are included, will be a rearrangement of perturbation theory, when all terms are included.

Both momentum translation and perturbation theory start at the same point: a complete set of basis states (the unperturbed states) and an interaction Hamiltonian. Power series are unique, and so any expansion of the transition amplitude in powers of the interaction term is the perturbation series. Any method of calculation other than perturbation theory will omit some portion of field dependence to the nth power in the nth term. If it doesn't, it is perturbation theory. Therefore, to say the MTA cannot be correct since it omits some part of that term[7] is tantamount to saying that no method of calculation other than perturbation theory can be considered.

A corollary to the uniqueness (once basis states and interaction term are selected) of perturbation theory is that any method which is not the perturbation series can appropriately be called nonperturbative. The momentum-translation approach is to carry out a unitary transformation (the momentum translation) to another system, do a perturbation expansion there, and then use the result back in the original system. In the original system, which is where the dynamical calculations are made, the technique certainly does not proceed in powers of the field strength, and thus it is

not the perturbation expansion. Nevertheless, it has been maintained[22,50] that it is improper to call the method nonperturbative because of the use of a perturbation series in the momentum-translated system. I disagree. As stated above, any method which is not the perturbation series is, *ipso facto*, nonperturbative.

The philosophy of momentum translation is that the effect of the field is asserted (in Coulomb gauge) by the field momentum in the equation of motion. The idea is to approximately transform this away, leaving a simple result. There is nothing iterative in this. It gives an answer at the first step, regardless of photon order.

VI. SINGLE-FIELD TRANSITIONS

The accuracy of the MTA is difficult to assess in general. In Ref. 1, two conditions were stated: $eaa_0\omega/E \ll 1$ and $\omega/E \ll 1$. The second of these conditions is not necessary. In fact, if $\omega/E = 1$, this corresponds to a problem treatable by first-order perturbation theory, where the MTA gives results essentially identical to perturbation theory. The other condition $eaa_0\omega/E \ll 1$ is more significant. However, it has been faulted by Decoster, who has attempted an elaborate error analysis.[5] Decoster's analysis does not consider the cancellations that occur among the terms beyond the basic MTA term, and it thus overestimates the errors. Some of the individual terms beyond the basic MTA can have magnitudes nearly as large as the MTA itself, but these terms have no physical meaning, and tend to be canceled by other terms. This is an important general point to stress. The momentum-translation method is not a successive approximation method. The basic MTA term has a physical meaning as a momentum-translated result. The sum of *all* the other terms has meaning as a correction, but none of the individual terms beyond the MTA has any physical significance in itself, nor can the MTA consistently be improved upon by including additional terms.

A pragmatic way to assess the accuracy of the MTA in single-field problems without near resonances is simply to compare results calculated from the MTA in the low-intensity limit with known perturbation theory results. Table I gives this comparison for $1S$-$2S$ and $1S$-$2P$ multiphoton transitions in hydrogen. These transitions are selected from the perturbative multiphoton calculations of Gontier and Trahin[51] because they clearly do not have intermediate near resonances.

The greatest discrepancy shown in Table I between perturbation theory and MTA is in the last entry, where the results differ by a factor of 2.25. This is excellent agreement for different methods of calculation of multiphoton processes.

Another comparison that can be made, related to the above results, is to consider the spontaneous two-photon decay of the metastable $2S$ state in hydrogen. In this process, all possible combinations of energies occur between the two emitted photons, as long as they satisfy the constraint that the sum of the photon energies equals the $2S$-$1S$ transition energy. The known perturbation result for the spontaneous transition rate is[52-55] 8.23 sec^{-1}, whereas the MTA gives[10] 3.23 sec^{-1}. The results differ by a factor of 2.5. The spectral distributions of emitted photons from perturbation theory,[54] from an exact solution,[55] and from the MTA[10] are very similar in shape, suggesting little frequency dependence in the accuracy of the MTA.

A different physical process where comparisons between perturbation theory and the MTA can be made is for two-photon detachment rates of negative ions. These rates have been calculated by Faisal[24] with the MTA, and may be compared with Geltman's[56] perturbation results. The comparisons are given in Table II.

The agreement shown in Tables I and II can be described as excellent. For example, several authors[15,57,58] have compared theoretical results for three-photon ionization of alkali-metal atoms as obtained by the approximate perturbation theory of Bebb and Gold,[59] the approximate perturbation theory of Morton,[60] and the Green's-function theory of Manakov *et al.*[61] Differences by factors of 10

TABLE I. Ratios of perturbation-theory cross sections (σ_{PT}) to MTA cross sections (σ_{MTA}) for multiphoton transitions in hydrogen with no possible intermediate near resonances.

$1S$-$2S$			
	$n=2$	$n=4$	$n=6$
σ_{PT}/σ_{MTA}	2.20	1.52	0.906
$1S$-$2P$			
	$n=3$	$n=5$	$n=7$
σ_{PT}/σ_{MTA}	0.911	0.760	0.444

TABLE II. Ratios of perturbation-theory transition probabilities (W_{PT}) to MTA transition probabilities (W_{MTA}) for two-photon detachment in negative ions.

	W_{PT}/W_{MTA}
I^-	0.78
Br^-	0.82
F^-	0.76

to 100 are commonplace between different theories, as well as between theory and experiment. Similar discrepancies occur in comparisons of calculations of other multiphoton processes.[15,58]

Although it is important to stress that the MTA should not be used when intermediate near resonances can occur, it is nevertheless instructive to see how near an intermediate resonance can be before causing difficulty for the MTA. This information can be inferred by comparisons of MTA results with the multiphoton calculations of Gontier and Trahin[51] for transitions from the ground state of hydrogen to the $n=3$ and 4 states. The $n=2$ intermediate state does not pose a problem for the MTA for two-photon transitions, but it does for three-photon transitions. From the specific energy differences in these transitions, it can be inferred that the nearest approach to an intermediate resonant state by an intermediate number of photons (e.g., one photon in a two-photon transition, or two photons out of a total of three) should be greater than perhaps 25 to 30% of the total transition energy for the MTA to give useful results.

The general conclusions implied by practical results from MTA calculations with no near resonances are several. First, a factor of 2 accuracy in transition probabilities seems to be the general case, although it is more conservative to assume no better than a factor of 10 accuracy when comparison results are not available. This accuracy might not be very impressive for two-photon processes, but for higher orders it is very useful indeed. This statement is buttressed by the conclusion that the MTA does not lose accuracy with increasing order of the process, and furthermore, that the MTA is as easy to apply for high-multiphoton orders as it is for low orders. This last quality is very important, since the MTA is generally very easy to apply, and it can often lead to analytical results where other methods can provide numerical output only.

VII. TWO-FIELDS TRANSITIONS

As explained earlier, a two-fields problem is one in which a transition in a system is effected through the agency of a second field in addition to the external electromagnetic field. The second field may be electromagnetic as in Raman scattering, or it may be of some other type. Intermediate near resonances can occur, as in single-field problems, but they are less likely, and they will not be discussed further. The remarks made about near resonances in the single-field case apply to the two-field case as well.

The basic condition, Eq. (23), is more easily satisfied in a two-fields problem than in a single-field problem. In particular, a process involving a single photon from the applied field constrains the field intensity to $eaa_0 \ll 1$ in the single-field case, but need not imply any such thing for a two-fields problem. If ω/E is small enough, true intense-field situations can be treated by the MTA. This is in contrast to the single-field case, where the MTA suffers from the same intensity constraint as perturbation theory. It was largely because of this extra flexibility about intensities that the two-fields problem was stressed in Refs. 1–3.

VIII. SUMMARY

A. Objections to the MTA

The most fundamental objection to the MTA, were the objection justified, is the assertion by Cohen-Tannoudji *et al.*[4] that the momentum-translation technique is nothing more than a gauge transformation, and hence devoid of physical content. This subject has been dealt with at length in Sec. II. The momentum-translation approach makes use of a unitary transformation *within* a Coulomb gauge, as shown unambiguously by the inference of the MTA from perturbation theory (Sec. II A), by the path-integral development (Sec. II B), and by the relationship of the MTA to the independent methods of Babiker[17,18] and of Campos and Krüger[19] (Sec. II D). The Coulomb-gauge character of the MTA is clearly in evidence in the transition matrix elements, which involve explicitly the Coulomb-gauge interaction Hamiltonian.

Another significant challenge to the momentum-translation method comes from Decoster[5] and others, who treat the single-field problem as if the leading (MTA) term in the momentum-translation expansion is no different in kind from the other terms which follow. Cohen-Tannoudji *et al.*[4] have done something similar. In fact (see Sec. VI), only the MTA term has physical significance as an approximation to the full result. Supplementary terms may be individually significant, but they tend to mutually cancel, so that the lead term is a good approximation to the complete series sum when Eqs. (22) and (23) are satisfied, and when no intermediate near resonances can occur. Adding supplementary terms to the MTA produces a hybrid of momentum translation and perturbation theory which has no physical or numerical meaning.

In his discussion of the two-fields problem within the MTA, Decoster[5] has confined his attention to the asymptotic-field case, where the asymptotic reference states contain the full effects of one of

the fields, but not the other. Under these circumstances, the MTA gives nontrivial results only when there is a change in charge state in the interaction. Decoster takes the point of view that the asymptotic-field case is the only type of two-fields problem possible, whereas it is, in fact, quite unusual. In the great majority of two-fields problems, the asymptotic reference states are free of interaction. This is discussed in Sec. IV C.

The simple MTA is known to fail badly when intermediate near resonances can occur in a transition. If a momentum-translation approach is to be used, then supplementary terms must be included. Useful results can be obtained this way, but then most of the simplicity, directness, and analytical utility of the MTA is lost.

The fact that the complete momentum-translation series can be shown explicitly to be a rearrangment of the complete perturbation series, and vice versa, has led to misunderstandings that have no relevance for the validity or utility of the MTA (see Sec. V).

Cohen-Tannoudji *et al.*[4] compare MTA results with known exact solutions for transitions in a harmonic oscillator.[62,63] They fault the MTA for predicting a multiphoton transition between adjacent states when, in fact, the exact solution has none. This inconsistency arises because of some properties peculiar to the harmonic oscillator. The harmonic oscillator appears to be the only problem in which multiphoton transitions between adjacent levels are not possible, nor are single-photon transitions between nonadjacent levels. The peculiar feature with respect to the lack of multiphoton transitions between adjacent levels shows up in the momentum-translation theory in the form of zero denominators in the supplementary terms. This means that the finite and nonvanishing result of the MTA term is meaningless for this problem, but for this problem only.

Cohen-Tannoudji *et al.*[4] have also criticized the MTA for failure to predict level shifts. The MTA is unlike perturbation theory in that transitions of all orders (from a perturbation-theory point of view) are calculated within the MTA in a fashion resembling first-order perturbation theory. Normally, one does not go beyond the leading term in momentum translation. It is for this reason that the level shift does not appear in the simple MTA. However, as shown in Sec. III, a level-shift calculation can be carried out in the context of momentum-translation solutions, but the calculation is formally identical to perturbation theory. Although the use of momentum translation confers no special advantage for calculating level shifts, the MTA does give a replica spectrum directly.

B. Strengths and limitations of the MTA

The MTA gives extremely simple and general analytical approximations for wave functions, transition amplitudes, and probabilities for multiphoton transitions. Above all, the simplicity is maintained for high-order transitions, which are as easy to calculate as low order, and for which accuracy is as good as for low order. Conditions for validity are $eaa_0\omega/E \ll 1$, $\omega a_0 \ll 1$, and the absence of intermediate near resonances.

The above comments hold true in general for two-fields problems, with the added remark that increasing values of eaa_0 can be treated as $n\omega/E$ declines.

A brief list of numerical experience to date with the MTA follows.

(a) Bound-bound transitions in hydrogen are given to within a factor of about 2 when no intermediate near resonances occur. See Table I.

(b) Two-photon detachment of halogenic negative ions is given accurately. See Table II.

(c) Two-photon spontaneous decay of the metastable $2S$ state in hydrogen is given to within a factor of about 2, and the emitted photon spectrum is given accurately.[10]

(d) The qualitative behavior of polarization effects in photoionization was deduced easily and correctly with the MTA.[64] The simple analytic MTA near-threshold results for polarization ratios agree numerically with the perturbation calculations by Lambropoulos[50] for four- and five-photon processes at the same near-threshold condition.

(e) MTA calculations by Jones[65] for polarization ratios in photoionization of hydrogen agree with perturbation calculations by Gontier and Trahin[66] for field frequencies which lie between the resonance-caused peaks in the dispersion curves.

The disadvantages of the MTA are that it can be very much in error when intermediate near resonances can occur, it does not lend itself to systematic correction, and it is no different from perturbation theory for calculation of level shifts.

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