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Quadrupole bremsstrahlung in the scattering of identical charged bosons and fermions

Robert J. Gould
Department of Physics, B-019, University of California, San Diego, La Jolla, California 92093

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In a nonrelativistic formulation employing noncovariant perturbation theory in the Born approximation and including exchange effects, the differential cross section for bremsstrahlung is derived for scattering by identical charged particles of arbitrary spin. The problem is first treated in a general manner, giving the expression for the total direct amplitude for the complete process (photon emission in scattering) for the case where the scattering potential is arbitrary. The derivation by means of perturbation theory demonstrates the critical role played by the small momentum associated with the emitted photon. Graviton bremsstrahlung is discussed briefly and the corresponding quadrupole cross section is given by a simple comparison with the photon bremsstrahlung formula.

I. INTRODUCTION

The special problem of bremsstrahlung in the scattering of two electrons has been considered by a number of authors. Most investigations have treated the relativistic problem by modern quantum electrodynamics with recent progress made with the help of computers for numerical evaluation of angular integrals over outgoing particle coordinates.[1] Unfortunately, because of its complexity, the general case requires these numerical evaluations and it is not possible to derive a formula for the cross section that is differential only in the outgoing photon energy. Actually, in the extreme relativistic limit, there had previously been some confusion associated with the problem, and several well-known textbooks[2] quote wrong formulas for the cross section. These questions now seem to have been cleared up.

Less attention has been paid to e-e bremsstrahlung in the nonrelativistic limit. Compared with "single-particle" bremsstrahlung, such as in electron-proton scattering, the e-e cross section is smaller by a factor $\sim v^2/c^2$. This is because e-p emission is of a dipole nature while e-e radiation is necessarily by the quadrupole process. The rate of e-e radiation is comparable to that from relativistic corrections to e-p bremsstrahlung. It is of interest for "practical" applications in the analysis of high-temperature laboratory and astrophysical plasmas. Because e-e bremsstrahlung is very small at low energies, its most important energy domain corresponds to scattering fast electrons. Thus, the most relevant nonrelativistic domain is that of the Born approximation which, Coulomb scattering, requires $E_e \gg 1$ Ry (or $v/c \gg \alpha = e^2/\hbar c \cong 1/137$).

Even in the nonrelativistic Born-approximation limit there has in the past been some question[3] of the appropriate formula for the e-e bremsstrahlung cross section. Neglecting exchange effects and taking a semiclassical approach starting from the classical quadrupole emission formula, Lifshitz[4] derived a formula for the e-e cross section; his essential result is contained in results of an alternative derivation given in this work. Fediushin[5] generalized the Lifshitz work to include exchange and obtained a formula for the cross-section differential in the outgoing photon energy, that is applicable to the e-e problem. Fediushin's result, which is also contained in a general formula derived herein for the special spin case $s = \frac{1}{2}$, has been found to be in agreement with the numerical evaluation by Haug[1] in the nonrelativistic limit of his more general treatment.

Because of the semiclassical foundation of the Lifshitz-Fediushin work, it is of interest to reinvestigate the e-e problem in a completely quantum-mechanical formulation. A rigorous treatment can be made simple, however, without introducing the formalism of quantum-field theory. The basic amplitude for the process is given by the second-order perturbation Hamiltonian resulting from the combined action of the scattering potential and the "photon-emission" perturbation. Further, there is a certain simplification inherent in quantum electrodynamics in the nonrelativistic limit. It is that the photon momentum is always small compared with particle momenta when the photon energy is smaller than or comparable with particle energies. The existence of this inequality provides the fundamental explanation why classical or semiclassical derivations often lead to correct results for some nonrelativistic processes even outside the domain of the classical limit. That is, in some nonrelativistic problems, even for general photon energies, the photon momentum is small and its particlelike character does not manifest itself in the kinematics. However, as we shall see, in e-e bremsstrahlung the effects of the photon momentum must be considered carefully; this is true *even in the soft-photon limit*. Thus, the only safe derivation of the e-e bremsstrahlung cross section is one which is purely quantum mechanical.

For the bremsstrahlung problem it is possible to derive a general result for the total direct amplitude for the process in terms of a general central scattering potential, and the derivation is outlined in the following section. Results are then applied in Sec. III to the special case where the scattering potential is Coulombic and where the particles have the same charge and mass but are somehow distinguishable. An expression for the photon-emission probability is obtained therein and is identified with a factor in a previous semiclassical treatment by Lifshitz. The general cross section for bremsstrahlung in Coulomb scattering is derived in Sec. IV; this formula, which includes exchange effects, is derived for the case of arbitrary spin s and is exhibited as an explicit function of s for the case where the incident particles are unpolarized. In the nonrelativistic limit spin interactions are negligible, but the value of s determines the number of symmetric and antisymmetric spatial-coordinate states and the associated coefficients for the squared respective total amplitudes. Graviton bremsstrahlung is discussed briefly in Sec. V.

II. PERTURBATION THEORY FORMULATION

A. Effective perturbation Hamiltonian

For the combined process of scattering with the emission of a photon the amplitude is essentially given by the perturbation Hamiltonian matrix element, which is the second-order expression

$$H'_{f0}=\sum_i V_{fi}\,\frac{1}{E_0-E_i}\,\Delta_{i0}^{(1,2)}+(V\leftrightarrow\Delta)\,. \qquad (1)$$

In this expression f, 0, and i refer to the final, initial, and intermediate states, respectively, and 1 and 2 are the charged-particle labels. The interparticle scattering potential is designated by the perturbation V and Δ denotes the photon-emission (electromagnetic) perturbation. The (1, 2) superscript on Δ means that the particle-photon interaction can involve either charged particle and the added terms denoted by $(V\leftrightarrow\Delta)$ mean that the photon can be emitted "before" and "after" scattering. Thus there are four terms in the matrix elements (1), each with a sum over intermediate states, and, although this formulation is noncovariant, the terms can be represented by Feynman-type diagrams (see Fig. 1) similar to those in a relativistic covariant development.[1,2] The V interaction in this nonrelativistic treatment can be represented by a "vertex" with two incoming and two outgoing charged particles. The four terms (1) then correspond to photon emission from each of the four legs of the diagram and, as in noncovariant relativistic formulations, the energy denominators correspond to Feynman propagators in covariant perturbation theory. Thus, in two of the four terms in (1) a final-state photon is present in the intermediate state. In addition to the four terms of the total direct amplitude (1) there are four exchange terms obtained by interchanging the particle final-state coordinates (momenta).

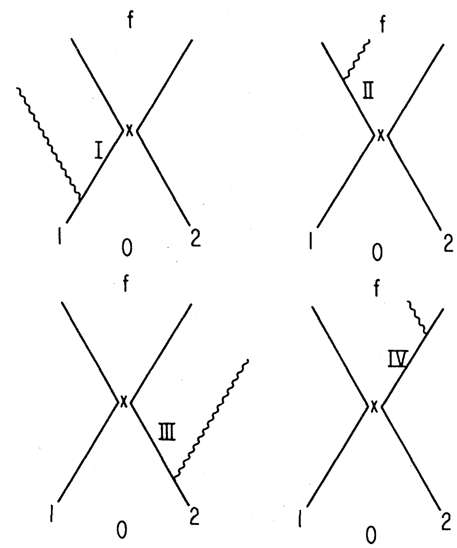

FIG. 1. Perturbation-theory diagrams for bremsstrahlung in two-particle collisions. The fixed Coulomb potential is designated by X, and I, II, III, and IV refer to the intermediate states. There are four additional exchange diagrams.

The amplitudes (1) and the corresponding diagrams (Fig. 1) correspond to bremsstrahlung in which photon emission is associated with the perturbation Δ acting only "in the external lines." This is a good approximation when the scattering is due to the action of a Coulomb potential, especially at nonrelativistic energies. Effects of "emission from internal lines" in Coulomb scattering are of higher order (in α) and of negligible magnitude. However, this would not necessarily be true for other scattering potentials. For example, in proton-proton scattering when the energy is sufficiently high that the nuclear forces contribute to the scattering, emission from these internal lines (exchanged virtual pions) must be included.

B. Photon-emission perturbation

The treatment of the bremsstrahlung problem in this paper is identical to a field-theory calculation. However, the basic results can be obtained in an elementary way without employing the detailed formalism of quantum-field theory. The simplified procedures are, at the same time, totally rigorous. Since there are relatively few systematic treatments of nonrelativistic quantum electrodynamics, it may be useful to give a brief outline of the derivation of the perturbation Hamiltonian associated with photon emission.

In the Coulomb gauge the Hamiltonian associated with the linear interaction of a system of charges with an electromagnetic radiation field is

$$H_{\rm rad} = \frac{i\hbar}{c} \sum_\alpha \frac{e_\alpha}{m_\alpha} \vec{A} \cdot \vec{\nabla}_\alpha , \tag{2}$$

where e_α and m_α are the particle charge and mass, respectively, and $\vec{A}$ is the radiation field vector potential. What is needed is essentially the vector potential $\vec{A} = \vec{a}_{\rm out}$ associated with a single outgoing photon. One approach that is convenient is to consider the *differential* perturbation $H_{\rm rad} = \Delta$ corresponding to the production of a photon in the direction defined by the solid angle $\Delta\Omega$ and having wave-vector magnitude within Δk. In the end, if required, an integration over these final-state variables can be performed. The amplitude for this process of spontaneous emission can be obtained through consideration of the closely related process of induced emission in which the system of charges is perturbed by an external field from a beam of photons defined by the same $\Delta\Omega$ and Δk. The vector potential for such a photon beam can be written

$$\vec{A} = \vec{\epsilon} A_0 \cos(\vec{k} \cdot \vec{r} - \omega t) , \tag{3}$$

where $\vec{\epsilon}$ is the polarization unit vector and A_0 is the the amplitude. The electric field of the wave is $\vec{E} = -(1/c)\partial\vec{A}/\partial t$ and A_0 can be related to the photon-beam occupation number $\bar{n}$ by writing the differential photon flux as

$$\Delta J = c(2\pi)^{-3} k^2 \Delta k \Delta\Omega \bar{n} = (\hbar k)^{-1} \Delta u \quad , \tag{4}$$

where Δu is the photon-beam energy density. Since

$$\Delta u = \langle (\vec{E})^2 \rangle / 4\pi = k^2 A_0^2 / 8\pi , \tag{5}$$

we have

$$A_0 = (ck \Delta k \Delta\Omega / \pi^2)^{1/2} \bar{n}^{1/2} \quad . \tag{6}$$

Since the probability for the overall process will be proportional to the square of factors in $H_{\rm rad}$, we can obtain the expression for $\vec{a}_{\rm out}$ associated with the spontaneous production of a single outgoing photon by simply taking the expressions (3) and (6) with the factor $\bar{n}^{1/2}$ *left out*. This is because for any process involving an outgoing photon the total rate for the induced plus spontaneous process is obtained from the induced rate with the replacement[6] $\bar{n} \to \bar{n} + 1$. Further, we can simplify the formulas by ignoring the time dependence in the expression (3), if we impose energy conservation and also rewrite the $\cos(\vec{k} \cdot \vec{r})$ factor in terms of complex exponentials. Then, since the perturbation matrix elements inherently require "momentum conservation at the vertices", we find that only the part $\frac{1}{2} \exp(-i\vec{k} \cdot \vec{r})$ contributes. Thus we obtain, in this very elementary but rigorous way, the following expression for the interaction Hamiltonian corresponding to photon production within Δk and $\Delta\Omega$ and with polarization $\vec{\epsilon}$:

$$\Delta = (i/2\pi)(k \Delta k \Delta\Omega)^{1/2} \sum_\alpha (e_\alpha / m_\alpha) \exp(-i\vec{k} \cdot \vec{r}_\alpha) \vec{\epsilon} \cdot \vec{\nabla}_\alpha \; . \tag{7}$$

This expression, in which units with $\hbar = c = 1$ are used, represents the differential Hamiltonian corresponding to a vertex with a single outgoing photon associated with all electromagnetic interactions involving "orbital" motion. Further, one can readily show that for particles with intrinsic magnetic moments $\vec{\mu}_\alpha$ there is an additional interaction term of the form ($\hbar = c = 1$):

$$\Delta' = (k/2\pi)(k \Delta k \Delta\Omega)^{1/2} \sum_\alpha \exp(-i\vec{k} \cdot \vec{r}_\alpha) \vec{\epsilon}\,' \cdot \vec{\mu}_\alpha \; ; \tag{8}$$

here $\vec{\epsilon}\,'$ is now the unit polarization vector in the direction of the photon's magnetic field [$\vec{\epsilon}$ in (7) is in the direction of the electric field]. The intrinsic magnetic moment interaction term (8) is much smaller than the interaction term (7) for nonrelativistic particles. However, for the calculation of radiation processes involving neutral particles with magnetic moments the term (8) would be of prime importance. There is, of course, an additional interaction term associated with a two-photon vertex that is of higher order and is important in photon scattering.[7]

C. Total direct amplitude

The amplitude for the two-particle bremsstrahlung process is determined by the second-order perturbation Hamiltonian martix element (1). In the Born approximation the particle wave functions are plane waves and the sum over intermediate states in (1) yields momentum-conservation[8] (Dirac or Kronecker, as is convenient) δ functions. The scattering potential matrix elements are, of course, just the Fourier transforms of V with, however, slightly different arguments in the four terms in (1). At this stage let us assume that the Fourier transform satisfies the condition $V(\vec{K}) = V(-\vec{K})$ which is guaranteed if the interparticle scattering force is central. In simplifying the resulting expression for H'_{f0} it is convenient to make use of the smallness of the photon momentum $\vec{k}$ compared with the particle momenta (e.g., $\vec{k}_\alpha$). The energy denominator terms in H'_{f0} can then be expanded and the terms involving the above-mentioned Fourier transforms can be expanded to express them in terms of transforms with a common argument.[9]

Further, it is convenient to express results in terms of center-of-mass frame (c.m.) particle momentum variables[10] $\vec{k}_0$, $\vec{k}_f$, and c.m. photon momentum $\vec{k}$. We then find, after some elementary manipulations,

$$H'_{f0}=\frac{(k\Delta k\Delta\Omega)^{1/2}}{\pi}\,\frac{e}{m^2k^2}\,V(\vec{q})\left((\vec{\epsilon}\cdot\vec{k}_0)(\vec{k}\cdot\vec{k}_0)-(\vec{\epsilon}\cdot\vec{k}_f)(\vec{k}\cdot\vec{k}_f)-\frac{1}{2q}\,\frac{\partial \ln V}{\partial q}\,k_-^2(\vec{\epsilon}\cdot\vec{q})(\vec{k}\cdot\vec{q})\right), \tag{9}$$

where

$$\vec{q}=\vec{k}_0-\vec{k}_f \tag{10}$$

and

$$k_-^2=k_0^2-k_f^2=mk\ . \tag{11}$$

We see how the amplitude for the process is determined by the form of the scattering potential through the third term in large parentheses in (9).

When the scattering is by means of a Coulomb field, $V(\vec{q})=4\pi e^2/q^2$, and

$$H'_{f0}=\frac{4e^3}{m^2}\,\frac{(\Delta k\Delta\Omega)^{1/2}}{k^{3/2}}A'\ , \tag{12}$$

with

$$\begin{aligned}A'=q^{-2}[&(\vec{\epsilon}\cdot\vec{k}_0)(\vec{k}\cdot\vec{k}_0)-(\vec{\epsilon}\cdot\vec{k}_f)(\vec{k}\cdot\vec{k}_f)\\&+(k_-^2/q^2)(\vec{\epsilon}\cdot\vec{q})(\vec{k}\cdot\vec{q})]\ .\end{aligned} \tag{13}$$

The expression (13) is the fundamental direct amplitude for the process.

III. PHOTON-EMISSION PROBABILITY IN DISTINGUISHABLE PARTICLE SCATTERING

Let us consider briefly the scattering of two particles which have the same mass and charge but which are somehow distinguishable. This simplified problem illustrates some basic differences between the two-particle (quadrupole) and one-particle radiation problems. The neglect of exchange simplifies the problem because of the complicating effect of the third term in large parentheses in (9) and in (13). This is because the exchange amplitude is obtained from the direct amplitude H'_{f0} by (see Ref. 10) the replacement $\vec{k}_f\to-\vec{k}_f$ for which the momentum transfer $\vec{q}$ becomes $\vec{q}_e=\vec{k}_0+\vec{k}_f$. Thus the factor in large parentheses in (9) and in brackets in (13) associated with photon emission is not symmetric under exchange and, moreover, is dependent on the character of the scattering potential.

For photon emission in distinguishable particle scattering, however, one can obtain a simple expression for the emission probability defined by

$$\Delta w=\frac{1}{|V_{f0}|^2}\sum_{\vec{\epsilon},\vec{k}}|H_{f0}|^2\ , \tag{14}$$

where the sum is over photon polarization and momentum states. If $\vec{i}_\alpha$ and $\vec{i}_\beta$ represent particle momentum direction unit vectors and $\vec{i}$ represents the direction of the photon momentum $\vec{k}$, the polarization sums in the individual terms in (14) are of the form

$$\sum_{\vec{\epsilon}}(\vec{\epsilon}\cdot\vec{i}_\alpha)(\vec{\epsilon}\cdot\vec{i}_\beta)=\vec{i}_\alpha\cdot\vec{i}_\beta-(\vec{i}\cdot\vec{i}_\alpha)(\vec{i}\cdot\vec{i}_\beta)\ . \tag{15}$$

If we integrate over all angles of emission of the photon, the results

$$\int(\vec{i}\cdot\vec{i}_\alpha)(\vec{i}\cdot\vec{i}_\beta)d\Omega/4\pi=\tfrac{1}{3}\vec{i}_\alpha\cdot\vec{i}_\beta\ , \tag{16}$$

$$\int(\vec{i}\cdot\vec{i}_\alpha)^2(\vec{i}\cdot\vec{i}_\beta)^2d\Omega/4\pi=\tfrac{1}{15}+\tfrac{2}{15}(\vec{i}_\alpha\cdot\vec{i}_\beta)^2\ , \tag{17}$$

can be employed. For Coulomb scattering of distinguishable particles of charge ze we then obtain for the differential probability of emitting a (quadrupole) photon of momentum (or energy) within Δk:

$$\Delta w_q^{(\mathrm{Coul})}=(8\alpha/15\pi)z^2(\Delta k/k)[4(\beta_0^2-\beta_f^2)^2+3\beta_0^2\beta_f^2\sin^2\theta]; \tag{18}$$

in this expression $\alpha=e^2/\hbar c$, $\beta=v/c$, and θ is the scattering angle (β_0, β_f, and θ are c.m. frame variables).

Although not identified as such, the expression (18) appears as a factor in one of the equations in the paper by Lifshitz.[4] It is well to emphasize again, however, that the formula holds for the special case of scattering by a Coulomb field. In single-particle scattering (by, say, an "external" potential), on the other hand, in which the emission is dipole in nature, the corresponding formula for the photon-emission probability is more general and is independent of the nature of the scattering potential. Defined as in (14), one readily obtains by the methods outlined in the previous section, the well-known result

$$\Delta w_q = (2\alpha/3\pi)z^2(\Delta k/k)(\vec{\beta}_0 - \vec{\beta}_f)^2 \quad . \tag{19}$$

Classical derivations also yield the dipole formula (14), but, because photon recoil plays a more critical role in quadrupole emission, a purely classical approach[11] does not give the corresponding formula (18).

IV. BREMSSTRAHLUNG CROSS SECTION

In terms of c.m. variables the cross section can be written in terms of a sum[12] over polarizations and directions of emission of the outgoing photon and an integration over the momentum-transfer variable q:

$$\Delta\sigma = \frac{1}{8\pi k_0^2}\int \sum_{\vec{\epsilon},\vec{k}} |H'|^2 q\, dq \quad . \tag{20}$$

In the integration over q the limits are, for indistinguishable particles, $q_{\min} = k_0 - k_f$; $q_{\max} = (k_0^2 + k_f^2)^{1/2}$. The squared Hamitonian in (20) is obtained from the direct amplitude (12) but with a proper mixture of an exchange amplitude with a sign determined by the spin state and intrinsic spin of the charged particles involved.

Since spin coordinates are not involved in the process, we can speak of a direct amplitude a [see Eqs. (12) and (13)] and an exchange amplitude a_e in terms of momentum variables with

$$a_e = a(\vec{k}_f \to -\vec{k}_f) \quad . \tag{21}$$

The weighting of the (squared) total amplitudes $a + a_e$ and $a - a_e$ is determined by the number of spin states of the colliding particles having the corresponding required spin-exchange symmetry. For unpolarized incident particles one then easily obtains an intensity or properly symmetrized squared amplitude given by[13]

$$S = a^2 + a_e^2 + 2(2s+1)^{-1}(-1)^{2s} a a_e \,. \tag{22}$$

This expression holds for both bosons ($2s$ = even integer) and fermions ($2s$ = odd integer).

The cross-section (20) differential in the magnitude of the photon momentum or energy is obtained by summing over polarization [using (15)], integrating over angles of emission employing (16) and (17), and then integrating over the momentum-transfer variable and can be expressed in a general form for arbitrary charge, mass, and spin. In terms of the electron charge (e) and mass (m) and $\alpha = e^2/\hbar c$ and $\Lambda = \hbar/mc$ we obtain, for unpolarized incident particles,

$$\Delta\sigma = \frac{4z^6}{15\mu^4}\alpha^3\Lambda^2\frac{\Delta k}{k}S \,, \tag{23}$$

where z and μ are the charge and mass in units of e and m; S is given by (22) with the corresponding terms

$$a^2 + a_e^2 = 6(1+\xi^2)\ln\frac{1+\xi}{1-\xi} + 20\xi \,, \tag{24}$$

$$a a_e = \frac{7(1-\xi^4)^2 + 3(1-\xi^2)^4}{2(1+\xi^2)^3}\ln\frac{1+\xi}{1-\xi} + \frac{6\xi(1+\xi^4)}{(1+\xi^2)^2} \,, \tag{25}$$

with

$$\xi = k_f/k_0 \,. \tag{26}$$

For the special case $s = \frac{1}{2}$, $z = 1$, $\mu = 1$ the result agrees with the formula derived by Fediushin[5] from the semiclassical formulation of Lifshitz.[4]

V. GRAVITON BREMSSTRAHLUNG

From the basic result (23) for photon bremsstrahlung, one can readily obtain the cross section for the production of a graviton in Coulomb scattering. If we compare the classical formulas for quadrupole photon and graviton production,[14] we find that the graviton formula is obtained by simply replacing $(ze)^2$ by $4G(\mu m)^2$ in the photon formula, where G is the gravitational constant. The two formulas are very similar; in one case the total energy radiation rate is proportional to the sum of the squared third time derivatives ($\dddot{D}_{ij}$) of the electric quadrupole moments, while in the other case the gravitational quadrupole moment is involved. Now the present paper has, through a purely quantum-mechanical derivation, provided a justification for the semiclassical approach of Lifshitz[4] to the quadrupole photon problem which begins from the classical formula. Lifshitz replaces $\dddot{D}_{ij}$ by three commutations with the Hamiltonian (excluding the photon interaction part). If this works for the photon problem it should also yield the correct result for graviton bremsstrahlung. As I have emphasized in this paper, for nonrelativistic problems, photons (or gravitons) are always soft in that their momenta are small even when their energies are comparable to the particle (electrons, etc.) energies; basically this is why classical radiation formulas often give correct results. Thus, even though we have not given a field-theory derivation of graviton interactions, as was done in Sec. II B, we can with confidence infer the graviton bremsstrahlung cross section. Simply replacing $(ze)^2$ by $4G(\mu m)^2$ in (23) we have

$$\Delta\sigma_{\text{graviton}} = \frac{16z^4}{15\mu^2}\alpha_g\alpha^2\Lambda^2\frac{\Delta k}{k}S \,, \tag{27}$$

where S is again given by the expressions (22), (24), and (25), and $\alpha_g \equiv Gm^2/\hbar c$ is the gravitational fine-structure constant.[15]

The graviton bremsstrahlung formula (27) holds for scattering by identical particles of arbitrary mass, charge, and spin, and is valid for arbitrary graviton energy. It agrees in the soft-graviton limit with a result given by Weinberg[14] to "logarithmic accuracy" (where the cross section has a logarithmic factor whose argument is large but not determined accurately). Quadrupole emission is, of course, the lowest-order mechanism for gravitational wave (graviton) emission. The result (27) could, for example, be used to compute the graviton luminosity of the sun[14] as a result of e-e collisions. However, it would also be necessary to include contributions from graviton production in e-p and e-α collisions.

VI. APPLICATIONS

The most important application of the results given here is to the correction from e-e bremsstrahlung to the total emission rate per unit volume in a hot plasma. Most of this emission is due to electron-ion bremsstrahlung and the e-e contribution is of relative magnitude $\sim kT/mc^2$. It is thus of the same order as the relativistic corrections to electron-ion bremsstrahlung. Because of the application to the interpretation of emission from cosmic x-ray sources such as galaxy clusters ($T\sim 10^8$ K), results for a thermal averaging of e-e bremsstrahlung and relativistic corrections to electron-ion bremsstrahlung are given in Ref. 16.

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APPENDIX: MAGNETIC-MOMENT INTERACTIONS

The Hamiltonian associated with photon emission through interaction with a particle's intrinsic magnetic moment was introduced in Sec. II B. In the nonrelativistic limit the effects of this interaction are small compared with that associated with the interaction with the particle's electrical charge. Although this may not be immediately obvious for the case where the lowest-order emission is electric quadrupole, the relative unimportance of the magnetic-moment effects can be demonstrated without difficulty.

The charge and magnetic-moment interaction Hamiltonians for photon production by particles of type α are, in the Coulomb gauge,

$$H'_{\rm el} = -(e_\alpha/m_\alpha c)\vec{A}\cdot\vec{p}_\alpha\,, \tag{A1}$$

$$H'_{\rm mag} = \vec{\mu}_\alpha\cdot \operatorname{curl}\vec{A}\,, \tag{A2}$$

where $\vec{A}$ is the vector potential associated with the outgoing photon. But $|\operatorname{curl}\vec{A}|\sim k|\vec{A}|$, where $k\,(=\omega/c)$ is the photon wave vector, and $|\vec{\mu}_\alpha|\sim e_\alpha\hbar/m_\alpha c$ for any sensible particle. Thus, one readily obtains the result

$$H'_{\rm mag}/H'_{\rm el}\sim\hbar\omega/p_\alpha c \ll 1\,, \tag{A3}$$

since $\hbar\omega \lesssim p_\alpha^2/m_\alpha$. That is, the magnitude of the interaction due to intrinsic magnetic moments is much smaller than that associated with the electric charge.

However, one might question whether these higher-order effects might still be important in the problem treated in this paper, namely, bremsstrahlung in identical particle scattering. In quadrupole bremsstrahlung the principal terms in the total amplitude, while surviving in the dipole process, cancel and it is necessary to expand the expressions in the energy denominators and the scattering potential matrix elements. This is how the expression in large parentheses in (9) arises.

A cancellation also occurs in the total amplitude associated with photon interaction with particle magnetic moments, so that the inequality (A3) still determines the unimportance of the magnetic-moment interactions. This can be demonstrated most simply by considering the amplitude from, for example, the top two diagrams in Fig. 1, where now the photon-emission vertex is associated with the $\vec{\mu}_\alpha\cdot\operatorname{curl}\vec{A}$ perturbation. Let $\vec{\sigma}$ denote the particle spin operator, $|m\rangle$ the spin state, and $\vec{\epsilon}\,'$ the photon polarization unit vector in the direction of curl $\vec{A}$. The amplitude associated with the two diagrams has the form

$$A_{\rm I}+A_{\rm II}\propto\sum_{\rm I}V_{f{\rm I}}\langle m_f|m_{\rm I}\rangle\frac{1}{E_0-E_{\rm I}}\langle m_{\rm I}|\vec{\sigma}\cdot\vec{\epsilon}\,'|m_0\rangle + \sum_{\rm II}\langle m_f|\vec{\sigma}\cdot\vec{\epsilon}\,'|m_{\rm II}\rangle\frac{1}{E_0-E_{\rm II}}\langle m_{\rm II}|m_0\rangle V_{{\rm II}0}\,. \tag{A4}$$

Because of the orthogonality of the spin eigenfunctions both terms yield the factor $\langle m_f|\vec{\sigma}\cdot\vec{\epsilon}\,'|m_0\rangle$ and to lowest order the scattering potential matrix elements are the same in both terms. However, the energy denominators are different:

$$\begin{aligned} E_0-E_{\rm I}&=-\hbar\omega+\vec{p}_0\cdot\vec{p}/m\,,\\ E_0-E_{\rm II}&=\hbar\omega-\vec{p}_f\cdot\vec{p}/m\,, \end{aligned} \tag{A5}$$

where $\vec{p}$ is the photon momentum. The term $\hbar\omega$ is much larger than the others in the denominators and, as a result, since it appears in one with the opposite sign, the two principal terms in A_{I} and A_{II} cancel. A similar cancellation would occur in amplitudes (A_{III} and A_{IV}) associated with the bottom two diagrams of Fig. 1 when representing the $\vec{\mu}_{\alpha}\cdot \mathrm{curl}\,\vec{A}$ perturbation. In bremsstrahlung associated with the scattering of nonrelativistic identical charged particles, spin is important only in exchange effects and (23) is a general formula for arbitrary spin.

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[1]E. Haug, Z. Naturforsch. 30a, 1099 (1975). This paper gives references to most of the previous work on, in particular, the relativistic problem. Additional work, not mentioned by Haug, is that of V. L. Lyuboshitz, Zh. Eksp. Teor. Fiz. 37, 1727 (1959) [Sov. Phys.—JETP 10, 1221 (1960)] and is devoted to polarization phenomena; see also, H. A. Olsen, *Applications of Quantum Electrodynamics* (Springer, Berlin, 1968), Vol. 44, pp. 83–201.

[2]W. Heitler, *The Quantum Theory of Radiation*, 3rd ed. (Clarendon, Oxford, 1954); J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955); A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics* (Interscience, New York, 1965). The comparison of the e-e and e-p bremsstrahlung cross sections in the ultrarelativistic limit was clarified by V. N. Baier, V. S. Fadin, and V. A. Khoze, Zh. Eksp. Teor. Fiz. 51, 1135 (1966) [Sov. Phys.—JETP 24, 760 (1967)] and K. Mork, Phys. Rev. 160, 1065 (1967); see also the brief remarks by R. J. Gould, Phys. Rev. 185, 72 (1969).

[3]Some idea of the confusion regarding the nonrelativistic cross section may be gained from the brief article by R. C. Stabler, Nature 206, 922 (1965); this paper gives a number of references to previous erroneous results.

[4]E. M. Lifshitz, Zh. Eksp. Teor. Fiz. 18, 562 (1948).

[5]B. K. Fediushin, Zh. Eksp. Teor. Fiz. 22, 140 (1952).

[6]This result is a consequence of the massless-boson nature of the photon.

[7]It is interesting to note that in Compton scattering by free electrons in the nonrelativistic limit the perturbation associated with the two-photon vertex gives the whole contribution to the cross section; the second-order contribution from the perturbation associated with the single-photon vertex (acting twice) is negligible. This was noticed long ago by E. Fermi, Rev. Mod. Phys. 4, 87 (1932); of course, Fermi did not then describe the perturbation effects in terms of interaction vertices. In the relativistic formulation for spin-$\frac{1}{2}$ particles, on the other hand, the whole contribution is due to the second-order effect of the single-photon vertex interaction. The "vertex" description does not have a direct correspondence between the nonrelativistic and relativistic formulations.

[8]The problem could also be formulated easily in the momentum representation.

[9]The necessity of expanding factors (energy denominators and scattering potential (Fourier transforms) with the expansion parameter proportional to k indicates the important role played by the photon momentum. That is, the kinematics must be treated carefully to obtain the fundamental surviving term in the total amplitude for the process, and the photon momentum, although small, cannot be neglected. In single-particle (dipole) bremsstrahlung the photon momentum does not play such a critical role; for example, in that case the corresponding H'_{f0} has only a proportionality dependence on $V(\vec{q})$ and not an additional dependence on the form $(\partial \ln V/\partial q)$ as in the brackets of (4). An equivalent statement concerning the intricacies of the kinematic effects in quadrupole bremsstrahlung could attribute effects to "retardation."

[10]Here $(\vec{k}_1)_0=-(\vec{k}_2)_0\equiv\vec{k}_0$ and $(\vec{k}_1)_f\approx-(\vec{k}_2)_f\equiv k_f$.

[11]In a purely classical approach in the soft-photon limit the term in (13) with $\sin^2\theta$ is obtained, but a different coefficient $(\frac{5}{6})$ is found for the $(\beta_0^2-\beta_f^2)^2$ term. This latter term does, however, approach zero in the soft-photon limit, but in the evaluation of the cross section for Coulomb bremsstrahlung it gives a contribution (~ 1), additive to a dominating (~ 10) logarithmic term. Thus, even in the soft-photon limit, for quadrupole bremsstrahlung quantum mechanics must be inserted into the problem at an early stage. In dipole bremsstrahlung these effects are not as critical.

[12]If the polarization and angular distributions of the outgoing photon are required, one would, of course, not sum over these variables.

[13]This formula is obtained by writing (in the case at hand the amplitudes are not complex) $S=c(a+a_e)^2+(1-c)(a-a_e)^2$, where c is the fraction of two-particle spin states having the required spin-exchange symmetry to go with an even spatial coordinate total amplitude. For bosons, $c=(s+1)/(2s+1)$; for fermions, $c=s/(2s+1)$ [cf. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics Nonrelativistic Theory*, 2nd ed. (Addison-Wesley, Reading, Mass., 1965)].

[14]Cf. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1962); S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[15]I have used the notation α_g in order not to confuse it with the other gravitational fine structure $\alpha_G=GM^2/hc$ in which M is the nucleon mass.

[16]R. J. Gould, Astrophys. J. 238, 1026 (1980); 243 (1981) (in press).