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Interaction of a composite system with the quantized radiation field in an approximately relativistic theory

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We show that in order to satisfy the condition of relativistic invariance for the one-photon radiative transition amplitudes of a composite system to relative order $1/c^2$ in an approximately relativistic canonical theory, the position operator in the theory should satisfy a condition which encompasses the world-line condition of Currie, Jordan, and Sudarshan. We also obtain the expressions for the $E1$ and the $M1$ single-photon transition amplitudes of the composite system in the center of momentum frame, including the relativistic corrections of leading order. These expressions show that the relativistic corrections are qualitatively different for the electrically charged and neutral composite systems. They also depend, especially in the case of the $M1$ amplitudes, rather sensitively on the binding interaction among the constituent particles. There are also relativistic corrections which depend simultaneously on the recoil momentum of the composite system and on the use of the relativistic center-of-mass variables in the problem and which so far have not been considered in the literature.

I. INTRODUCTION

In this paper we like to consider the interaction of a composite system with the quantized radiation field within the framework of an approximate relativistic theory[1-8] (correct to relative order $1/c^2$) of the composite system. In this approximate relativistic theory all the operators involved are constructed out of the position ($\vec{r}_\mu$), the momentum ($\vec{p}_\mu$) and the spin ($\vec{s}_\mu$) variables of the N constituent particles making up the composite system. The constituent particles have charges e_μ and masses m_μ ($\mu = 1, 2, \ldots, N$). To be specific we also assume that all the constituent particles have spin $\frac{1}{2}$. The case when some or all the particles have no spin can be studied by setting the spin terms involving those particles equal to zero in the various expressions. Two of the characteristic features of our theory are that the position and momentum variables of the constituent particles satisfy the canonical commutation rules and that the wave functions representing the state of the composite system involve only one time, namely, the time of the observer. So the formalism is very much similar to that of the Schrödinger theory, except that now the theory is correct to a higher order in $1/c$. The basic idea[4] of the theory is to make the Hilbert space of the composite system a reducible representation of the Poincare group, correct to relative order $1/c^2$, by constructing the ten generators of the group, namely, the Hamiltonian, the total momentum operator, the total angular-momentum operator, and the Lorentz-boost operators in terms of the position, momentum, and spin variables of the system such that the Lie algebra of the Poincare group is satisfied among them to order $1/c^2$.

There are several advantages in considering this approximately relativistic theory which is correct only to relative order $1/c^2$. First of all the "no-interaction theorem" of Currie, Jordan, and Sudarshan[9] will not be valid in an approximately relativistic theory. According to this theorem, in an exactly relativistic classical-mechanical theory of N particles, there can be no interaction among the constituent particles provided the space-time coordinates of the events on the world line of a particle transform in the required manner under Lorentz transformations, and the position and the momentum variables are canonical. The condition that the points on the world line of a particle transform correctly under a Lorentz transformation gives rise to an equation[9] which involves the Poisson brackets of the position variable with the Lorentz-boost generator and the Hamiltonian. This equation is usually[9] referred to as the world-line condition. If we take it to be valid also in quantum mechanics (although there is no rigorous reason[9] for it) as an operator equation with the Poisson brackets replaced in the usual manner by the commutators, then the no-interaction theorem will also be valid in an exactly relativistic quantum-mechanical theory. On the other hand, in an approximately relativistic theory, the world-line condition and the canonical commutation rules are consistent with a nonzero internal interaction among the constituent particles. We consider this to be an attractive feature of the approximate theory. Another attractive feature of the present theory is that the interaction terms can be introduced into the Hamiltonian in such a way that the resulting formalism is consistent with the results of the relativistic quantum-field theory to order $1/c^2$.

For example, by expanding the scattering amplitude of two charged particles (corresponding to the one-photon-exchange diagrams) in powers of $1/c$, and keeping only terms of order up to $1/c^2$, then, by determining what kind of interaction terms in the Hamiltonian of our theory will reproduce this approximate scattering amplitude in the Born approximation, we can derive the Darwin-Breit Hamiltonian[10,11] of two charged particles. Since the theory is correct to relative order $1/c^2$, we can use it to derive the relativistic corrections of leading order. For example, the hyperfine-structure splitting of the energy levels of the positronium can be derived in this way.[12] We can also derive the Lamb shifts and other radiative effects by considering the appropriate Feynman diagrams and then deriving the corresponding effective interaction terms[12] in the Hamiltonian. Needless to say, the theory is also useful in situations where there is no underlying field theory.

In this paper we like to discuss the interaction of the composite system with the quantized radiation field within the framework of the above approximately relativistic theory. As we will see later this discussion is very useful in calculating the relativistic corrections of leading order to the electric- ($E1$) and magnetic-dipole ($M1$) one-photon transition amplitudes of the composite system. In the radiation or the Coulomb gauge the quantized radiation field can be represented by the transverse vector potential $\vec{A}(\vec{r},t)$. Moreover in the interaction picture this vector potential satisfies the free-wave equation so that it can be expanded in terms of the plane waves. The coefficients of expansion are proportional to the photon creation and annihilation operators. If we also impose the periodic boundary conditions in a box of volume V, the vector potential $\vec{A}(\vec{r},t)$ can be written as

$$\vec{A}(\vec{r},t)=\frac{c}{\sqrt{V}}\left(\frac{2\pi\hbar}{\omega}\right)^{1/2}\times\sum_{\vec{k},\alpha}\hat{\epsilon}_\alpha[a_{\vec{k},\alpha}(0)e^{i\vec{k}\cdot\vec{r}}e^{-i\omega t}+a^\dagger_{\vec{k},\alpha}e^{-i\vec{k}\cdot\vec{r}}e^{+i\omega t}]\,, \tag{1}$$

where

$$\hat{\mathcal{E}}_\alpha\cdot\vec{k}=0\ . \tag{2}$$

We will derive the Hamiltonian representing the interactions of the composite system with this quantized radiation field (correct to order $1/c^4$) by means of two principles: (1) The Schrödinger equation in the presence of a vector potential should be gauge invariant in the sense that the equations should retain the same mathematical form under the simultaneous replacements

$$\vec{A}(\vec{r}_\mu,t)\to\vec{A}'(\vec{r}_\mu,t)=\vec{A}(\vec{r}_\mu,t)+\vec{\nabla}_\mu\chi_\mu(\vec{r}_\mu,t) \tag{3}$$

and

$$\psi(\vec{r}_1,\vec{r}_2,\ldots,\vec{r}_N,t)\to\psi'(\vec{r}_1,\ldots,\vec{r}_N,t)=(\Pi_\mu e^{(ie_\mu/c)\chi_\mu(\vec{r}_\mu,t)})\psi(\vec{r}_1,\ldots,\vec{r}_N,t)\ . \tag{4}$$

In Eqs. (3) and (4), χ_μ ($\mu=1,2,\ldots,N$) represents a gauge function of the variables $\vec{r}_\mu$ and t, and $\vec{\nabla}_\mu$ represents the gradient operator with respect to the variable $\vec{r}_\mu$. (2) In the limit when the internal interaction among the constituent particles goes to zero the Hamiltonian of the composite system in the presence of the vector potential should be a simple sum of N terms, each term being the Foldy-Wouthuysen (FW) reduced (to order $1/c^4$) Hamiltonian of a single spin-$\frac{1}{2}$ particle in the presence of a vector potential. The first principle can be satisfied if we generate the Hamiltonian H' in the presence of the vector potential from the original Hamiltonian H (without the vector potential) by the minimal replacement,

$$\vec{p}_\mu\to\vec{p}_\mu-\frac{e_\mu}{c}\vec{A}_\mu\ ,$$

where

$$\vec{A}_\mu=\vec{A}(\vec{r}_\mu,t)\ . \tag{5}$$

That is,

$$H'=H\left(\vec{p}_\mu-\frac{e_\mu}{c}\vec{A}_\mu\right), \tag{6}$$

if $H=H(\vec{p}_\mu)$ and the interaction Hamiltonian is

$$H_I=H'-H\ . \tag{7}$$

If the particles have no spin, we find that the second principle is then automatically satisfied. If they have spin, there are some additional spin-dependent terms which involve the electric and magnetic fields and they cannot be obtained by the method of minimal replacement in the approximately relativistic Hamiltonian.[13] But these additional terms are all known from the known results[14] on the FW reduced (to order $1/c^4$) Hamiltonian of a single spin-$\frac{1}{2}$ particle in the presence of time-dependent external electromagnetic fields. Thus for the composite system made up of N spin-$\frac{1}{2}$ point particles (Dirac particles) the interaction Hamiltonian satisfying the two principles is

$$H_I = H\left(\vec{p}_\mu - \frac{e_\mu}{c}\vec{A}_\mu\right) - H(\vec{p}_\mu) - \sum_{\mu=1}^{N} \frac{e_\mu}{m_\mu c}\vec{s}_\mu \cdot \vec{B}_\mu + \sum_{\mu=1}^{N} \frac{e_\mu}{4m_\mu^2 c^2}\vec{s}_\mu \cdot (\vec{p}_\mu \times \vec{E}_\mu - \vec{E}_\mu \times \vec{p}_\mu) - i\sum_{\mu=1}^{N} \frac{e_\mu}{4m_\mu^2 c^2}\vec{s}_\mu \cdot (\vec{\nabla}_\mu \times \vec{E}_\mu) + \sum_{\mu=1}^{N} \frac{e_\mu}{4m_\mu^3 c^3}[\vec{p}_\mu^{\,2}, \vec{s}_\mu \cdot \vec{B}_\mu]_+ . \quad (8)$$

Since in this paper we are only interested in the one-photon transitions, in Eq. (8) we have kept only those spin-dependent terms which are linear in the fields. The fields $\vec{E}$ and $\vec{B}$ are related to the vector potential $\vec{A}$ by the equations

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

and

$$\vec{B} = \vec{\nabla} \times \vec{A} . \quad (9)$$

It should be mentioned that the spin-dependent terms in Eq. (8) are applicable only to spin-$\frac{1}{2}$ point constituent particles (Dirac particles) with no anomalous magnetic moment. If the constituent particles have anomalous magnetic moments there will be additional spin-dependent terms, some of which will involve the derivatives of the electric and magnetic fields. We would also like to emphasize that Eq. (8) is correct to order $1/c^4$ if the original Hamiltonian $H = H(\vec{p}_\mu)$ is given correct to order $1/c^2$. The reason for this is the following: Since H is expanded in powers of $1/c^2$, the next higher-order terms in H beyond those of order $1/c^2$ will be the terms of order $1/c^4$. Because we generate the interaction Hamiltonian H_I by the gauge-invariant method of minimal substitution, the terms of order $1/c^4$ in $H(\vec{p}_\mu)$ can give rise to only terms of order $1/c^5$ or higher in H_I. Also the spin-dependent terms in Eq. (8) are given correct to order $1/c^4$ by means of the FW transformations of the original gauge-invariant Dirac Hamiltonian of a single particle in the presence of an external electromagnetic (em) field. Since the lowest-order electric ($E1$) and magnetic-dipole ($M1$) one-photon transition amplitudes are due to terms of order $1/c$ and $1/c^2$, respectively, in the interaction Hamiltonian H_I, Eq. (8) should be capable of providing us with the relativistic corrections of leading order (of relative order $1/c^2$) to the $E1$ and the $M1$ amplitudes of the composite system.

It is important to notice that there are terms in H_I of Eq. (8) which depend on the internal or binding interaction among the constituent particles. This is due to the fact that the terms in $H(\vec{p}_\mu)$ representing the internal interaction depend, in general, on the constituent momenta ($\vec{p}_\mu$). As a a consequence, in general, H_I of Eq. (8) is not a simple sum of the FW reduced Hamiltonians of the individual particles. There will also be non-FW terms in H_I which depend directly on the internal interaction. For the purpose of later calculations it is better to rewrite Eq. (8) in another form. In Eq. (8) let us Taylor expand $H(\vec{p}_\mu - e_\mu \vec{A}_\mu/c)$ about the point $\vec{p}_\mu$. Then

$$H\left(\vec{p}_\mu - \frac{e_\mu}{c}\vec{A}_\mu\right) = H(\vec{p}_\mu) - \sum_{\mu=1}^{N} \frac{e_\mu}{2c}\left(A_\mu^i \frac{\partial}{\partial p_{\mu,i}} H + \frac{\partial H}{\partial p_{\mu,i}} A_\mu^i\right) + O(A_\mu^2) . \quad (10)$$

Next we notice that

$$\frac{\partial H}{\partial p_{\mu,i}} = -i[r_{\mu,i}, H] . \quad (11)$$

Therefore using Eqs. (10) and (11), Eq. (8) becomes

$$H_I = i\sum_{\mu=1}^{N} \frac{e_\mu}{2c}(\vec{A}_\mu \cdot [\vec{r}_\mu, H] + [\vec{r}_\mu, H] \cdot \vec{A}_\mu) - \sum_{\mu=1}^{N} \frac{e_\mu}{m_\mu c}\vec{s}_\mu \cdot \vec{B}_\mu + \sum_{\mu=1}^{N} \frac{e_\mu}{4m_\mu^2 c^2}\vec{s}_\mu \cdot [(\vec{p}_\mu \times \vec{E}_\mu) - (\vec{E}_\mu \times p_\mu)] - i\sum_{\mu=1}^{N} \frac{e_\mu}{4m_\mu^2 c^2}\vec{s}_\mu \cdot (\vec{\nabla}_\mu \times \vec{E}_\mu) + \sum_{\mu=1}^{N} \frac{e_\mu}{4m_\mu^3 c^3}[p_\mu^2, \vec{s}_\mu \cdot \vec{B}_\mu]_+ . \quad (12)$$

It should be emphasized that in Eq. (12) we have kept only those terms which are linear in the fields and in the potential and that Eq. (12) is correct to order $1/c^4$.

There are at least two reasons for believing that we are correct in dealing with the Hamiltonian of Eq. (12). First of all, Eq. (12) is consistent with the two principles we listed before, which are of course the necessary conditions for the correctness of any Hamiltonian. We may note that since the spin-dependent terms depend only on the $\vec{E}$ and $\vec{B}$ fields they obviously do not spoil the gauge-invariance property required by the first principle. Of course this is not at all surprising since the spin-dependent terms are obtained by the FW reduction of the original gauge-invariant Dirac Hamiltonian of a single spin-$\frac{1}{2}$

particle in the presence of a vector potential. Second, when we apply Eq. (12) or Eq. (8) to the special case of a two-particle composite system, bound by internal em forces, the results obtained coincide with those obtained from calculations[15,16] based more directly on quantum electrodynamics. But Eq. (12) has the virtue that it is also applicable to situations where there is no underlying well-established relativistic quantum-field theory.

Before applying Eq. (12) to specific examples, there is one more thing we should do to make sure that it is indeed correct relativistically to the required order. In the relativistically invariant quantum electrodynamics the interaction Lagrangian density and hence the interaction Hamiltonian density (since the interaction Lagrangian density does not involve derivatives of fields) is relativistically invariant. Hence the S operator is also relativistically invariant. So the S-matrix elements can be written as the product of an invariant matrix element and some kinematical factors involving the energies and the momenta of particles. From the structure of these kinematical factors, we can immediately infer that the S-matrix elements have the required Lorentz-transformation property. For example, it is a trivial matter to verify that the lifetimes calculated from the decay matrix elements in a relativistic quantum-field theory transform correctly under Lorentz transformations. We should note that this situation comes about because the formalism involved is manifestly covariant. On the contrary our approximate relativistic theory is, of course, not manifestly covariant. We know that our theory in the absence of an external em field is relativistically correct of relative order $1/c^2$, because we have constructed the ten generators of the Poincaré group in terms of the variables $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$ ($\mu = 1, 2, \ldots, N$), such that their commutators satisfy the Lie algebra of the Poincaré group to order $1/c^2$. With the introduction of the quantized radiation field, in order to verify the relativistic invariance of the theory to relative order $1/c^2$, we should again verify that the generators of the Poincaré group constructed out of the basic variables $\vec{r}_\mu$, $\vec{p}_\mu$, $\vec{s}_\mu$, and the vector potential $\vec{A}(r,t)$, satisfy the Lie algebra in the required approximation. Since this has not been accomplished we should explicitly demonstrate that the transition amplitudes calculated from the Hamiltonian of Eq. (12) has the required Lorentz-transformation properties. For this purpose we will split the one-photon transition amplitude calculated from Eq. (12) into two parts, the parity-odd part T_o, which is nonvanishing only when the initial and the final internal states of the composite system have opposite parities, and the parity-even part T_e which is nonzero only when the initial and the final states of the composite system have the same internal parities. The parity-odd (T_o) and parity-even (T_e) amplitudes defined in the above manner do not mix under Lorentz transformations because the operation of space reflection commutes with the operation of Lorentz boosts. When expanded in terms of multipoles the parity-odd part T_o will contain the electric-dipole amplitude (T_{E1}) and the parity-even part T_e will contain the magnetic-dipole amplitude (T_{M1}). We will explicitly demonstrate that the one-photon transition amplitudes T_o and T_e separately satisfy the required relativistic condition (to be derived later) on the one-photon transition amplitude to relative order $1/c^2$. This result gives us one more reason to believe that the interaction Hamiltonian of Eq. (12) will indeed correctly give us the relativistic corrections of leading order (of relative order $1/c^2$) to the $E1$ and $M1$ amplitudes. The above investigation is also important for another reason. The twin requirements of gauge invariance from which we have derived Eqs. (8) and (12) and the relativistic condition on the one-photon transition amplitude may put some interesting constraints on the structure of the theory, which, needless to say, are worth understanding. In fact we will find that in order to satisfy the relativistic condition on the one-photon amplitudes T_o and T_e, the position operator of the theory should satisfy the world-line condition of Currie, Jordan, and Sudarshan at least within the matrix elements taken between the eigenstates of the Hamiltonian with different energy eigenvalues. We think this is an interesting result especially because of the fact that the world-line condition need not be valid[9] in quantum mechanics, in contrast to the situation in classical mechanics,[9] where it has to be valid. As a result of our investigation, we also find that the interaction dependent part of the Lorentz-boost operator should satisfy certain constraint equations. We would like to emphasize that our results are applicable to any composite system made up of an arbitrary (but fixed) number of constituent spin-$\frac{1}{2}$ or spin-0 particles and bound by arbitrary internal interactions.

In the course of this investigation we also derive two interesting expressions [Eqs. (87) and (88)], which are valid to relative order $1/c^2$, for the one-photon electric- ($E1$) and magnetic-dipole ($M1$) transition amplitudes in the center-of-mass (c.m.) frame of the composite system. They can be used to calculate the relativistic corrections of relative order $1/c^2$ to the $E1$ and $M1$ transition amplitudes of any composite system bound by any kind of internal forces. These expressions will clearly show that the leading-order relativistic corrections are qualitatively different for the electrically charged and neutral composite systems. It is interesting to observe that there are relativistic correction terms which depend simultaneously on the recoil momentum of the composite system and on the use of the relativistic c.m. variables in the calculation. To the best of our knowledge these terms were not considered previously in the literature.[15-22] On the other hand, from our discussions to follow, it will become obvious that the use of the relativistic c.m. variables is absolutely necessary in a consistent approximate relativistic theory of the composite system to relative order $1/c^2$. There are many applications of our results—especially with regard to the relativistic $M1$ transitions.[21,22] They will be explored in future publications. The application of our expression for the $E1$ amplitude [Eq. (87)] to calculate the relativistic corrections to the electric-dipole transition rates of charmonium is discussed in another paper.[23]

The format of the rest of the paper is as follows: In Sec. II we describe the relativistic condition on the one-photon transition amplitude of the composite system. In Sec. III we investigate the conditions under which this relativistic condition can be satisfied for the parity-odd (T_o) and parity-even (T_e) amplitudes in the first-order time-dependent perturbation theory, and in Sec. IV we discuss the results of our investigation. In Sec. V we write the expressions for the electric- and magnetic-dipole one-photon transition amplitudes of the stationary composite system and point out some of their interesting features. We also mention some of the possible applications. Finally, in Sec. VI we make some concluding remarks.

II. RELATIVISTIC CONDITION ON THE ONE-PHOTON TRANSITION AMPLITUDE

Let us consider two Lorentz frames L and L', the second one moving with a relative velocity $-\vec{v}$ with respect to the first. The origins of the two frames coincide at $t=0$. To the observer O stationary in the first frame (L), the composite system is stationary, whereas to the observer O' (who is stationary in the second frame (L')], it is moving with a velocity $\vec{v}$, or with total momentum equal to $\vec{P}$. The observer O in L makes a set of measurements and finds that at $t=0$ and composite system is in state $|B\rangle$, an eigenstate of H, with no photon being present. Subsequently he makes another set of measurements and finds the probability of finding the composite system in another eigenstate $|A\rangle$ at time $t=t_0$, with the photon being emitted with its energy and momentum lying in the small interval d^3k, the average energy and momentum being ω and $\vec{k}$. We assume that t_0 is large enough so that $\omega \approx \omega_{BA} = E_B - E_A$. To the observer O', the time interval between the two sets of measurements is $t_0' = t_0/(1-v^2/c^2)^{1/2}$. Also, the corresponding sets of measurements conducted by O' will determine the probability of finding the composite system at $t' = t_0' = t_0/(1-v^2/c^2)^{1/2}$, in the energy eigenstate $|A'\rangle$, with the simultaneous presence of a photon of average energy and momentum ω' and $\vec{k}'$ with a spread given by d^3k', if originally at time $t = t' = 0$ the composite system was in the energy eigenstate $|B'\rangle$ with no photons being present. The primed quantities are obtained from the unprimed ones by a Lorentz transformation of relative velocity $-\vec{v}$. Since we take the states to be simultaneous eigenstates of H and $\vec{P}$ we can write

$$|B\rangle = |B\rangle_I \otimes |0\rangle_{\text{c.m.}} \tag{13}$$

and

$$|A\rangle = |A\rangle_I \otimes |-\vec{k}\rangle_{\text{c.m.}}, \tag{14}$$

where $|A\rangle_I$ and $|B\rangle_I$ are eigenstates of the internal Hamiltonian h related to the Hamiltonian H by the relation

$$H = (h^2 + c^2P^2)^{1/2}. \tag{15}$$

In Eqs. (13) and (14), $|0\rangle_{\text{c.m.}}$ and $|-\vec{k}\rangle_{\text{c.m.}}$ are eigenstates of the total momentum operator $\vec{P}$ with the eigenvalues zero and $-\vec{k}$, respectively. The momentum $-\vec{k}$ is the recoil momentum of the composite system in the state $|A\rangle$. Since the states $|A'\rangle$, $|B'\rangle$, and $|\vec{k}', \hat{\epsilon}_\alpha'\rangle_{\text{ph}}$ are obtained from the states $|A\rangle$, $|B\rangle$, and $|\vec{k}, \hat{\epsilon}_\alpha\rangle_{\text{ph}}$ by a Lorentz transformation, we can write,

$$|A'\rangle = e^{i\vec{K}\cdot\vec{u}}|A\rangle = |A\rangle_I \otimes |\vec{P} - \vec{k}'\rangle_{\text{c.m.}}, \tag{16}$$

$$|B'\rangle = e^{i\vec{K}\cdot\vec{u}}|B\rangle = |B\rangle_I \otimes |\vec{P}\rangle_{\text{c.m.}}, \tag{17}$$

and

$$|\vec{k}', \hat{\epsilon}_\alpha'\rangle_{\text{ph}} = e^{i\vec{K}_R\cdot\vec{u}}|\vec{k}, \hat{\epsilon}_\alpha\rangle_{\text{ph}}, \tag{18}$$

where $\vec{K}$ and $\vec{K}_R$ are the Lorentz-boost operators of the composite system and the free radiation field, respectively, and

$$u = \tanh^{-1}(v/c). \tag{19}$$

It is interesting to note that the internal states remain invariant under the Lorentz boosts because of the relativistic separation between the internal and the c.m. variables. There is not even the "Wigner spin rotation[24]" of the state $|A'\rangle$ since it can be neglected to relative order $1/c^2$. To relative order $1/c^2$ the energy, momentum, and polarization vectors of the photon in the two Lorentz frames are related by the equations,[25]

$$\omega' = \omega + \vec{v}\cdot\vec{k} + \tfrac{1}{2} v^2/c^2\omega\,, \tag{20}$$

$$\vec{k}' = \vec{k} + k\frac{\vec{v}}{c} + \frac{1}{2}\,\frac{(\vec{k}\cdot\vec{v})}{c^2}\,\vec{v}\,, \tag{21}$$

$$\hat{\epsilon}'_\alpha = \hat{\epsilon}_\alpha - \frac{1}{c}\,\hat{\epsilon}_\alpha\cdot\vec{v}\left[\left(1 - \frac{\vec{v}\cdot\hat{k}}{c}\right)\hat{k} + \frac{1}{2}\,\frac{\vec{v}}{c}\right]. \tag{22}$$

The probabilities measured by the observers O and O' should be the same since both of them are observing the same physical events but frum different Lorentz frames. If we define $T(t_o)$ to be the probability amplitude, according to observer O, of observing the composite system in state $|A\rangle$ with the simultaneous presence of a photon of energy ω, momentum $\vec{k}$, and polarization vector $\hat{\epsilon}_\alpha$, at time t_o, if the composite system were originally (at time $t=0$) in the state $|B\rangle$ with no photon being present, and T' $[t'_o = t_o/(1-v^2/c^2)^{1/2}]$ to be the corresponding probability amplitude according to observer O', we obtain

$$|T(t_o)|^2 d^3k = |T'(t'_o)|^2 d^3k'\,. \tag{23}$$

Since

$$d^3k' = \left(\frac{\omega'}{\omega}\right) d^3k\,, \tag{24}$$

we obtain from Eq. (23)

$$\left(\frac{\omega'}{\omega}\right)^{1/2} |T'(t'_o)| = |T(t_o)|\,. \tag{25}$$

Because of the continuous nature of the Lorentz group, the phases of the amplitudes can be chosen so that Eq. (25) can be rewritten as

$$\left(\frac{\omega'}{\omega}\right)^{1/2} T'(t'_o) = T(t_o)\,, \tag{26}$$

where

$$t'_o = t_o/(1-v^2/c^2)^{1/2}\,.$$

Equation (26) is the necessary and sufficient condition for the relativistic invariance of the one-photon transition amplitude. In the next section we will seek the conditions under which this equation will be satisfied for the composite system.

III. VERIFICATION OF THE RELATIVISTIC CONDITION FOR THE ONE-PHOTON TRANSITION AMPLITUDE

For reasons of clarity and for setting up the notation we will first quote the results of previous authors,[4-8,26] especially those of Krajcik and Foldy,[8] which will be pertinent to our calculations. To order $1/c^2$, we can write the Hamiltonian of the composite system as

$$H = \sum_{\mu=1}^{N}\left(\frac{p_\mu^2}{2m_\mu} - \frac{p_\mu^4}{8m_\mu^3 c^2}\right) + U^{(0)} + U^{(1)}\,, \tag{27}$$

where $U^{(0)}$ and $U^{(1)}$ represent the internal interaction terms of the zeroth and first order in $1/c^2$, respectively. The other generators of the Poincaré group, except for the Lorentz-boost operator $\vec{K}$, will have the simple additive or free-particle forms.[8] When expressed in terms of the c.m. variables the generators should have the single-particle forms.[8] The single-particle form of the Hamiltonian is given by Eq. (15). If we expand the internal Hamiltonian h in powers of $1/c^2$ as

$$h = Mc^2 + h^{(0)} + h^{(1)} + \cdots\,, \tag{28}$$

where $h^{(i)}$ is of the ith order in $1/c^2$, and Eq. (15) to order $1/c^2$, takes the form

$$H = Mc^2 + h^{(0)} + \frac{p^2}{2M} + h^{(1)}$$
$$- h^{(0)}\frac{p^2}{2M^2c^2} - \frac{p^4}{8M^3c^2}\,. \tag{29}$$

Krajcik and Foldy[8] have shown that in order to convert the constituent forms of the generators into the single-particle forms, correct to order $1/c^2$, the nonrelativistic relations between the constituent and the c.m. variables should be modified to[27]

$$\vec{r}_\mu = \vec{\rho}_\mu + \vec{R} - \frac{1}{2c^2}\left[\frac{\vec{\rho}_\mu\cdot\vec{p}}{M}\left(\frac{\vec{\pi}_\mu}{m_\mu} + \frac{\vec{P}}{2M}\right) + \text{H.c.}\right] - \frac{1}{2c^2}\sum_\nu\left(\frac{\pi_\nu^2\vec{\rho}_\nu}{2m_\nu M} + \text{H.c.}\right) + \sum_\nu \frac{(\vec{\rho}_\nu\times\vec{\pi}_\nu)\times\vec{P}}{2M^2c^2}$$
$$- \frac{\vec{\sigma}_\mu\times\vec{P}}{2m_\mu Mc^2} + \sum_\nu \frac{\vec{\sigma}_\nu\times\vec{\pi}_\nu}{2m_\nu Mc^2} + \sum_\nu \frac{\vec{\sigma}_\nu\times\vec{P}}{2M^2c^2} - \frac{1}{M}\,\vec{W}^{(1)} - \frac{i}{M}\left[\int_0^{\vec{P}} d\vec{p}\cdot\vec{W}^{(1)}, \vec{\rho}_\mu\right], \tag{30}$$

$$\vec{p}_\mu = \vec{\pi}_\mu + \frac{m_\mu}{M}\,\vec{P} + \left(\frac{\pi_\mu^2}{2m_\mu} - \frac{m_\mu}{M}\sum_\nu\frac{\pi_\nu^2}{2m_\nu} + \frac{\vec{\pi}_\mu\cdot\vec{P}}{2M}\right)\frac{\vec{P}}{Mc^2} - \frac{i}{M}\left[\int_0^{\vec{P}} d\vec{P}\cdot\vec{W}^{(1)}, \vec{\pi}_\mu\right], \tag{31}$$

$$\vec{s}_\mu = \vec{\sigma}_\mu - \frac{\vec{\sigma}_\mu\times(\vec{\pi}_\mu\times\vec{P})}{2m_\mu Mc^2} - \frac{i}{M}\left[\int_0^{\vec{P}} d\vec{P}\cdot\vec{W}^{(1)}, \vec{\sigma}_\mu\right]. \tag{32}$$

The internal variables in Eqs. (30)–(32) satisfy the following constraint equations[8]:

$$\sum_{\mu=1}^{N}\vec{\pi}_\mu=0\,,$$

$$\sum_{\mu=1}^{N}m_\mu\vec{\rho}_\mu=0\,. \tag{33}$$

In terms of the above internal variables, the internal Hamiltonians $h^{(0)}$ and $h^{(1)}$ take the forms[8]

$$h^{(0)}=\sum_{\mu=1}^{N}\frac{\pi_\mu^2}{2m_\mu}+U^{(0)}\,,$$

$$h^{(1)}=-\sum_{\mu=1}^{N}\frac{\pi_\mu^4}{8m_\mu^3c^2}+U^{(1)}\big|_{P=0}\,. \tag{34}$$

The operator $W^{(1)}$ entering into Eqs. (30)–(32) is an interaction dependent part[8] of the Lorentz-boost operator. Since $W^{(1)}$ will turn out to be quite important in our discussions we list here the conditions it should fulfill so that the Lie algebra of the Poincaré group is satisfied to order $1/c^2$. In order to satisfy the commutation rule

$$[K_i,H]=iP_i \tag{35}$$

to order $1/c^2$ we should have

$$[\vec{W}^{(1)},h^{(0)}]=-\frac{i}{Mc^2}U^{(0)}\vec{P}-iM\vec{\nabla}_pU^{(1)}-\frac{1}{2c^2}\sum_\mu\left[\left\{\vec{\rho}_\mu\left(\frac{\pi_\mu^2}{2m_\mu}+\frac{\vec{\pi}_\mu\cdot\vec{P}}{M}\right)+\text{H.c.}\right\},U^{(0)}\right]$$
$$+\frac{1}{2c^2}\sum_\mu\left[\frac{\vec{\sigma}_\mu\times\vec{\pi}_\mu}{m_\mu},U^{(0)}\right]+\frac{1}{2Mc^2}\sum_\mu[\vec{\sigma}_\mu\times\vec{P},U^{(0)}]\,. \tag{36}$$

From the commutation rule

$$[K_i,K_j]=-i\epsilon ijk\,J_k/c^2 \tag{37}$$

we also find that

$$[R_i,W_j^{(1)}]-[R_j,W_i^{(1)}]=0\,. \tag{38}$$

With these results in mind, we will now turn to the calculation of the one-photon transition amplitudes of the stationary and the moving composite systems. Using the interaction picture and the first-order perturbation theory we can write the one-photon transition amplitude of the stationary composite system as

$$T(t_0)=-i_{\text{ph}}\left\langle\vec{k},\hat{\epsilon}_\alpha\right|\otimes\left\langle A\right|\int_0^{t_0}H_I(t')\,dt'\left|B\right\rangle\otimes\left|0\right\rangle_{\text{ph}}\,. \tag{39}$$

Using Eqs. (12) and (1), Eq. (39) can be rewritten as

$$T(t_0)=\frac{1}{\sqrt{V}}\,c\left(\frac{2\pi}{\omega}\right)^{1/2}\left\langle A\right|\sum_{\mu=1}^{N}\frac{e_\mu}{2c}\left([\vec{r}_\mu,H]\cdot\hat{\epsilon}_\alpha e^{-i\vec{k}\cdot\vec{r}_\mu}+e^{-i\vec{k}\cdot\vec{r}_\mu}\hat{\epsilon}_\alpha[\vec{r}_\mu,H]\right)+\sum_{\mu=1}^{N}\frac{e_\mu}{m_\mu c}\,\vec{s}_\mu\cdot(\vec{k}\times\hat{\epsilon}_\alpha)e^{-i\vec{k}\cdot\vec{r}_\mu}$$
$$-k\sum_{\mu=1}^{N}\frac{e_\mu}{4m_\mu^2c^2}\,\vec{s}_\mu\cdot[(\vec{p}_\mu\times\hat{\epsilon}_\alpha)e^{-i\vec{k}\cdot\vec{r}_\mu}+e^{-\vec{k}\cdot\vec{r}_\mu}(\vec{p}_\mu\times\hat{\epsilon}_\alpha)]$$
$$+k\sum_{\mu=1}^{N}\frac{e_\mu}{4m_\mu^2c^2}\,\vec{s}_\mu\cdot(\vec{k}\times\hat{\epsilon}_\alpha)e^{-i\vec{k}\cdot\vec{r}_\mu}$$
$$-\sum_{\mu=1}^{N}\frac{e_\mu}{4m_\mu^3c^3}\,[p_\mu^2,\,\vec{s}_\mu\cdot(\vec{k}\times\hat{\epsilon}_\alpha)e^{-i\vec{k}\cdot\vec{r}_\mu}]_+\left|B\right\rangle\int_0^{t_0}e^{i(\omega-\omega_{BA})t'}dt'\,. \tag{40}$$

In order to verify the relativistic condition of Eq. (26) we must express all the constituent variables in Eq. (40) in terms of the internal and c.m. variables by means of Eqs. (13), (14), and (29)–(32). In the nonrelativistic limit the parity-even part of $T(t_0)$ is of relative order $1/c$ compared to the parity-odd part. So if we want to express both parts to relative order $1/c^2$ we have to express all the operators in the matrix element of Eq. (40) to relative order $1/c^3$. In doing this it is crucial to note that the photon momentum $\vec{k}$ is of order $1/c$ compared to the internal momentum $\vec{\pi}$. We also note that, to order $1/c^3$,

$$[\vec{r}_\mu, H] = [\vec{\rho}_\mu, h^{(0)} + h^{(1)}]\left(1 - \frac{P^2}{2M^2c^2}\right) + \frac{i}{M}\vec{P} - \frac{i}{2M^3c^2}P^2\vec{P} - \frac{i}{M^2c^2}h^{(0)}\vec{P}$$
$$+\left[-\frac{1}{2c^2}\sum_\nu\left(\frac{\pi_\nu^2\vec{\rho}_\nu}{2m_\nu M} + \text{H.c.}\right) + \sum_\nu \frac{\vec{\sigma}_\nu\times\vec{\pi}_\nu}{2m_\nu Mc^2} - \frac{1}{M}\vec{W}^{(1)}, h^{(0)}\right]$$
$$+\left[-\frac{1}{2c^2}\left\{\frac{\vec{\rho}_\mu\cdot\vec{P}}{M}\left(\frac{\vec{\pi}_\mu}{m_\mu} + \frac{\vec{P}}{2M}\right) + \text{H.c.}\right\} + \sum_\nu \frac{(\vec{\rho}_\nu\times\vec{\pi}_\nu)\times\vec{P}}{2M^2c^2}\right.$$
$$\left.-\frac{\vec{\sigma}_\mu\times\vec{P}}{2m_\mu Mc^2} + \sum_\nu \frac{\vec{\sigma}_\nu\times\vec{P}}{2M^2c^2} - \frac{i}{M}\left[\int_0^{\vec{P}} d\vec{P}\cdot\vec{W}^{(1)}, h^{(0)}\right], h^{(0)}\right] \tag{41}$$

and

$$e^{-i\vec{k}\cdot\vec{r}_\mu} \simeq e^{-i\vec{k}\cdot\vec{R}}\left(1 - i\vec{k}\cdot\vec{\rho}_\mu - \tfrac{1}{2}(\vec{k}\cdot\vec{\rho}_\mu)^2 + \frac{i}{6}(\vec{k}\cdot\vec{\rho}_\mu)^3\right.$$
$$+\frac{i}{2c^2}\vec{k}\cdot\left\{\left[\frac{\vec{\rho}_\mu\cdot\vec{P}}{M}\left(\frac{\vec{\pi}_\mu}{m_\mu} + \frac{\vec{P}}{2M}\right) + \text{H.c.}\right] + \sum_\nu\left(\frac{\pi_\nu^2\vec{\rho}_\nu}{2m_\nu M} + \text{H.c.}\right) - \sum_\nu \frac{(\vec{\rho}_\nu\times\vec{\pi}_\nu)\times\vec{P}}{M^2}\right.$$
$$\left.\left.-\sum_\nu \frac{\vec{\sigma}_\nu\times\vec{P}}{M^2} + \frac{\vec{\sigma}_\mu\times\vec{P}}{m_\mu M} - \sum_\nu \frac{\vec{\sigma}_\nu\times\vec{\pi}_\nu}{m_\nu M}\right\} + \frac{i}{M}\vec{k}\cdot\vec{W}^{(1)} - \frac{1}{M}\vec{k}\cdot\left[\int_0^{\vec{P}} d\vec{P}\cdot\vec{W}^{(1)}, \vec{\rho}_\mu\right]\right). \tag{42}$$

Also,

$$e^{-i\vec{k}\cdot\vec{R}}|0\rangle_{\text{c.m.}} = |-\vec{k}\rangle_{\text{c.m.}},$$
$${}_{\text{c.m.}}\langle-\vec{k}|e^{-i\vec{k}\cdot\vec{R}} = {}_{\text{c.m.}}\langle 0|. \tag{43}$$

Using Eqs. (13), (14), (29)–(32), and (41)–(43), Eq. (40) now takes the form

$$T(t_0) = \frac{1}{\sqrt{V}}\,c\left(\frac{2\pi}{\omega}\right)^{1/2} {}_I\!\left\langle A\right|\sum_{\mu=1}^N \frac{e_\mu}{2c}\left(\hat{\epsilon}_\alpha\cdot[\vec{\rho}_\mu, h^{(0)} + h^{(1)}]\right.$$
$$\times\left\{1 - i\vec{k}\cdot\vec{\rho}_\mu - \tfrac{1}{2}(\vec{k}\cdot\vec{\rho}_\mu)^2 + \frac{i}{6}(\vec{k}\cdot\vec{\rho}_\mu)^3\right.$$
$$\left.+\frac{i}{2c^2}\vec{k}\cdot\left[\sum_\nu\left(\frac{\pi_\nu^2\vec{\rho}_\nu}{2m_\nu M} + \text{H.c.}\right) - \sum_\nu \frac{\vec{\sigma}_\nu\times\vec{\pi}_\nu}{m_\nu M}\right] + \frac{i}{M}\vec{k}\cdot\vec{W}^{(1)}(0)\right\}$$
$$+\left\{1 - i\vec{k}\cdot\vec{\rho}_\mu - \tfrac{1}{2}(\vec{k}\cdot\vec{\rho}_\mu)^2 + \frac{i}{6}(\vec{k}\cdot\vec{\rho}_\mu)^3\right.$$
$$\left.+\frac{i}{2c^2}\vec{k}\cdot\left[\sum_\nu\left(\frac{\pi_\nu^2\vec{\rho}_\nu}{2m_\nu M} + \text{H.c.}\right) - \sum_\nu \frac{\vec{\sigma}_\nu\times\vec{\pi}_\nu}{m_\nu M}\right] + \frac{i}{M}\vec{k}\cdot\vec{W}^{(1)}(0)\right\}\hat{\epsilon}_\alpha\cdot[\vec{\rho}_\mu, h^{(0)} + h^{(1)}]$$
$$+\hat{\epsilon}_\alpha\cdot\left[-\frac{1}{2c^2}\sum_\nu\left(\frac{\pi_\nu^2\vec{\rho}_\nu}{2m_\nu M} + \text{H.c.}\right) + \sum_\nu \frac{\vec{\sigma}_\nu\times\vec{\pi}_\nu}{2m_\nu Mc^2} - \frac{1}{M}\vec{W}^{(1)}_{(0)}, h^{(0)}\right](1 - i\vec{k}\cdot\vec{\rho}_\mu)$$
$$\left.+(1 - i\vec{k}\cdot\vec{\rho}_\mu)\hat{\epsilon}_\alpha\cdot\left[-\frac{1}{2c^2}\sum_\nu\left(\frac{\pi_\nu^2\vec{\rho}_\nu}{2m_\nu M} + \text{H.c.}\right) + \sum_\nu \frac{\vec{\sigma}_\nu\times\vec{\pi}_\nu}{2m_\nu Mc^2} - \frac{1}{M}\vec{W}^{(1)}(0), h^{(0)}\right]\right)$$
$$+\sum_{\mu=1}^N \frac{e_\mu}{2c}\hat{\epsilon}_\alpha\cdot\left[\frac{1}{M}(\vec{k}\cdot\vec{\nabla}_P)\vec{W}^{(1)}\Big|_{P=0} + \frac{1}{2c^2}\left(\frac{(\vec{\rho}_\mu\cdot\vec{k})}{M}\frac{\vec{\pi}_\mu}{m_\mu} + \text{H.c.}\right) + \frac{i}{M}[\vec{k}\cdot\vec{W}^{(1)}_{(0)}, \vec{\rho}_\mu], h^{(0)}\right]$$
$$+\sum_{\mu=1}^N \frac{e_\mu}{m_\mu c}\vec{\sigma}_\mu\cdot(\vec{k}\times\hat{\epsilon}_\alpha)\left(1 + \frac{k}{2Mc} + \frac{k}{4m_\mu c} - \frac{\pi_\mu^2}{2m_\mu^2c^2} - i\vec{k}\cdot\vec{\rho}_\mu - \tfrac{1}{2}(\vec{k}\cdot\vec{\rho}_\mu)^2\right)$$
$$-k\sum_{\mu=1}^N \frac{e_\mu}{4m_\mu^2c^2}\vec{\sigma}_\mu\cdot[(\vec{\pi}_\mu\times\hat{\epsilon}_\alpha)(1 - i\vec{k}\cdot\vec{\rho}_\mu) + (1 - i\vec{k}\cdot\vec{\rho}_\mu)(\vec{\pi}_\mu\times\hat{\epsilon}_\alpha)]\Big|B\Big\rangle_I$$
$$\times\int_0^{t_0} e^{i(\omega-\omega_{BA})t'}dt'. \tag{44}$$

In arriving at Eq. (44) we have also made the assumption that we can Taylor expand $\vec{W}^{(1)}(P)$ about $\vec{P}=0$ so that

$$\vec{W}^{(1)}(-\vec{k}) \simeq \vec{W}^{(1)}(0) - (\vec{k}\cdot\vec{\nabla}_P)\vec{W}^{(1)}|_{P=0}$$

and (45)

$$\int_0^{\vec{P}} d\vec{P}\cdot\vec{W}^{(1)} \simeq -\vec{k}\cdot\vec{W}^{(1)}(0).$$

Next, as mentioned in Sec. I we split the transition amplitude $T(t_0)$ into two parts, the parity-odd part $T_0(t_0)$, which is nonvanishing only when the internal states $|A\rangle_I$ and $|B\rangle_I$ have opposite parities, and the parity-even part $T_e(t_0)$, which is nonvanishing only when the states $|A\rangle_I$ and $|B\rangle_I$ have the same parities. That is,

$$T(t_o) = T_o(t_o) + T_e(t_o)\,. \tag{46}$$

In order to achieve this splitting we write

$$\vec{W}^{(1)} = \vec{W}_e^{(1)} + \vec{W}_o^{(1)}, \tag{47}$$

where $\vec{W}_e^{(1)}$ and $\vec{W}_o^{(1)}$ are the parts of $\vec{W}^{(1)}$ which are, respectively, even and odd under the inversion of the internal coordinates. Even though $\vec{W}^{(1)}$ is an odd-parity operator with respect to the inversion of $\vec{r}_\mu$ or equivalently the combined inversion of the internal and the c.m. coordinates, we do not know, in general, its transformation property with respect to the inversion of the internal coordinates alone. Using Eqs. (34) and (36) we also note that

$$\left[-\frac{1}{2c^2}\sum_\nu\left(\frac{\pi_\nu^2}{2m_\nu M}\vec{\rho}_\nu+\vec{\rho}_\nu\frac{\pi_\nu^2}{2m_\nu M}\right)+\sum_\nu\frac{\vec{\sigma}_\nu\times\vec{\pi}_\nu}{2m_\nu Mc^2}-\frac{1}{M}\vec{W}^{(1)}(0),\,h^{(0)}\right]=i\vec{\nabla}_P U^{(1)}|_{P=0}-\frac{i}{2Mc^2}\sum_\mu^N\frac{\pi_\mu^2}{m_\mu^2}\vec{\pi}_\mu\,. \tag{48}$$

$\vec{\nabla}_P U^{(1)}|_{P=0}$ is necessarily an odd-parity operator with respect to the internal variables since $\vec{\nabla}_P U^{(1)}|_{P=0}$ cannot depend on $\vec{R}$ or $\vec{P}$. Therefore the right-hand side of Eq. (48) is an odd-parity operator with respect to the internal variables. Using Eqs. (47) and (48) we now find, to relative order $1/c^2$,

$$\begin{aligned}T_o(t_o)=\frac{1}{\sqrt{V}}\,c\left(\frac{2\pi}{\omega}\right)^{1/2}{}_I\Big\langle A\Big|&\sum_{\mu=1}^N\frac{e_\mu}{2c}\{\hat{\epsilon}_\alpha\cdot[\vec{\rho}_\mu,h^{(0)}+h^{(1)}][1-\tfrac{1}{2}(\vec{k}\cdot\vec{\rho}_\mu)^2]+[1-\tfrac{1}{2}(\vec{k}\cdot\vec{\rho}_\mu)^2]\hat{\epsilon}_\alpha\cdot[\vec{\rho}_\mu,h^{(0)}+h^{(1)}]\}\\&-i\sum_{\mu=1}^N\frac{e_\mu}{m_\mu c}\vec{\sigma}_\mu\cdot(\vec{k}\times\hat{\epsilon}_\alpha)(\vec{k}\cdot\vec{\rho}_\mu)-k\sum_{\mu=1}^N\frac{e_\mu}{2m_\mu^2c^2}\vec{\sigma}_\mu\cdot(\vec{\pi}_\mu\times\hat{\epsilon}_\alpha)\\&+i\frac{Q}{c}\hat{\epsilon}_\alpha\cdot\left(\vec{\nabla}_P U^{(1)}\Big|_{P=0}-\frac{1}{2Mc^2}\sum_{\mu=1}^N\frac{\pi_\mu^2}{m_\mu^2}\vec{\pi}_\mu\right)\Big|B\Big\rangle_I\int_0^{t_o}e^{i(\omega-\omega_{BA})t'}dt'\end{aligned} \tag{49}$$

and

$$\begin{aligned}T_e(t_o)=\frac{1}{\sqrt{V}}\,c\left(\frac{2\pi}{\omega}\right)^{1/2}{}_I\Big\langle A\Big|&\sum_{\mu=1}^N\frac{e_\mu}{2c}\hat{\epsilon}_\alpha\cdot[\vec{\rho}_\mu,h^{(0)}+h^{(1)}]\\&\times\left\{-i\vec{k}\cdot\vec{\rho}_\mu+\frac{i}{6}(\vec{k}\cdot\vec{\rho}_\mu)^2+\frac{i}{2c^2}\vec{k}\cdot\left[\sum_\nu\left(\frac{\pi_\nu^2\vec{\rho}_\nu}{2m_\nu M}+\text{H.c.}\right)-\sum_\nu\left(\frac{\vec{\sigma}_\nu\times\vec{\pi}_\nu}{m_\nu M}\right)\right]\right.\\&\left.+\frac{i}{M}\vec{k}\cdot\vec{W}^{(1)}(0)\right\}+\hat{\epsilon}_\alpha\cdot\left(\vec{\nabla}_P U^{(1)}\Big|_{P=0}-\frac{1}{2Mc^2}\sum_\nu\frac{\pi_\nu^2}{m_\nu^2}\vec{\pi}_\nu\right)(\vec{k}\cdot\vec{\rho}_\mu)+\text{H.c.}\\&+\sum_{\mu=1}^N\frac{e_\mu}{m_\mu c}\vec{\sigma}_\mu\cdot(\vec{k}\times\hat{\epsilon}_\alpha)\left(1+\frac{k}{2Mc}+\frac{k}{4m_\mu c}-\frac{\pi_\mu^2}{2m_\mu^2c^2}-\tfrac{1}{2}(\vec{k}\cdot\vec{\rho}_\mu)^2\right)\\&+ik\sum_{\mu=1}^N\frac{e_\mu}{4m_\mu^2c^2}\vec{\sigma}_\mu\cdot\left[(\vec{\pi}_\mu\times\hat{\epsilon}_\alpha)(\vec{k}\cdot\vec{\rho}_\mu)+(\vec{k}\cdot\vec{\rho}_\mu)(\vec{\pi}_\mu\times\hat{\epsilon}_\alpha)\right]\\&+k\sum_{\mu=1}^N\frac{e_\mu}{2}\hat{\epsilon}_\alpha\cdot\left[\frac{1}{2c^2}\left(\frac{(\vec{\rho}_\mu\cdot\vec{k})}{M}\frac{\vec{\pi}_\mu}{m_\mu}+\text{H.c.}\right)+\frac{i}{M}[\vec{k}\cdot\vec{W}_o^{(1)}(0),\vec{\rho}_\mu]\right]\\&+k\frac{Q}{2M}\hat{\epsilon}_\alpha\cdot[(\vec{k}\cdot\vec{\nabla}_P)\vec{W}^{(1)}|_{P=0}]\Big|B\Big\rangle_I\int_0^{t_o}e^{i(\omega-\omega_{BA})t'}dt'\,,\end{aligned} \tag{50}$$

where $Q=\sum_{\mu=1}^{N} e_\mu$ is the net charge of the composite system.

In order to verify the relativistic condition on the one-photon transition amplitude [Eq. (26)] we should next consider the analogous expressions for $T'(t_o')$ (the transition amplitude of the moving composite system as defined in Sec. II). In the first-order perturbation theory

$$T(t_o')=-i\ {}_{\rm ph}\Big\langle \vec{k}',\hat{\epsilon}_\alpha'\Big|\otimes\Big\langle A'\Big|\int_0^{t_o'} H_I(t')dt'\Big|B'\Big\rangle\otimes\Big|0\Big\rangle_{\rm ph}\ . \tag{51}$$

Using the expression for H_I given by Eq. (12) and the equation for the vector potential given by Eq. (1), Eq. (51) can be written in the form

$$\begin{aligned}\left(\frac{\omega'}{\omega}\right)^{1/2}T'(t_o')=\frac{1}{\sqrt{V}}\,c\left(\frac{2\pi}{\omega}\right)^{1/2}\Big\langle A'\Big|&\sum_{\mu=1}^{N}\frac{e_\mu}{2c}\,([\vec{r}_\mu,H]\cdot\hat{\epsilon}_\alpha' e^{-i\vec{k}'\cdot\vec{r}_\mu}+e^{-i\vec{k}'\cdot\vec{r}_\mu}\hat{\epsilon}_\alpha'\cdot[\vec{r}_\mu,H])\\&+\sum_{\mu=1}^{N}\frac{e_\mu}{m_\mu c}\,\vec{s}_\mu\cdot(\vec{k}'\times\hat{\epsilon}_\alpha')e^{-i\vec{k}'\cdot\vec{r}_\mu}\\&-k'\sum_{\mu=1}^{N}\frac{e_\mu}{4m_\mu^2c^2}\,\vec{s}_\mu\cdot[(\vec{p}_\mu\times\hat{\epsilon}_\alpha')e^{-i\vec{k}'\cdot\vec{r}_\mu}+e^{-i\vec{k}'\cdot\vec{r}_\mu}(\vec{p}_\mu\times\hat{\epsilon}_\alpha')]\\&+k'\sum_{\mu=1}^{N}\frac{e_\mu}{4m_\mu^2c^2}\,\vec{s}_\mu\cdot(\vec{k}'\times\hat{\epsilon}_\alpha')e^{-i\vec{k}'\cdot\vec{r}_\mu}\\&-\sum_{\mu=1}^{N}\frac{e_\mu}{4m_\mu^3c^3}\,[p_\mu^2,\vec{s}_\mu\cdot(\vec{k}'\times\hat{\epsilon}_\alpha')e^{-i\vec{k}'\cdot\vec{r}_\mu}]_+\Big|B'\Big\rangle\int_0^{t_o'}e^{i(\omega'-\omega_{BA}')t'}dt'\,.\end{aligned} \tag{52}$$

In order to compare $T'(t_o')$ with $T(t_o)$ we have to express the right-hand side of Eq. (52) in terms of the internal and the c.m. variables and also in terms of the unprimed photon variables $\hat{\epsilon}_\alpha$, $\vec{k}$, and ω. We do this by making use of Eqs. (16), (17), (20–22), (41–43) and the equation

$$\int_0^{t_o'}e^{i(\omega'-\omega_{BA}')t'}dt'=(1+\tfrac{1}{2}v^2/c^2)\int_0^{t_o}e^{i(\omega-\omega_{BA})t'}dt'\ . \tag{53}$$

After some straightforward but lengthy calculations and by splitting $T'(t_o')$ into the parity-odd and parity-even parts by the equation

$$T'(t_o')=T_o'(t_o')+T_e'(t_o')\ , \tag{54}$$

we finally find that

$$\begin{aligned}\left(\frac{\omega'}{\omega}\right)^{1/2}T_o'(t_o')=T_o(t_o)+\frac{1}{\sqrt{V}}c\left(\frac{2\pi}{\omega}\right)^{1/2}{}_{\rm c.m.}\Big\langle P\Big|\otimes{}_I\Big\langle A\Big|\sum_{\mu=1}^{N}\frac{e_\mu}{2c}\Big(&-\frac{2}{M}\hat{\epsilon}_\alpha\cdot[[\vec{W}_o^{(1)}(\vec{P})-\vec{W}_o^{(1)}(0)],h^{(0)}]\\&-\frac{2i}{M}\Big[\Big[\int_0^{P}d\vec{P}\cdot\vec{W}_e^{(1)},\vec{p}_\mu\Big],h^{(0)}\Big]\cdot\hat{\epsilon}_\alpha\Big)\Big|B\Big\rangle_I\otimes\Big|P\Big\rangle_{\rm c.m.}\end{aligned} \tag{55}$$

and

$$\begin{aligned}\left(\frac{\omega'}{\omega}\right)^{1/2}T_e'(t_o')=T_e(t_o)+\frac{1}{\sqrt{V}}\,c\left(\frac{2\pi}{\omega}\right)^{1/2}\\\times\Big\{{}_{\rm c.m.}\Big\langle P\Big|\otimes{}_I\Big\langle A\Big|\sum_{\mu=1}^{N}\frac{e_\mu}{2c}\Big\{&\left(-\frac{ik}{Mc}\right)\{(\vec{P}\cdot\vec{p}_\mu)\hat{\epsilon}_\alpha\cdot[\vec{p}_\mu,U^{(0)}]+\hat{\epsilon}_\alpha\cdot[\vec{p}_\mu,U^{(0)}](\vec{P}\cdot\vec{p}_\mu)\}\\&-\frac{2}{M}ck\hat{\epsilon}_\alpha\cdot[\vec{W}_e^{(1)}(P)-\vec{W}_e^{(1)}(0)]-\frac{2i}{M}ck\Big[\int_0^{\vec{P}}d\vec{P}\cdot\vec{W}_o^{(1)},\vec{p}_\mu\Big]\cdot\hat{\epsilon}_\alpha\Big\}\Big|B\Big\rangle_I\otimes\Big|P\Big\rangle_{\rm c.m.}\\+{}_{\rm c.m.}\Big\langle P\Big|\otimes{}_I\Big\langle A\Big|\sum_{\mu=1}^{N}\frac{e_\mu}{2c}\Big(\Big\{&\hat{\epsilon}_\alpha\cdot[\vec{p}_\mu,h^{(0)}]\Big(\frac{i}{M}\vec{k}\cdot[\vec{W}_o^{(1)}(P)-\vec{W}_{(o)}^{(1)}]-\frac{1}{M}\vec{k}\cdot\Big[\int_0^{\vec{P}}d\vec{P}\cdot\vec{W}_e^{(1)},\vec{p}_\mu\Big]\Big)\\&+\hat{\epsilon}_\alpha\cdot\Big[[\vec{W}_o^{(1)}(P)-\vec{W}_o^{(1)}(0)]+i\Big[\int_0^{\vec{P}}d\vec{P}\cdot\vec{W}_e^{(1)},\vec{p}_\mu\Big],h^{(0)}\Big]\frac{i}{M}(\vec{k}\cdot\vec{p}_\mu)+\text{H.c.}\Big\}\\&+\hat{\epsilon}_\alpha\cdot\Big[\frac{1}{M}(\vec{k}\cdot\vec{\nabla}_P)\vec{W}_e^{(1)}|,h^{(0)}\Big]\\&+\frac{i}{M}\hat{\epsilon}_\alpha\cdot[[\vec{k}\cdot[\vec{W}_o^{(1)}(P)-\vec{W}_o^{(1)}(0)],\vec{p}_\mu],h^{(0)}]\Big)\Big|B\Big\rangle_I\otimes\Big|P\Big\rangle_{\rm c.m.}\Big\}\,,\\\times\int_0^{t_o}e^{i(\omega-\omega_{BA})t'}dt'\,.\end{aligned} \tag{56}$$

In order that the parity-odd amplitude T_o satisfies the relativistic condition of Eq. (26) the second term on the right-hand side of Eq. (55) should vanish. This means that the matrix element in the second term should vanish for any two internal states whose energy eigenvalues differ and whose parities are opposite. Since the operator involved has odd parity, its matrix elements are automatically zero when the internal states have the same parities. Also since the operator involved is a commutator with $h^{(0)}$ and is of order $1/c^3$, the matrix element vanishes when the internal states have the same energy. Now, if we assume that the Hamiltonian is invariant under space reflection, the energy eigenstates should have a definite parity, either even or odd; so the matrix element involved should vanish for all eigenstates of the internal Hamiltonian h. Since the result should be true for arbitrary values of the charges e_μ, and the photon parameters $\hat{\epsilon}_\alpha$ and $\vec{k}$,

$$\left[[\vec{W}_o^{(1)}(P)-\vec{W}_o^{(1)}(0)]+i\left[\int_0^{\vec{P}} d\vec{P}\cdot\vec{W}_e^{(1)},\ \vec{\rho}_\mu\right],\ h^{(0)}\right]=0\ , \tag{57}$$

or equivalently,

$$-i[\vec{W}_o^{(1)}(P)-\vec{W}_o^{(1)}(0)]+\left[\int_0^{\vec{P}} d\vec{P}\cdot\vec{W}_e^{(1)},\ \vec{\rho}_\mu\right]=\vec{Z}_{o,\mu}\ , \tag{58}$$

where $\vec{Z}_{o,\mu}$ is a vector operator of odd parity with respect to the internal variables and

$$[\vec{Z}_{o,\mu},\ h^{(0)}]=0\ . \tag{59}$$

Differentiating Eq. (59) with respect to P_i, we obtain

$$-i\ \frac{\partial\vec{W}_o^{(1)}}{\partial P_i}+[W_{e,i}^{(1)},\vec{\rho}_\mu]=\frac{\partial\vec{Z}_{o,\mu}}{\partial P_i}\ . \tag{60}$$

In order that the relativistic condition should hold true for the parity-even amplitude T_e, the second term on the right-hand side of Eq. (56) should be zero. There are two matrix elements in the second term, the second matrix element being of relative order $1/c$ with respect to the first. So the two matrix elements must be separately zero. Since the matrix element has to be zero for any two internal eigenstates with different energy eigenvalues and since e_μ and $\hat{\epsilon}_\alpha$ are arbitrary, we obtain, from the vanishing of the first matrix element, the operator condition,

$$\frac{1}{2c^2}\ \{(\vec{P}\cdot\vec{\rho}_\mu)[\vec{\rho}_\mu,U^{(0)}]+[\vec{\rho}_\mu,\ U^{(0)}](\vec{P}\cdot\vec{\rho}_\mu)\}$$
$$-i[\vec{W}_e^{(1)}(P)-\vec{W}_e^{(1)}(0)]+\left[\int_0^{\vec{P}} d\vec{P}\cdot\vec{W}_o^{(1)},\vec{\rho}_\mu\right]=\vec{Z}_{e,\mu}\ , \tag{61}$$

where $\vec{Z}_{e,\mu}$ is a vector operator of even internal parity which comutes with the internal Hamiltonian $h^{(0)}$. Differentiating Eq. (61) with respect to P_i,

$$\frac{1}{2c^2}\,(\rho_{\mu,i}[\vec{\rho}_\mu,U^{(0)}]+[\vec{\rho}_\mu,U^{(0)}]\rho_{\mu,i})$$
$$-i\ \frac{\partial\vec{W}_e^{(1)}}{\partial P_i}+[W_{o,i}^{(1)},\vec{\rho}_\mu]=\frac{\partial\vec{Z}_{e,\mu}}{\partial P_i}\ . \tag{62}$$

In order to obtain the operator condition corresponding to the vanishing of the second matrix element we first notice that this matrix element can be simplified considerably by making use of Eqs. (57) and (58). After the simplification we will obtain the operator condition

$$(\rho_{\mu,i}Z_{o,\mu,j}+Z_{o,\mu,j}\rho_{\mu,i})-\frac{\partial}{\partial P_j}\ W_{e,i}^{(1)}(P)$$
$$+\frac{1}{i}[W_{o,j}^{(1)}(P)-W_{o,j}^{(1)}(0),\ \rho_{\mu,i}]=Q_{ij}\ , \tag{63}$$

where Q_{ij} is a tensor operator which commutes with $h^{(0)}$. Using Eq. (62), Eq. (63) can also be written in another form, namely,

$$(\rho_{\mu,i}Z_{o,\mu,j}+Z_{o,\mu,j}\rho_{\mu,i})+\frac{i}{2c^2}\ (\rho_{\mu,j}[\rho_{\mu,i}U^{(0)}]$$
$$+[\rho_{\mu,i},U^{(0)}]\rho_{\mu,j})+[W_{o,j}(0),\ \rho_{\mu,i}]=\eta_{ij}\ , \tag{64}$$

where η_{ij} is a tensor operator of even internal parity, which commutes with $h^{(0)}$.

IV. DISCUSSION OF THE RESULTS

In order to satisfy the relativistic condition of Eq. (26), Eqs. (60), (62), and (64) should hold true. Adding Eqs. (60) and (62) we obtain

$$\frac{1}{2c^2}\ (\rho_{\mu,i}[\vec{\rho}_\mu,U^{(0)}]+[\vec{\rho}_\mu,U^{(0)}]\rho_{\mu,i})$$
$$-i\ \frac{\partial\vec{W}^{(1)}}{\partial P_i}+[W_i^{(1)},\vec{\rho}_\mu]=\frac{\partial\vec{Z}_\mu}{\partial P_i}\ , \tag{65}$$

where $\vec{Z}_\mu$ ($\vec{Z}_\mu=\vec{Z}_{e,\mu}+\vec{Z}_{o,\mu}$) is an operator which commutes with $h^{(0)}$. In arriving at Eq. (65) we have made use of Eq. (47).

We will now show that Eq. (65) is very closely related to the classical world-line condition of Currie, Jordan, and Sudarshan.[9] In fact, if we choose the right-hand side $\partial\vec{Z}_\mu/\partial P_i$ equal to zero (which we can), Eq. (65) is the same as the world-line condition. In order to show this let us first describe the classical world-line condition. Currie, Jordan, and Sudarshan[9] have shown that in a classical mechanical Hamiltonian theory, if the space-time coordinates on the classical world line of a particle transformed in the usual manner under Lorentz transformations, the position variable in the theory should satisfy the following equation:

$$[r_{\mu,j},K_i]=\frac{1}{c^2}\, r_{\mu,i}[r_{\mu,j},H]\ . \tag{66}$$

In Eq. (66), the square brackets denote the Poisson brackets and $r_{\mu,i}$ is the ith component of the position variable of the μth particle. Using Eq. (66) they have then shown[9] that in an exactly relativistic classical mechanical theory of two particles there can be no interaction. Lewtwyler[28] has generalized it to the general case of any finite number (N) of particles. In a quantum-mechanical theory the value of the position of the particle at a particular time has to be replaced by the expectation value of the position at that time. If we then insist that the expectation value as a function of time should transform in the expected manner under Lorentz transformations we cannot derive Eq. (66) with the Poisson brackets replaced by the corresponding commutators. But we can take another point of view that we can always get the corresponding quantum-mechanical equation from a classical equation if we replace the Poisson brackets and the products of dynamical variables in the latter by the commutators and the symmetrized products of operators, respectively, in the former. Then, in a quantum-mechanical theory of spinless particles (remember that the classical particles are necessarily spinless) we should have

$$[r_{\mu,j},\ K_i]=\frac{1}{2c^2}\ (r_{\mu,i}[r_{\mu,j},H]+[r_{\mu,j},H]r_{\mu,i}), \tag{67}$$

where the square brackets now represent the commutators. For particles with spin there could be additional spin-dependent terms on the right-hand side of Eq. (67). Coester and Havas[29] have derived the classical world-line condition for particles with spin in quantum mechanics by demanding that the world-line condition required (by definition) that the commutation relations of the individual particle positions $\vec{r}_\mu$ with the Lorentz generator $\vec{K}$ be the same with or without the interaction. They thus obtain

$$[r_{\mu,j},K_i]=\frac{1}{2c^2}\ (r_{\mu,i}[r_{\mu,j},H]+[r_{\mu,j},H]r_{\mu,i}) + i\ \frac{\epsilon_{ijk}\sigma_{\mu,k}}{H_\mu+m_\mu c^2}\ +[r_{\mu,j},\ H]\ \frac{(\vec{\sigma}_\mu\times\vec{P}_\mu)_i}{(H_\mu+m_\mu c^2)^2}. \tag{68}$$

Equations (67) or (68) now demand that in an exactly relativistic quantum-mechanical theory of a system of N particles there can be no interaction among the particles. The proof will be exactly similar to the ones given by Currie, Jordan, and Sudarshan[9] and by Lewtwyler.[28] But this proof will not go through for an approximately relativistic theory which is relativistic only up to a particular order in $1/c$. So in approximately relativistic theories the world-line condition and nonzero interaction are mutually compatible.

Next we define

$$H=H_f+U \tag{69}$$

and

$$\vec{K}=\vec{K}_f+\vec{V}\ , \tag{70}$$

where H_f and $\vec{K}_f$ are the forms of H and $\vec{K}$ when there is no interaction among the particles (free-particles). If we now substitute Eqs. (69) and (70) in Eq. (68) and then make use of the fact that for free-particles, Eq. (68) is exactly satisfied with K_i and H replaced by K_i^f and H_f, we obtain

$$[r_{\mu,j},V_i]=\frac{1}{2c^2}(r_{\mu,i}[r_{\mu,j},U]+[r_{\mu,j},U]r_{\mu,i}) + \frac{[r_{\mu,j},U](\vec{\sigma}_\mu\times\vec{p}_\mu)_i}{(H_\mu+m_\mu c^2)^2}\ . \tag{71}$$

To relative order $1/c^2$, Eq. (71) becomes

$$[r_{\mu,j},V_i^{(1)}]\simeq\frac{1}{2c^2}(r_{\mu,i}[r_{\mu,j},U^{(0)}]+[r_{\mu,j},U^{(0)}]r_{\mu,i})\ , \tag{72}$$

where $\vec{V}^{(1)}$ is the term of order $1/c^2$ in an expansion of $\vec{V}$ in powers of $1/c^2$. Krajcik and Foldy[8] have shown that

$$\vec{V}^{(1)}=\frac{1}{2c^2}(\vec{R}U^{(0)}+U^{(0)}\vec{R})+\vec{W}^{(1)}\ . \tag{73}$$

Since Eq. (72) is to be satisfied only to order $1/c^2$, and since $\vec{V}^{(1)}$ is already of order $1/c^2$, $\vec{r}_\mu$ which occurs on both sides of Eq. (72), can be replaced by its nonrelativistic expression, namely,

$$\vec{r}_\mu\simeq\vec{\rho}_\mu+\vec{R}. \tag{74}$$

Substituting Eqs. (73) and (74) in Eq. (72) we then obtain

$$[R_j+\rho_{\mu,j},W_i^{(1)}]\simeq\frac{1}{2c^2}(\rho_{\mu,i}[\rho_{\mu,j},U^{(0)}]+[\rho_{\mu,j},U^{(0)}]\rho_{\mu,i})\ . \tag{75}$$

We see that Eq. (75) is the same as Eq. (65) if we choose $\partial\vec{Z}_\mu/\partial P_i=0$ and when we recognize Eq. (38) and the result that

$$[W_j^{(1)},R_i]=-i\ \frac{\partial W_j^{(1)}}{\partial P_i}. \tag{76}$$

We are free to make the choice $\partial\vec{Z}_\mu/\partial P_i=0$, since the only requirement on $\vec{Z}_\mu$ is that it commutes with $h^{(0)}$. In other words, if Eq. (75) is satisfied, the relativistic condition on the one-photon transition amplitude is also satisfied although the relativistic condition on the one-photon amplitude does not necessarily require the validity of Eq. (75), but only the validity of the weaker condition given by Eq. (65).

The result that the twin requirements of the relativistic invariance of the one-photon transition amplitude and of the validity of the Hamiltonian of Eq. (12) lead to Eq. (65) which encompasses the classical world-line condition, is quite interesting because in general[9] the world-line condition need not be valid in quantum-mechanical theories. We see that the theory based on the Hamiltonian of Eq. (12) has several appealing features, the gauge invariance in the sense defined in the introduction, the relativistic invariance to relative order $1/c^2$, and the consistency of these conditions with the classical world-line condition to order $1/c^2$.

Coester and Havas[29] have shown that if the classical world-line condition is satisfied to order $1/c^2$, the nonrelativistic limit of the potential is either independent of the internal momentum or if it depends on the internal momentum it should be of the form

$$U^{(0)}=f_1(\vec{\rho}_\nu,\vec{\sigma}_\nu)+\vec{\mathrm{I}}\cdot\vec{\mathcal{J}}f_2(\vec{\rho}_\nu,\vec{\sigma}_\nu)\,, \tag{77}$$

where

$$\vec{\mathrm{I}}=\sum_\mu(\vec{\rho}_\mu\times\vec{\pi}_\mu) \tag{78}$$

and

$$\vec{\mathcal{J}}=\sum_\mu\vec{\sigma}_\mu\,. \tag{79}$$

We will now show that Eqs. (65) and (36) determine the interaction dependent term in the relativistic relation of Eq. (31) between the constituent momentum variable $\vec{\mathrm{p}}_\mu$ and the c.m. variables, at least in the special case when $U^{(0)}$ is independent of momenta. From Eq. (65) we can get a more useful equation (which will have no reference to the unknown quantity $\vec{\mathrm{Z}}_\mu$) by taking the commutator of both sides of Eq. (65) with $h^{(0)}$. Using Eqs. (76) and (38) we then obtain the equation

$$[h^{(0)},[W_i^{(1)},R_j+\rho_{\mu,j}]]=0\,, \tag{80}$$

provided $U^{(0)}$ commutes with $\vec{\rho}_\mu$. Using the Jacobi identity and the equation

$$[R_j+\rho_{\mu,j},h^{(0)}]=i\frac{\pi_{\mu,j}}{m_\mu}\,, \tag{81}$$

we then get from Eq. (80),

$$[W_i^{(1)},i\pi_{\mu,j}/m_\mu]=[R_j+\rho_{\mu,j},[W_i^{(1)},h^{(0)}]]\,. \tag{82}$$

As a result of Eq. (36), after a simple integration, Eq. (82) then leads to

$$-\frac{i}{M}\left[\int_0^{\vec{P}}d\vec{\mathrm{P}}\cdot\vec{\mathrm{W}}^{(1)},\vec{\pi}_\mu\right]=-\frac{m_\mu}{M^2c^2}U^{(0)}\vec{\mathrm{P}}+im_\mu[\vec{\rho}_\mu+\vec{\mathrm{R}},U^{(1)}(P)-U^{(1)}(0)]$$
$$+\frac{i}{2Mc^2}[[(\vec{\rho}_\mu\cdot\vec{\mathrm{P}})\vec{\pi}_\mu+\vec{\pi}_\mu(\vec{\rho}_\mu\cdot\vec{\mathrm{P}})]+\vec{\sigma}_\mu\times\vec{\mathrm{P}},U^{(0)}]\,. \tag{83}$$

Equation (83) gives the interaction dependent term in the relativistic relation of Eq. (31).

V. ONE-PHOTON $E1$ AND $M1$ TRANSITION AMPLITUDES IN THE CENTER OF MOMENTUM FRAME OF THE COMPOSITE SYSTEM

The parity-odd and parity-even amplitudes T_0 and T_e of the stationary composite system given by Eqs. (49) and (50) are not of pure multipole types. When expanded in terms of multipole fields, $T_0(t_0)$ contains the electric-dipole ($E1$) amplitude $T_{E1}(t_0)$ in addition to higher multipoles of odd parity, whereas $T_e(t_0)$ contains the magnetic-dipole ($M1$) amplitude $T_{M1}(t_0)$ in addition to other higher multipoles of even parity. But we can extract the $E1$ part from $T_0(t_0)$ and the $M1$ part from $T_e(t_0)$ by noting that the $E1$ and $M1$ transition operators are vector operators of odd and even parities, respectively, in the Hilbert space of the internal states of the composite system. After some algebra we obtain the expressions below for the $E1$ and the $M1$ one-photon transition amplitudes.

A. $E1$ transition amplitude

$$T_{E1}(t_0)=\frac{1}{\sqrt{V}}c\left(\frac{2\pi}{\omega}\right)^{1/2}k\hat{\epsilon}_\alpha\cdot{}_I\langle A|\vec{\mathrm{X}}_0+\vec{\mathrm{X}}_1|B\rangle_I\cdot\int_0^{t_0}e^{i(\omega-\omega_{BA})t'}dt'\,, \tag{84a}$$

where

$$\vec{\mathrm{X}}_0=\vec{\mathrm{D}}=\sum_{\mu=1}^N e_\mu\vec{\rho}_\mu \tag{84b}$$

and

$$\vec{X}_1 = -\frac{ik}{20}\sum_{\mu=1}^{N}\frac{e_\mu}{m_\mu c}\{2(\rho_\mu^2\vec{\pi}_\mu+\vec{\pi}_\mu\rho_\mu^2)-[\vec{\rho}_\mu(\vec{\rho}_\mu\cdot\vec{\pi}_\mu)+(\vec{\pi}_\mu\cdot\vec{\rho}_\mu)\vec{\rho}_\mu]\}+i\frac{Q}{\omega_{BA}}\left(\vec{\nabla}_P U^{(1)}|_{P=0}-\frac{1}{2Mc^2}\sum_{\nu=1}^{N}\frac{\pi_\nu^2}{m_\nu^2}\vec{\pi}_\nu\right). \quad (84c)$$

The operator $\vec{X}_1$ gives the relativistic corrections of relative order $1/c^2$ to the nonrelativistic $E1$ transition operator $\vec{X}_0$. The last term in the expression for $\vec{X}_1$ of Eq. (84c) comes specifically from the use of the relativistic c.m. variables. This term depends explicitly on the internal interaction $U^{(1)}$ and vanishes when the net charge Q of the composite system is zero. So the relativistic corrections to the $E1$ transition amplitudes are qualitatively different for the electrically charged and neutral composite systems. From Eq. (84c) we also find that the relativistic corrections of order $1/c^2$ to the $E1$ transition operator are spin independent except for any possible spin dependence of $U^{(1)}$.

B. $M1$ transition amplitude

The $M1$ transition amplitude, extracted out of Eq. (50), is

$$T_{M1}(t_0)=\frac{1}{\sqrt{V}}c\left(\frac{2\pi}{\omega}\right)^{1/2}k(\hat{k}\times\hat{\epsilon}_\alpha)\cdot{}_I\langle A|\vec{Y}_0+\vec{Y}_1|B\rangle_I\cdot\int_0^{t_0}e^{i(\omega-\omega_{BA})t'}dt', \quad (85a)$$

where

$$\vec{Y}_0=\vec{\mu}=\sum_{\nu=1}^{N}\frac{e_\nu}{2m_\nu c}(\vec{\rho}_\nu\times\vec{\pi}_\nu+2\vec{\sigma}_\nu) \quad (85b)$$

and

$$\begin{aligned}\vec{Y}_1=&\sum_{\mu=1}^{N}\frac{e_\mu}{4m_\mu c}\Bigg[\vec{\sigma}_\mu\left(\frac{k}{m_\mu c}-\frac{2\pi_\mu^2}{m_\mu^2c^2}\right)+im_\mu([\vec{\rho}_\mu,h^{(1)}]\times\vec{\rho}_\mu-\vec{\rho}_\mu\times[\vec{\rho}_\mu,h^{(1)}])\\&+\frac{ik}{2m_\mu c}[\vec{\rho}_\mu\times(\vec{\sigma}_\mu\times\vec{\pi}_\mu)-(\vec{\sigma}_\mu\times\vec{\pi}_\mu)\times\vec{\rho}_\mu]\\&-\frac{k^2}{5}(\rho_\mu^2[(\vec{\rho}_\mu\times\vec{\pi}_\mu)+4\vec{\sigma}_\mu]-\{\vec{\rho}_\mu[\vec{\rho}_\mu\cdot(\tfrac{1}{6}\vec{\rho}_\mu\times\vec{\pi}_\mu+\vec{\sigma}_\mu)]+[(\tfrac{1}{6}\vec{\rho}_\mu\times\vec{\pi}_\mu+\vec{\sigma}_\mu)\cdot\vec{\rho}_\mu]\vec{\rho}_\mu\})\Bigg]\\&-\sum_{\mu=1}^{N}\frac{e_\mu}{4m_\mu c^3}\left[\sum_\nu\left(\frac{\pi_\nu^2\vec{\rho}_\nu}{2m_\nu M}+\frac{\vec{\rho}_\nu\pi_\nu^2}{2m_\nu M}\right)-\sum_\nu\frac{\vec{\sigma}_\nu\times\vec{\pi}_\nu}{m_\nu M}\right]\times\vec{\pi}_\nu+\frac{1}{4c}\Bigg[\vec{D}\times\left(\vec{\nabla}_P U^{(1)}|_{P=0}-\frac{1}{2Mc^2}\sum_\nu\frac{\pi_\nu^2}{m_\nu^2}\vec{\pi}_\nu\right)\\&-\left(\vec{\nabla}_P U^{(1)}|_{P=0}-\frac{1}{2Mc^2}\sum_\nu\frac{\pi_\nu^2}{m_\nu^2}\vec{\pi}_\nu\right)\times\vec{D}\Bigg]\\&+\frac{k}{2Mc}\vec{\mu}+\frac{ik}{4M}(\vec{W}_o^{(1)}\times\vec{D}+\vec{D}\times\vec{W}_o^{(1)})+\frac{i}{4Mc}\{\vec{W}_o^{(1)}(0)\times[\vec{D},h^{(0)}]-[\vec{D},h^{(0)}]\times\vec{W}_o^{(1)}(0)\}. \end{aligned} \quad (85c)$$

The operator $\vec{Y}_1$ gives the relativistic corrections of relative order $1/c^2$ to the nonrelativistic $M1$ transition operator $\vec{Y}_o$. In the expression for $\vec{Y}_1$ of Eq. (85c) the second and third terms would have vanished had we neglected the $1/c^2$ terms in the relativistic relation of Eq. (30) between $\vec{r}_\mu$ and the c.m. variables. The last three terms on the right-hand side of Eq. (85c) would have vanished if we had neglected either the recoil of the composite system or the $1/c^2$ terms in Eqs. (30)–(32). The first term in Eq. (85c) would have survived even if we had used the nonrelativistic c.m. variables. We should also note that the leading relativistic corrections to the $M1$ amplitude depend on the internal interaction among the constituent particles even when the net electric charge of the composite system is zero. However, most of the interaction dependent relativistic corrections vanish when the electric-dipole moment $\vec{D}$ of the composite system vanishes. The correction terms, arising specifically due to the use of the relativistic c.m. variables and the recoil momentum of the composite system, are especially interesting since, to the best of our knowledge, they were neglected in the literature so far. But these corrections become negligible when any one of the constituent masses is large compared to all the others. In fact, the new corrections are of order m/M compared to the old ones. So they can be neglected for electronic atoms. But they are crucial for composite systems such as positronium, charmonium, deuteron, and mesic atoms.

The $E1$ and the $M1$ decay rates coming from Eqs. (84) and (85) are given by

$$W_{BA}=\tfrac{4}{3}k_o^2k|{}_I\langle A|\vec{\eta}_o+\vec{\eta}_1|B\rangle_I|^2, \quad (86a)$$

where

$$k_o=\frac{E_B^I-E_A^I}{c}, \quad (86b)$$

and $\vec{\eta}$ is $\vec{X}$ or $\vec{Y}$ depending on whether we are dealing with the $E1$ or the $M1$ transitions. In Eq. (86a), $|A\rangle_I$ and $|B\rangle_I$ are the eigenstates of the nonrelativistic Hamiltonian $h = h^{(0)} + h^{(1)}$. It may be useful to write

$$|A\rangle_I = |A\rangle_0 + |A\rangle_1 \quad \text{and} \quad |B\rangle_I = |B\rangle_0 + |B\rangle_1 , \tag{87}$$

where $|A\rangle_0$ and $|B\rangle_0$ are the eigenstates of the nonrelativistic Hamiltonian $h^{(0)}$, and $|A\rangle_1$ and $|B\rangle_1$ are their relativistic corrections of order $1/c^2$ due to $h^{(1)}$. In the first-order perturbation theory, for example,

$$|A\rangle_1 = \sum_{k \neq A} |k\rangle_o \frac{{}_o\langle k|h^{(1)}|A\rangle_o}{(E_A^{(0)} - E_k^{(0)})} . \tag{88}$$

One can think of many applications of Eqs. (84)–(87), the calculation of the relativistic corrections to the $E1$ and the $M1$ transition amplitudes, and especially the calculation of the so-called "relativistic $E1$ and $M1$ transitions[21]" (transitions which proceed only because of the relativistic effects since the matrix element of $\vec{X}_0$ or $\vec{Y}_0$ between the eigenstates of $h^{(0)}$ vanishes) in positronium, charmonium, and mesic atoms, the computation of the relativistic corrections to the photodisintegration cross section of deuteron, etc., just to mention a few. Using Eqs. (84), (86), (87), and (88) we have estimated the relativistic correction to the one-photon electric-dipole transition amplitudes of charmonium. They are discussed in another paper.[23]

VI. CONCLUDING REMARKS

Since we were concerned only with the calculation of the one-photon transition amplitudes, in the interaction Hamiltonian H_I of Eq. (12) we had included only those terms which were linear in the vector potential and in the fields; but in order to calculate the processes of the composite system involving two photons and for calculating the Zeeman effect, etc., we need the expression for H_I including the quadratic and the higher-order terms in the potential and the fields. To relative order $1/c^2$, the expression for H_I which can be used to calculate the one- and two-photon processes and the Zeeman effect of the composite system to relative order $1/c^2$, is

$$\begin{aligned} H_I = {} & i\sum_{\mu=1}^{N} \frac{e_\mu}{2c}\{[\vec{r}_\mu, H]\cdot\vec{A}_\mu + \vec{A}_\mu\cdot[\vec{r}_\mu, H]\} \\ & - \frac{1}{4c^2}\sum_{\mu=1}^{N}\sum_{\nu=1}^{N}\sum_{i=1}^{3}\sum_{j=1}^{3} e_\mu e_\nu([r_{\mu,i},[r_{\nu,j},H]]A_{\mu,i}A_{\nu,j} + A_{\mu,i}A_{\nu,j}[r_{\mu,i},[r_{\nu,j},H]]) \\ & - \sum_{\mu=1}^{N}\frac{e_\mu}{m_\mu c}\vec{s}_\mu\cdot\vec{B}_\mu + \sum_{\mu=1}^{N}\frac{e_\mu}{4m_\mu^2c^2}\vec{s}_\mu\cdot[(\vec{p}_\mu\times\vec{E}_\mu) - (\vec{E}_\mu\times\vec{p}_\mu)] - i\sum_{\mu=1}^{N}\frac{e_\mu}{4m_\mu^2c^2}\vec{s}_\mu\cdot(\vec{\nabla}_\mu\times\vec{E}_\mu) + \sum_{\mu=1}^{N}\frac{e_\mu}{4m_\mu^3c^3}[p_\mu^2, \vec{s}_\mu\cdot\vec{B}_\mu]_+ \\ & - \sum_{\mu=1}^{N}\frac{e_\mu^2}{4m_\mu^3c^4}[(\vec{A}_\mu\cdot\vec{p}_\mu + \vec{p}_\mu\cdot\vec{A}_\mu)\vec{s}_\mu\cdot\vec{B}_\mu + \text{H.c.}] - \sum_{\mu=1}^{N}\frac{e_\mu^2}{2m_\mu^3c^4}(\vec{s}_\mu\cdot\vec{B}_\mu)^2 - \sum_{\mu=1}^{N}\frac{e_\mu^2}{8m_\mu^2c^3}\vec{s}_\mu\cdot(\vec{A}_\mu\times\vec{E}_\mu - \vec{E}_\mu\times\vec{A}_\mu) . \end{aligned} \tag{89}$$

We have shown elsewhere[30] that the two-photon transition amplitude calculated in the second-order perturbation theory on the basis of Eq. (89) satisfies the relativistic condition on the two-photon amplitude,

$$\left(\frac{\omega_2'}{\omega_2}\right)^{1/2}\left(\frac{\omega_1'}{\omega_1}\right)^{1/2} T'^{(t_0')}_{B';A'\vec{k}_1'\hat{\epsilon}_\alpha'(1),\ \vec{k}_2'\hat{\epsilon}_\beta'(2)} = T^{(t_0)}_{B;A\vec{k}_1\hat{\epsilon}_\alpha(1),\vec{k}_2\hat{\epsilon}_\beta(2)} \tag{90}$$

where the notation is self-evident from our discussion in Section II. Using Eq. (89) it is especially interesting to calculate the leading relativistic corrections of the Zeeman effect of any composite system since the accuracy of the experimental results here can be very high.

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[1]P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949).

[2]L. H. Thomas, Phys. Rev. 85, 868 (1952).

[3]B. Bakamjian and L. H. Thomas, Phys. Rev. 92, 1300 (1953).

[4]L. L. Foldy, Phys. Rev. 122, 275 (1961).

[5]H. Osborn, Phys. Rev. 176, 1514 (1968); 176, 1523 (1968).

[6]F. E. Close and L. A. Copley, Nucl. Phys. B 19, 477 (1970).

[7]F. E. Close and H. Osborn, Phys. Rev. D 2, 2127 (1970).

[8]R. A. Krajcik and L. L. Foldy, Phys. Rev. D 10, 1777 (1974).

[9]D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. 35, 350 (1963).

[10]C. G. Darwin, Philos. Mag. 39, 537 (1920).

[11]G. Breit, Phys. Rev. 34, 553 (1929).

[12]J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1967), p. 263.

[13]In the case where the internal interaction in the composite system is electromagnetic, the correct interaction Hamiltonian including the spin-dependent terms of Eq. (12), can be derived by the method of the minimal substitution provided we make the minimal substitution in the covariant Breit Hamiltonian and then make the FW transformation of this equation to order $1/c^4$. Drake (Ref. 17) showed this in 1971. But, as pointed out by Sucher (Ref. 21), it is not conceptually satisfactory to start from the covariant Breit equation since this equation has several serious problems connected with its negative energy solutions. See Ref. 21 (p. 1795) for a brief discussion of the problems of the covariant Breit equation. See also, G. E. Brown and D. G. Ravenhall, Proc. R. Soc. London A 208, 552 (1951); J. Sucher, University of Maryland Report No. 77-018; L. Armstrong Jr., in *Structure and Collisions of Ions and Atoms*, edited by I. A. Sellin (Springer, Berlin, 1978), pp. 79–81. The approximate Darwin-Briet Hamiltonian (which is correct only to relative order $1/c^2$) does not have these problems. This is one of the reasons why we work within the framework of approximately relativistic Hamiltonians in this paper.

[14]J. D. Bhorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), pp. 48–52.

[15]G. W. F. Drake, Phys. Rev. A 5, 1979 (1972).

[16]G. Feinberg and J. Sucher, Phys. Rev. Lett. 26, 681 (1971).

[17]G. W. F. Drake, Phys. Rev. A 3, 908 (1971).

[18]D. L. Lin and G. Feinberg, Phys. Rev. A 10, 1425 (1974).

[19]G. W. F. Drake, Phys. Rev. A 9, 2799 (1974).

[20]E. J. Kelsey, Ann. Phys. (N.Y.) 98, 462 (1976).

[21]J. Sucher, Rep. Prog. Phys. 41, 1781 (1978).

[22]J. Sucher, *Proceedings of the Fifth International Conference on Atomic Physics, Berkeley* (Plenum, New York, 1976), p. 415.

[23]K. J. Sebastian (unpublished).

[24]E. P. Wigner, in *Physicist's Conception of Nature*, edited by J. Mehra (Reidel, Boston, 1973), p. 320.

[25]K. J. Sebastian, Phys. Rev. A 19, 1398 (1979).

[26]K. J. Sebastian and D. Yun, Phys. Rev. D 19, 2509 (1979).

[27]It should be noted that in our Eqs. (30)–(32) we have chosen the arbitrary function (Ref. 1) of Ref. 8 to be zero. The appropriateness of this choice is discussed in footnote 13 of Ref. 25.

[28]H. Lewtwyler, Il Nuovo Cimento 37, 556 (1965).

[29]F. Coester and P. Havas, Phys. Rev. D 14, 2556 (1976).

[30]K. J. Sebastian (unpublished).