Statistical characterization of periodic, area-preserving mappings

John R. Cary, James D. Meiss, and A. Bhattacharjee
Institute for Fusion Studies, University of Texas, Austin, Texas 78712
(Received 2 February 1981)

A method of statistically characterizing an area-preserving, doubly periodic mapping is presented. This method allows one to calculate the characteristic functions, and the joint probabilities of the mapping.

The standard mapping (Ref. 1, Sec. 5) has been studied extensively as a model of stochastic behavior. Early work included calculations of the perturbed invariants for small values of the nonlinearity parameter and the study of the transition to stochasticity.[2] More recently, there has been considerable interest in the study of the statistical description of the stochastic regime with the calculation of the diffusion coefficient.[3,4] Here we present a method for statistically characterizing the standard mapping

$$p_{n+1}=p_n+\epsilon\sin(x_n),\quad x_{n+1}=x_n+p_{n+1}\ , \tag{1}$$

and other doubly periodic mappings (Ref. 1, Sec. 5.1). The essence of our method is the calculation of the characteristic functions of the mapping. These characteristic functions allow us to determine various statistical quantities, such as the diffusion coefficient, the correlation functions, and the joint probability distributions.

The characteristic functions are defined with integer argument as follows:

$$\begin{aligned}\chi_0(m_0)&\equiv\langle\exp(im_0x_n)\rangle\ ,\\ \chi_1(m_0,m_1)&\equiv\langle\exp(im_0x_n+im_1x_{n+1})\rangle\ ,\\ \chi_2(m_0,m_1,m_2)&\equiv\langle\exp(im_0x_n+im_1x_{n+1}+im_2x_{n+2})\rangle\end{aligned}\tag{2}$$

The angular brackets in these definitions are left unspecified for the moment, except that they are time-translation invariant. An example of a time-translation invariant average is the average of the initial conditions over an invariant region R of phase space,[5] i.e., a region which is mapped onto itself:

$$\langle\exp(im_0x_n+im_1x_{n+1}+im_2x_{n+2})\rangle\equiv\left[\int_R dp_0\,dx_0\right]^{-1}\int_R dp_0\,dx_0\exp[im_0x_n(x_0,p_0)+im_1x_{n+1}(x_0,p_0)+im_2x_{n+2}(x_0,p_0)]\ . \tag{3}$$

The crucial point of this analysis is that the mapping equations (1) in the form

$$x_{n+k}=2x_{n+k-1}-x_{n+k-2}+\epsilon\sin(x_{n+k-1}) \tag{4}$$

allow one to express χ_{n+1} in terms of χ_n for $n\geq 1$. By using the Bessel-function identity $\exp[i\epsilon\sin(x)]=J_l(\epsilon)\exp(ilx)$, we find

$$\begin{aligned}\chi_k(m_0,m_1,\ldots,m_k)&\equiv\langle\exp(im_0x_n+im_1x_{n+1}+im_2x_{n+2}+\cdots+im_kx_{n+k})\rangle\\ &\equiv\langle\exp[im_0x_n+\cdots+im_{k-3}x_{n+k-3}+i(m_{k-2}-m_k)x_{n+k-2}\\ &\qquad+i(m_{k-1}+2m_k)x_{n+k-1}+im_k\epsilon\sin(x_{n+k-1})]\rangle\\ &=\sum_l J_l(m_k\epsilon)\chi_{k-1}(m_0,\ldots,m_{k-3},m_{k-2}-m_k,m_{k-1}+2m_k+l)\ .\end{aligned}\tag{5}$$

Moreover, χ_1 itself may be calculated by Eq. (3):

$$\chi_1(m_0,m_1)=\left[\int_R dp_0\,dx_0\right]^{-1}\int_R dp_0\,dx_0\exp[i(m_0+m_1)x_0-im_0p_0]\ . \tag{6}$$

The interpretation of this method is that we must first select a class of orbits and, thereby, an invariant region R. Performing the integrals in Eq. (6) we obtain the characteristic functions averaged over that class of orbits. All subsequent quantities derived from the characteristic functions are therefore averages over that class of orbits.

Let us take the region R in Eq. (6) to be all of phase space. Equations (5) and (6) immediately

 ©1981 The American Physical Society

yield

$$\chi_1(m_0,m_1)=\delta_{m_0,0}\delta_{m_1,0} \ , \tag{7a}$$

$$\chi_2(m_0,m_1,m_2)=\delta_{m_0,m_2}J_{-m_1-2m_0}(m_2\epsilon) \ , \tag{7b}$$

and

$$\chi_3(m_0,m_1,m_2,m_3)=J_{m_3-m_1-2m_0}(m_0\epsilon)J_{m_0-m_2-2m_3}(m_3\epsilon) \ . \tag{7c}$$

Knowledge of the characteristic functions allows us to calculate the averaged diffusion constant

$$D\equiv\left\langle \lim_{T\to\infty}\frac{1}{2T}(p_f-p_i)^2\right\rangle=\left\langle \lim_{T\to\infty}\frac{1}{2T}\sum_{i=1}^{T}\sum_{j=1}^{T}\Delta p_i\Delta p_j\right\rangle \ . \tag{8}$$

To reduce the double sum in the last equation, we rewrite it in the form

$$D=\lim_{T\to\infty}\frac{1}{2T}\sum_{i=1}^{T}\sum_{j=1-i}^{T-i}\langle\Delta p_i\Delta p_{i+j}\rangle \ . \tag{9}$$

Next we assume that the terms in the sum fall off rapidly with j, so that as $T\to\infty$, we can sum over all j:

$$D=\lim_{T\to\infty}\frac{1}{2T}\sum_{i=1}^{T}\sum_{j=-\infty}^{\infty}\langle\Delta p_i\Delta p_{i+j}\rangle \ . \tag{10}$$

Then we use the time-translation invariance of the average to sum over i:

$$D=\tfrac{1}{2}\sum_{j=-\infty}^{\infty}\langle\Delta p_i\Delta p_{i+j}\rangle \ . \tag{11}$$

Finally, we use the Fourier expansion,

$$\Delta p_j=\epsilon\sin(x_j)=(-i/2)[\exp(ix_j)-\exp(-ix_j)] \ ,$$

to obtain the following formula for the diffusion coefficient

$$D=\tfrac{1}{4}\epsilon^2\,\mathrm{Re}\left\{\chi_0(0)-\chi_0(2)+2\sum_{j=1}^{\infty}[\chi_j(1,0,\ldots,0,-1)-\chi_j(1,0,\ldots,0,1)]\right\} \ . \tag{12}$$

Inserting the special solution of Eqs. (7), we find

$$D=\tfrac{1}{4}\epsilon^2\left\{1-2J_2(\epsilon)+2J_3^2(\epsilon)-2J_1^2(\epsilon)+2\sum_{n=-\infty}^{\infty}J_n(-\epsilon)J_{n-4}(\epsilon)J_{-2(n-2)}[(n-2)\epsilon]\right.$$
$$\left.-2\sum_{n=-\infty}^{\infty}J_n^2(\epsilon)J_{-2(n+1)}[(n+2)\epsilon]+2\sum_{j=5}^{\infty}[\chi_j(1,0,0,\ldots,0,-1)-\chi_j(1,0,0,\ldots,0,1)]\right\} \ . \tag{13}$$

We note that the first four terms of this series are exactly those found by Rechester and White.[3] (We also note that Karney *et al.*[6] found those same terms by computing the force correlation functions.)

A knowledge of the characteristic functions allows one also to calculate the correlation functions

$$C_m\equiv\langle \mathrm{fr}(x_n)\,\mathrm{fr}(x_{n+m})\rangle \ , \tag{14}$$

where $\mathrm{fr}(x)$ is the number between zero and 2π which equals x modulo 2π. The function $\mathrm{fr}(x)$ can be written as a Fourier series

$$\mathrm{fr}(x)=\sum_{k=-\infty}^{\infty}r_k e^{ikx} \ , \tag{15}$$

where $r_0=\pi$ and $r_k=i/k$ for $k\neq 0$. By straightforward algebraic manipulation we find

$$C_m=\pi^2+\mathrm{Re}\left\{4\pi\sum_{k=1}^{\infty}\frac{i\chi_0(k)}{k}+2\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\chi_m(k,0,0,\ldots,0,-l)-\chi_m(k,0,0,\ldots,0,l)}{lk}\right\} \ . \tag{16}$$

For the solution of Eq. (7) we obtain

$$C_0=\tfrac{4}{3}\pi^2 \ , \tag{17a}$$

$$C_1=\pi^2 \ , \tag{17b}$$

$$C_2=\pi^2-2\sum_{k=1}^{\infty}\frac{J_{2k}(k\epsilon)}{k^2} \ , \tag{17c}$$

$$C_3=\pi^2-2\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{J_{l-2k}(k\epsilon)J_{k-2l}(l\epsilon)-(-1)^{l+k}J_{l+2k}(k\epsilon)J_{k+2l}(l\epsilon)}{lk} \ . \tag{17d}$$

It should be noted that we have made an error in our application of this method to the calculation of the diffusion coefficient by neglecting the accelerator modes (Ref. 1, Sec. 5.5; Ref. 6). Accelerator modes exist throughout a region of finite measure in phase space. They have the property that p_n continuously increases or decreases roughly as n. Such modes contribute an infinite diffusion when folded into the average (8) and will cause the series (13) not to converge.

To correct for the presence of accelerator modes, we should choose the region R of Eqs. (3) and (6) to be the stochastic region of phase space. Our value for $\chi_1(0,0)$ remains unchanged by this choice, but now $\chi_1(m,n)$ for m or n not equal to zero no longer vanishes. Instead it is of the order of $1/\epsilon^2$ since the relative measure of the accelerator regions is of that order (Ref. 1, Sec. 5.5). This will cause corrections to Eq. (13) of order equal to that of the last sum.

To illustrate the generality of this method, we consider its application to the mapping

$$u_{j+1}=u_j+\mu-\nu\cos v_j \quad , \quad v_{j+1}=v_j+\mu+\nu\sin u_{j+1} \quad ,$$

which has been discussed by Karney.[7] If we define the characteristic functions to be

$$\chi_{M,N}(m_0,\ldots,m_M,n_0,\ldots,n_N)\equiv\langle\exp(im_0u_j+im_1u_{j+1}+\cdots+im_Mu_{j+M}+in_0v_j+in_1v_{j+1}+\cdots+in_Nv_{j+N})\rangle \quad ,$$

we immediately find the recursion relations

$$\chi_{M,M}=\sum_l J_l(n_M\nu)e^{in_M\mu}\chi_{M,M-1}(m_0,m_1,\ldots,m_{M-1},m_M+l,n_0,n_1,\ldots,n_{M-2},n_{M-1}+n_M)$$

and

$$\chi_{M,M-1}=\sum_l J_l(m_M\nu)\exp[i(m_M\mu-l\pi/2)]\chi_{M-1,M-1}(m_0,\ldots,m_{M-1}+m_M,n_0,\ldots,n_{M-1}+l) \quad .$$

Finally, we would like to note that this technique allows one to calculate the fractionally reduced joint probability distribution $P(\bar{x}_0,\bar{p}_0,\bar{x}_n,\bar{p}_n)$ which has been discussed by Grebogi *et al.*[8] For example,

$$P(\bar{x}_0,\bar{p}_0,\bar{x}_k,\bar{p}_k)=\langle\delta[\bar{x}_0-\mathrm{fr}(x_j)]\delta[\bar{p}_0-\mathrm{fr}(p_j)]\delta[\bar{x}_k-\mathrm{fr}(x_{j+k})]\delta[\bar{p}_k-\mathrm{fr}(p_{j+k})]\rangle$$

$$=\left(\frac{1}{2\pi}\right)^4\sum_{m_0,m_k,n_0,n_k}\exp(-im_0\bar{x}_0-im_k\bar{x}_k-in_0\bar{p}_0-in_k\bar{p}_k)\,\chi_{k,k}(m_0,0,0,\ldots,m_k,n_0,0,0,\ldots,n_k) \quad .$$

Obviously this method can be used to find the characteristic functions of any doubly periodic mapping. Furthermore, in the stochastic regime, where the higher-order characteristics are small, one can use this method to obtain an approximate expression for the diffusion coefficient, the correlation functions, and the joint probabilities.

[1]Boris V. Chirikov, Phys. Rep. 52, 263 (1979).

[2]J. M. Greene, J. Math. Phys. 20, 1183 (1979).

[3]A. B. Rechester and R. B. White, Phys. Rev. Lett. 44, 1586 (1980).

[4]A. B. Rechester, M. N. Rosenbluth, and R. B. White, Phys. Rev. A 23, 2664 (1981).

[5]V. I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics* (Benjamin, New York, 1968), pp. 15–19.

[6]C. F. F. Karney, A. B. Rechester, and R. B. White, Princeton University, Plasma Physics Report No. PPPL-1752, 1981 (unpublished).

[7]Charles F. F. Karney, Phys. Fluids 22, 2188 (1979).

[8]Celso Grebogi and Allan N. Kaufman, Lawrence Berkeley Laboratory Report No. LBL-12329, 1981 (unpublished).