PHYSICAL REVIEW A VOLUME 23, NUMBER 5 MAY 1981

Propagation narrowing in the transmission of a light pulse through a spectral hole

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(Received 20 October 1980)

We comment on the spectral modification and temporal reshaping of a narrow-band light pulse propagating through an atomic absorber that has a still narrower spectral hole. The regime of propagation narrowing is distinguished from the regime of Beer's law.

I. INTRODUCTION

Hole burning is the term for the selective photoexcitation of atoms or molecules in a small portion of an inhomogeneously broadened optical absorption line. In this region of the line, following the photoexcitation, there will be fewer than the normal number of atoms or molecules in their ground states. A subsequent narrow-band weak probe pulse will suffer less than normal attenuation if its center frequency lies within the excitation region. The resulting absorption curve mapped out by scanning the frequency of the probe beam has a hole in it. Figure 1 shows an example. All of this is well known.[1]

In the present note we address a question that comes up when the weak probe pulse has a spectral width greater than the width of the hole, as shown in Figure 2. This question is: What is the temporal form of the transmitted pulse? The uncertainty relation $\Delta\nu\Delta\tau \geqslant 1$ suggests that $\Delta\tau$ increases because $\Delta\nu$ decreases, but in order to answer the question properly it turns out to be necessary to consider the propagation of the probe pulse through a finite depth of absorber. If the probe pulse is weak enough to have negligible effect on the hole, then the question can be answered analytically in closed form.

II. PROPAGATION OF WEAK OPTICAL PULSES THROUGH BROADBAND INHOMOGENEOUSLY BROADENED ABSORBERS

In the familiar two-level-atom model for an absorber,[2] transmission and absorption of light are governed by the equation

$$(\partial/\partial z + \partial/\partial ct)\mathcal{E}(t,z) = 2\pi i(\omega/c)\mathcal{P}(t,z), \qquad (1)$$

where $\mathcal{E}$ and $\mathcal{P}$ are the complex envelopes of the electric field strength and the atomic polarization density. An expression for $\mathcal{P}$ involves the off-diagonal part of the two-level atom's density

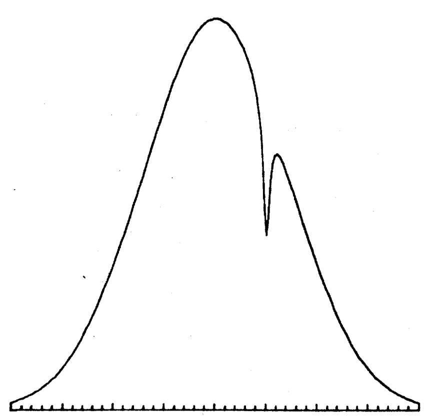

FIG. 1. A Doppler-broadened absorption line with a homogeneously broadened hole.

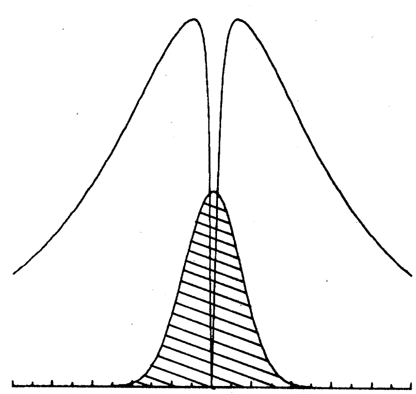

FIG. 2. The initial probe pulse spectrum (shaded) overlaid on an absorption line with a much narrower spectral hole.

matrix in the rotating wave approximation, averaged over the Doppler distribution of atomic velocities:

$$\mathcal{P}(t,z) = Nd\langle r_{12}(t,z;v)\rangle_v . \quad (2)$$

Here, N is the density of atoms and d is the magnitude of the atomic transition dipole matrix element along the direction of the electric field vector.

If, as we assume, the electric field of the probe pulse is too weak to alter the shape of the inhomogeneous line (with the hole already burned into it), then the equation obeyed by $r_{12}(t,z;v)$ is simply

$$\partial r_{12}/\partial t = -(\beta + i\Delta)r_{12} + (i/\hbar)d\mathcal{E}. \quad (3)$$

Here, $\Delta = \Delta(v)$ is the detuning between pulse frequency and the Doppler-shifted atomic resonance frequency and β is the halfwidth of the homogeneous component of the atomic absorption line. An average over Δ's is equivalent to an average over v's.

Because Eqs. (1) and (3) are linear they are trivially solved by Fourier transform methods. We define

$$\mathcal{E}(t,z) = \int e(\nu,z)e^{-i\nu t}d\nu/2\pi , \quad (4a)$$

$$r_{12}(t,z) = \int \rho(\nu,z)e^{-i\nu t}d\nu/2\pi , \quad (4b)$$

and find the equation

$$[\partial/\partial z - i\nu/c - if(\nu) + \tfrac{1}{2}g(\nu)]e(\nu,z) = 0 , \quad (5)$$

where f and g are the dispersive and absorptive parts of the dipole reaction field:

$$f(\nu) = -\tfrac{1}{2}G\,\mathrm{Im}\left\langle \frac{1}{\beta + i(\Delta - \nu)}\right\rangle , \quad (6a)$$

$$\tfrac{1}{2}g(\nu) = \tfrac{1}{2}G\,\mathrm{Re}\left\langle \frac{1}{\beta + i(\Delta - \nu)}\right\rangle , \quad (6b)$$

Here, G is the primitive attenuation parameter for the problem

$$G = 4\pi Nd^2\omega/\hbar c , \quad (7)$$

and the angular brackets denote the Doppler average.

The solution to Eq. (5) is immediate:

$$e(\nu,z) = \mathcal{E}_0 e^{-1/2(\nu/\delta\nu_0)^2} e^{if(\nu)z} e^{-1/2g(\nu)z}. \quad (8)$$

We have assumed that the incident pulse is Gaussian in time, with bandwidth $\delta\nu_0$ and temporal length $2\pi/\delta\nu_0$.

The pulse Fourier energy spectrum obviously changes with propagation:

$$|e(\nu,z)|^2 = |\mathcal{E}_0|^2 e^{-(\nu/\delta\nu_0)^2} e^{-g(\nu)z}. \quad (9)$$

However, in the usual case, when the Doppler linewidth is appreciably greater than $\delta\nu_0$, we can take $g(\nu) = g(0)$ for all significant frequencies ($|\nu| \lesssim \delta\nu_0$). Then the only significant frequency dependence is due to the Gaussian factor that came from the incident pulse. Thus the pulse retains its Gaussian shape and is simply attenuated uniformly at all frequencies according Beer's law:

$$|e(\nu,z)|^2 \cong |\mathcal{E}_0|^2 e^{-(\nu/\delta\nu_0)^2} e^{-g(0)z}. \quad (10)$$

All this is well known, and Crisp[3] has given interesting examples showing dramatic departures from the kind of behavior implied by (10) when the pulse bandwidth is much greater, rather than less, than the Doppler linewidth.

III. PROPAGATION THROUGH A SPECTRAL HOLE

With the background formulas derived in Sec. II, it is easy to take into account a narrow hole in the Doppler line. For simplicity we will take the shape of the Doppler curve, as well as the hole, to be Lorentzian. This is realistic as far as the hole is concerned and not a bad approximation for the Doppler curve because we will be concerned only with its center portion. Another simplication will be to take the centers of the spectral curves of the incident pulse, the Doppler line, and the hole all to coincide at $\nu = 0$, as shown in Fig. 2. Under these conditions the angular bracket in Eqs. (6) is to be interpreted as an average over Doppler detuning Δ with the normalized weight function

$$p(\Delta) = \frac{\beta^*}{\pi}\frac{1}{\Delta^2 + (\beta^*)^2} ,$$

where β^* is obviously now the Doppler width. Then we can evaluate the brackets in Eqs. (6) by explicit integration and find

$$f(\nu) = -\tfrac{1}{2}\alpha\left(\frac{\nu/\beta^*}{1 + (\nu/\beta^*)^2} - \frac{\nu/\beta_H}{1 + (\nu/\beta_H)^2}\right) , \quad (11a)$$

$$\tfrac{1}{2}g(\nu) = \tfrac{1}{2}\alpha\left(\frac{1}{1 + (\nu/\beta^*)^2} - \frac{1}{1 + (\nu/\beta_H)^2}\right) , \quad (11b)$$

where α is the normal Doppler absorption coefficient at line center:

$$\alpha = 4\pi Nd^2\omega/\hbar c\beta^* , \quad (12)$$

and β^* and β_H are the half-widths of the Dopper line and the hole, respectively. We are principally interested in the limits

$$\beta^* \gg \delta\nu_0 \gg \beta_H \gg \beta . \quad (13)$$

In other words, we assume the Doppler line to be much broader than any other spectral feature, and the width of the incident pulse to be much broader than the hole in the Doppler line. The underlying homogeneous width is taken to be negligibly small compared to all the other widths. These conditions are qualitatively met by the curves in Fig. 2.

Given relation (13), there are two distinct regions of the spectrum:

(a) The region of *propagation narrowing,* where $\nu \lesssim \beta_H$.

(b) The region of *Beer's law,* where $\nu \gg \beta_H$.

We have discussed region (b) briefly in Sec. II. Region (a) is characterized by a simple approximate expression for $g(\nu)$:

$$g(\nu) \simeq \alpha (\nu/\beta_H)^2 . \tag{14}$$

Thus, in region (a) we have

$$| e(\nu,z)|^2 = | \mathcal{E}_0|^2 e^{-\nu^2/[\delta\nu(z)]^2} , \tag{15}$$

where

$$\delta\nu(z) = \beta_H/[\alpha z + (\beta_H/\delta\nu_0)^2]^{1/2} . \tag{16}$$

That is, in the region of propagation narrowing the spectrum is Gaussian but with a variable width. Note that at $\alpha z = 0$ the width is equal to the initial pulse width $\delta\nu_0$ and only becomes smaller due to propagation. The decrease can, however, be rapid because $\beta_H/\delta\nu_0$ can be much less than unity. The threshold for propagation narrowing occurs at $\alpha z \approx (\beta_H/\delta\nu_0)^2$.

In the region of Beer's law the spectrum is also Gaussian, but with width $\delta\nu_0$, independent of αz.

To interpolate between regions (a) and (b) requires the full expression (11b) for $g(\nu)$. In Fig. 3 we show the full $|e(\nu,z)|^2$ as a function of ν for several values of αz, given the ratios $\beta^*/\delta\nu_0 = \sqrt{10}$ and $\beta_H/\delta\nu_0 = 1/10$. The two distinct Gaussians appropriate to spectral regions (a) and (b) are evident. Figure 4 shows the effect of the hole alone by plotting $|e(\nu,z)|^2$ without the first term of $g(\nu)$. In effect, Fig. 4 shows that the relative importance of the hole grows with propagation, but, as expected, only near line center. The broad base of all of the curves is simply $\exp[-(\nu/\delta\nu_0)^2]$.

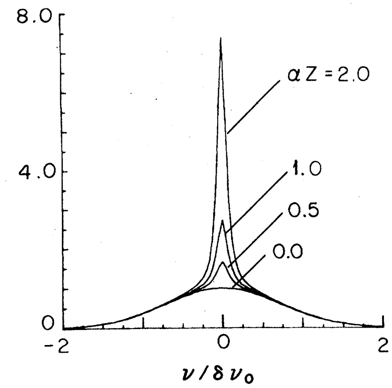

FIG. 4. The spectrum of the probe pulse, as in Fig. 3, except that the normal component of absorption has been removed.

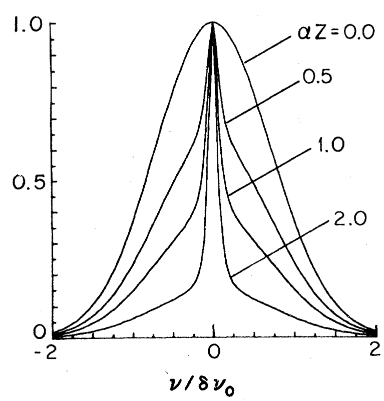

FIG. 3. The spectrum of the probe pulse $|e(\nu,z)|^2$ at a succession of propagation depths $\alpha z = 0.0$, 0.5, 1.0, and 2.0. The horizontal axis is in units of $\delta\nu_0$.

IV. TIME-DEPENDENT FEATURES

The exact expressions obtained in Sec. III for $|e(\nu,z)|^2$, as well as the approximate relations valid in the regions of propagation narrowing and Beer's law, permit good qualitative estimates of the time dependence of the transmitted pulse. However, $|e(\nu,z)|^2$ does not provide everything, because the consequences of $f(\nu)$ are not included. In this section we present the results of numerical computation of the Fourier transform (4a), thereby giving the full space-time behavior of $\mathcal{E}(t,z)$.

Figure 5 shows the temporal behavior of the transmitted pulse after propagation to $\alpha z = 2$. A clear departure from the Gaussian shape of the

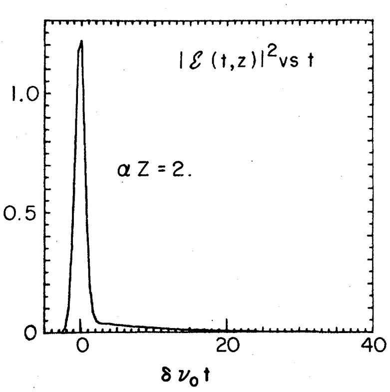

FIG. 5. The probe pulse intensity $|\mathcal{E}(t,z)|^2$ in arbitrary units as a function of time, after propagating two absorption depths into the medium. The horizontal axis is in units of $(\delta\nu_0)^{-1}$.

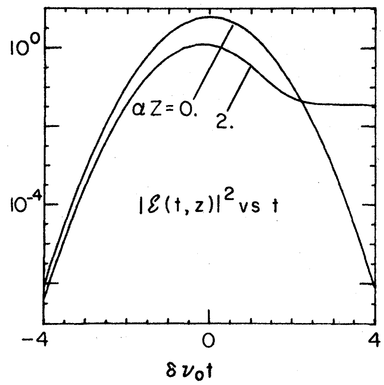

FIG. 6. The probe pulse intensity $|\mathcal{E}(t,z)|^2$ as a function of time at two different spatial positions, $\alpha z=0$ and 2. The vertical axis is logarithmic, but to the same scale as in Fig. 5, and the units of time are $(\delta\nu_0)^{-1}$.

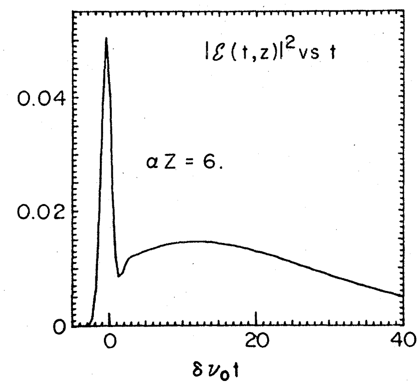

FIG. 7. The probe pulse intensity $|\mathcal{E}(t,z)|^2$ as a function of time after propagating six absorption depths into the medium. The vertical axis is to the same scale as Fig. 5, and the units of time are $(\delta\nu_0)^{-1}$.

input pulse is evident in the trailing portions of the transmitted pulse. In Fig. 6 we show the data of Fig. 5 again, but with the $z=0$ curve added for comparison. At $\delta\nu_0 t\simeq 2$ the trailing edge of the transmitted pulse becomes stronger than the tail of the incident pulse, although it is two orders of magnitude weaker than the incident pulse peak.

Finally, Fig. 7 shows the transmitted pulse at $\alpha z=6$. It is seen that the bulk of the pulse energy is now in the long tail, and one can begin to say that a single frequency-time uncertainty relation again describes the pulse reasonably well. The temporal width of the tail is roughly $\sqrt{3}$ times the width shown in Fig. 5, in accord with (16) since $(\beta_H/\delta\nu_0)^2$ is negligible. Moreover, we note that the resonant interaction of pulse and atoms, during the pulse transmission, is fully coherent in our model. This is responsible for the slight interference minimum that occurs between the two components of the pulse.

V. DISCUSSION

The spectral and temporal changes predicted by solution (8) are illustrated in the figures. They are a consequence of purely linear dispersion theory and are novel only in the sense that classic discussions[4] of light propagation in dispersive media do not appear to have included treatments either of purely inhomogeneous line broadening or of lines with holes in them. Experimental observations of the predicted peak delay and pulse lenghtening do not appear to have been reported either.

Another view of these results is obtained by regarding our model of an absorption line with a hole as a continuous-band interference filter. In one sense it is a pure interference filter; no energy loss mechanisms are included in the model. However, conversion of pulse energy into loss-free atomic dipole oscillations all across the Doppler line will nevertheless lead to pulse decay, and it is more realistic to speak of an interference filter, as opposed to an absorption filter, when the effect of $f(\nu)$ is much more important than that of $g(\nu)$. Formulas (11) show that for $\nu\ll\beta_H$ this condition is well satisfied:

$$f(\nu)\simeq \tfrac{1}{2}\,\alpha\nu/\beta_H\,, \tag{17a}$$

$$\tfrac{1}{2}g(\nu)\simeq \tfrac{1}{2}\,\alpha(\nu/\beta_H)^2\,. \tag{17b}$$

This is a consequence of the hole in the line, of course. For a normal line without a hole, one has $g\gg f$ for all frequencies near the center.

Finally, because of the linearity of the model, it is also possible to treat the pulse as if it were interacting with two entirely separate groups of atoms. The first group comprises the usual dipole oscillators in the full Doppler line, and the second group is a set of "negative" oscillators occupying a region of width β_H at line center. The negative oscillators emit rather than absorb light. It is the emission of the "negative" oscillators that causes the growth, with increasing propagation distance, of the peaks of the curves in Fig. 4. In this view of the model it is the interference between the positive and negative oscillators at line center that causes the pulse lengthening. In classical theory the negative dipoles can never be more "negative" than the real dipoles are "positive," because they are designed only to cancel the real dipoles in a certain spectral region. In the quantum theory the negative atoms are not so severely restricted. For example, the hole has been constructed in our treatment so that $g(0)=0$. That is, we have assumed the absence of absorbing dipoles at line center. However, the original preparation of the hole could have been arranged to include a degree of atomic inversion at line center. In this case $g(0) < 0$, and gain rather than loss would be expected. The "negative" oscillators in the model would then more than cancel the positive ones. We will not explore this case.

ACKNOWLEDGMENTS

This research was partially supported by the Department of Energy, Contract No. DE-AC03-76DP01118, the Joint Services Electronics Program, Contract No. DAAG29-79-C-0079, the National Research Council of Canada, the National Science Foundation and Army Research Office, Grant No. NSF-DMR80-06966, and the Office of Naval Research, Contract No. N00014-79-0666. We are grateful to J. J. Yeh and B. J. Herman for assistance with the numerical integration required for Figs. 5–7, and we acknowledge conversations with M. A. Johnson, B. W. Shore, S. Stenholm, A. Szöke, and J. J. Yeh.

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[1]See, for example, discussions in A. Yariv, *Quantum Electronics* (Wiley, New York, 1967), Chap. 8; M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, Mass., 1974), Chap. 10; and L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975), Chap. 6.

[2]We follow the conventions of L. Allen and J. H. Eberly, see Ref. 1.

[3]M. D. Crisp, Phys. Rev. A 1, 1604 (1970). See also Ref. 2, Sec. 1.8.

[4]A. Sommerfeld, *Optics* (Academic, New York, 1972); L. Brillouin, *Wave Propagation and Group Velocity* (Academic, New York, 1960).