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Analytic solutions to the two-state problem for a class of coupling potentials

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A class of pulse functions is found for which analytic solutions to the problem of two levels coupled by these pulse functions is obtained. The hyperbolic-secant coupling pulse is included in this class of functions leading to the Rosen-Zener solution, but all other pulses belonging to the class function are asymmetric. The asymmetric pulses lead to qualitatively new features in the solutions; in general, it is impossible to have a zero-transition probability with such asymmetric pulses.

I. INTRODUCTION

A problem of considerable interest in physics is to determine the time evolution of a two-level system whose levels are coupled by a time-dependent potential. The probability amplitudes for the two levels in the interaction representation denoted by $a_1(t)$ and $a_2(t)$, obey the coupled differential equations

$$da_1/dt = -i\chi(t)e^{-i\omega t}a_2 , \tag{1a}$$

$$da_2/dt = -i\chi(t)e^{i\omega t}a_1 , \tag{1b}$$

where ω is the frequency separation of levels 2 and 1 and $\chi(t)$ is the coupling parameter (assumed real). By introducing a characteristic time scale T and defining dimensionless parameters

$$\tau = t/T , \tag{2a}$$

$$\alpha = \omega T , \tag{2b}$$

$$\beta = S/\pi, \quad S = \int_{-\infty}^{\infty} \chi(t)\,dt , \tag{2c}$$

$$f(\tau) = \chi(\tau T)T/\beta , \tag{2d}$$

one can transform Eqs. (1) into

$$\dot{a}_1 = -i\beta f(\tau)e^{-i\alpha\tau}a_2 , \tag{3a}$$

$$\dot{a}_2 = -i\beta f(\tau)e^{i\alpha\tau}a_1 , \tag{3b}$$

where a dot indicates $d/d\tau$. Owing to Eqs. (2c) and (2d), the function $f(\tau)$ is normalized as

$$\int_{-\infty}^{\infty} f(\tau)\,d\tau = \pi \tag{4}$$

and the parameter S is the pulse area.

Equations (3) arise in any semiclassical two-state calculation in which the two levels, separated in energy by $\hbar\alpha/T$, are coupled by a potential $\hbar\beta f(\tau)/T$. These equations also arise in two-state problems in which the levels are coupled by a nearly resonant oscillating field. In that case, assuming the "antiresonant" component of the field can be neglected, the quantity $\hbar\beta f(\tau)/T$ takes on the role of an envelope function for the field while α/T represents the atom-field detuning. Since Eqs. (3) are of such fundamental importance in many branches of physics, it is useful to have analytic solutions of these equations for various envelope functions $f(\tau)$. Of course, one can numerically integrate Eqs. (3), but such procedures can be costly (especially for large α) and do not necessarily yield the more general qualitative features of the solutions.

If $\alpha = 0$, a simple solution can be found for arbitrary $f(\tau)$. The probability amplitude a_1 or a_2 is given by

$$a_i = A_i \cos\theta(\tau) + B_i \sin\theta(\tau) , \tag{5a}$$

$$\theta(\tau) = \beta \int_{-\infty}^{\tau} f(\tau')\,d\tau' , \tag{5b}$$

where A_i and B_i are constants. For $\alpha \neq 0$, however, there are, to our knowledge, only two *smooth* envelope functions $f(\tau)$ for which an analytic solution of Eqs. (3) has been obtained. One such function is $f(\tau) = \text{const} = 1$ for which the solution[1] is

$$a_{1,2} = A_{1,2}\cos\sigma_{1,2}\tau + B_{1,2}\sin\sigma_{2,1}\tau , \tag{6a}$$

$$\sigma_{1,2} = \tfrac{1}{2}[-\alpha \pm (\alpha^2 + 4\beta^2)^{1/2}] . \tag{6b}$$

It should be noted, however, that this envelope function does not vanish at $\tau = \pm\infty$ implying that it cannot represent a physical pulse of finite duration. The other function for which an analytic solution of Eqs. (3) is known is $f(\tau) = \text{sech}\,\tau$. By employing the change of variable

$$z = \tfrac{1}{2}\int_{-\infty}^{\tau} \text{sech}^2\tau'\,d\tau' , \tag{7}$$

Rosen and Zener[2] were able to show that the gen-

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eral solution in this case could be given in terms of hypergeometric functions.

It is the purpose of this note to indicate that analytic solutions to Eqs. (3) may be found for an entire class of positive definite functions $f(\tau)$. The hyperbolic secant is included in this class of functions as a special case, but the rest of the functions are *not* symmetric about any given τ. This asymmetry leads to new features in the solutions.

II. SOLUTION FOR A CLASS OF FUNCTIONS

Equations (3a) and (3b) may be combined to yield the following second-order linear differential equation for $a_1(\tau)$:

$$\ddot{a}_1 + (i\alpha - \dot{f}/f)\dot{a}_1 + \beta^2 f^2 a_1 = 0\,. \tag{8}$$

The amplitude a_2 obeys a similar equation with $-\alpha$ replacing α. In order to determine a class of functions $f(\tau)$ for which analytic solutions of Eq. (8) exist, we introduce the change of variable

$$z = z(\tau) \geq 0\,, \tag{9a}$$

subject to the restriction that z is real and that

$$z(-\infty) = 0\,, \quad z(\infty) = 1\,; \tag{9b}$$

the transformation $z(\tau)$ changes the range of the independent variable from $(-\infty, \infty)$ to $[0,1]$.

In terms of the variable z, one may write Eq. (8) in the form

$$a_1'' + \frac{\frac{d}{d\tau}(\ln\dot{z}) + (i\alpha - \dot{f}/f)}{\dot{z}} a_1' + \frac{\beta^2 f^2}{[\dot{z}]^2} a_1 = 0\,, \tag{10}$$

where a prime indicates differentiation with respect to z. The general idea is to see whether or not Eq. (10) can be cast into the form of a standard equation of mathematical physics. In this paper, we determine the conditions under which Eq. (10) becomes the hypergeometric equation[3]

$$z(1-z)a_1'' + (Az+B)a_1' + Da_1 = 0\,, \tag{11}$$

where

$$A = -(a+b+1)\,, \tag{12a}$$

$$B = c\,, \tag{12b}$$

$$D = -ab\,, \tag{12c}$$

and a, b, c are the constants appearing in the hypergeometric equation of standard form.[3] We could determine equally well those conditions under which Eq. (10) becomes a generalized hypergeometric equation (equation of gauss); however, the hypergeometric equation is the only equation of gauss that yields physical solutions which are nonidentically zero at $z=0$ and 1.

By equating Eqs. (10) and (11), one may obtain

$$\beta^2 f^2/(\dot{z})^2 = D/z(1-z) \tag{13}$$

and

$$\dot{z} = i\alpha z(1-z)/[(1+A)z + (B-\tfrac{1}{2})]\,. \tag{14}$$

In order to have a one to one mapping of τ onto z, we require that $z(\tau)$ is a monotonically increasing function, implying that $\dot{z}$ is real and positive. This requirement used in conjunction with Eqs. (13) and (14) implies the following restrictions:

$$A+1 = i\alpha\lambda\,, \quad \lambda \text{ real} \tag{15a}$$

$$B - \tfrac{1}{2} = i\alpha\mu\,, \quad \mu \text{ real} \tag{15b}$$

$$\mu > 0\,, \quad \lambda/\mu > -1\,, \tag{16}$$

$$D \text{ real}, \quad D > 0\,. \tag{17}$$

In terms of these new variables, Eqs. (3) take the form

$$a_1' = -i\left(\frac{D}{z(1-z)}\right)^{1/2} e^{-i\alpha\tau(z)} a_2\,, \tag{18a}$$

$$a_2' = -i\left(\frac{D}{z(1-z)}\right)^{1/2} e^{i\alpha\tau(z)} a_1\,. \tag{18b}$$

Equation (10) becomes

$$z(1-z)a_1'' + [c - (a+b+1)z]a_1' - aba_1 = 0 \tag{19}$$

with

$$a = i\alpha\lambda[-1 + (1 - 4D/\alpha^2\lambda^2)^{1/2}]/2\,, \tag{20a}$$

$$b = i\alpha\lambda[-1 - (1 - 4D/\alpha^2\lambda^2)^{1/2}]/2\,, \tag{20b}$$

$$c = \tfrac{1}{2} + i\alpha\mu\,, \tag{20c}$$

and Eq. (14) may be rewritten

$$\dot{z} = z(1-z)/(\mu + \lambda z)\,. \tag{21}$$

The general solution of Eq. (19) is[3]

$$a_1 = A_1 F(a,b,c;z) + A_2 z^{1-c} F(a-c+1, b-c+1, 2-c; z)\,, \tag{22}$$

where $F(a,b,c;z)$ is the hypergeometric function, and A_1 and A_2 are integration constants. The time variable τ as a function of z may be obtained by integrating Eq. (21); one finds

$$\tau = \ln[z^{\mu}/(1-z)^{\mu+\lambda}]\,. \tag{23}$$

The upper-state amplitude may now be calculated by combining Eqs. (23), (18), (20), and (12c) to give

$$a_2 = i(-ab)^{-1/2} z^{c}(1-z)^{1-c+a+b} a_1'\,. \tag{24}$$

By differentiating Eq. (22) and using some simple properties of F functions,[3] one finds

$$a_2 = i(-ab)^{-1/2}[(ab/c)z^c F(c-a, c-b, 1+c; z)A_1 + (1-c)(1-z)^{1-c+a+b}F(a-c+1, b-c+1, 1-c; z)A_2] . \tag{25}$$

The constant A_1 and A_2 appearing in Eqs. (22) and (25) may be evaluated by imposing initial conditions $a_1(z=0)$ and $a_2(z=0)$. In terms of $a_1(0)$ and $a_2(0)$, Eqs. (22) and (25) become

$$a_1(z) = F(a, b, c; z)a_1(0) - \frac{i(-ab)^{1/2}z^{1-c}}{1-c}F(a-c+1, b-c+1, 2-c; z)a_2(0), \tag{26a}$$

$$a_2(z) = -\frac{i(-ab)^{1/2}z^c}{c}F(c-a, c-b, 1+c; z)a_1(0) + (1-z)^{1-c+a+b}F(a-c+1, b-c+1, 1-c; z)\,a_2(0) , \tag{26b}$$

which together with Eqs. (20), (12c), (13), and (21) provide a complete solution to the problem.

III. NATURE OF THE PULSE

In this section, we describe the pulse shapes for which the solution (26) is valid and in Sec. IV, we present an analysis of the solution in light of these pulse shapes. The pulse shape $P(t)$, as defined by

$$P(t) = (\beta/T)f(\tau), \qquad \tau = t/T , \tag{27}$$

is obtained from Eqs. (13), (21), (4), (23), and (2c) to be

$$P(t) = \frac{S}{\pi T}\frac{[z(1-z)]^{1/2}}{1+\lambda z} , \tag{28a}$$

$$t = T\ln[z/(1-z)^{1+\lambda}] , \tag{28b}$$

where S is the pulse area defined in Eq. (2c). In arriving at Eq. (28), we used the normalization condition (4) to obtain

$$D = \beta^2 = S^2/\pi^2 \tag{29}$$

and set

$$\mu = 1 \tag{30}$$

without loss of generality.

The pulse is characterized by its area S, its time-scale parameter T, and the parameter $\lambda(-1<\lambda<\infty)$. Various properties of the pulse may now be listed as follows:

Pulse amplitude. The pulse maximum A_0 occurring at

$$z_{max} = 1/(2+\lambda) , \tag{31a}$$

$$t_{max} = T[\lambda\ln(2+\lambda) - (1+\lambda)\ln(1+\lambda)] , \tag{31b}$$

is given by

$$A_0 = \frac{S}{\pi T}\frac{1}{2(1+\lambda)^{1/2}} . \tag{32}$$

Pulse area. The pulse area is $\int_{-\infty}^{\infty}P(t)\,dt = S$.

Pulse asymmetry. For any value $\lambda \neq 0$, the pulse is not symmetric. Defining

$$P_- = \int_{-\infty}^{t_{max}} P(t)\,dt , \quad P_+ = \int_{t_{max}}^{\infty} P(t)\,dt , \tag{33}$$

and an asymmetry parameter

$$A = \frac{P_+ - P_-}{P_+ + P_-} , \tag{34}$$

one can use Eqs. (28a), (21), and (31a) to obtain

$$A = 1 - (4/\pi)\tan^{-1}[1/(1+\lambda)^{1/2}] . \tag{35}$$

As λ varies from -1 to 0 to ∞, A varies from -1 to 0 to 1.

If $\lambda = 0$, $A = 0$ and the pulse is symmetric. In this limit one obtains from Eqs. (28) and (21)

$$P(t) = (S/2\pi T)\,\mathrm{sech}(t/2T) , \tag{36}$$

$$\dot{z} = dz/d\tau = \tfrac{1}{4}\,\mathrm{sech}^2(t/2T) , \tag{37}$$

which corresponds to both the pulse and transformation [Eq. (7)] used to arrive at the Rosen-Zener solution.

Pulse width. To find the full width at half maximum (FWHM) of the pulse, we seek those values of z, labeled $z_{1/2}$, for which $P(z) = \frac{1}{2}A_0$ and then calculate the corresponding $t_{1/2}$ values using Eq. (28b). From Eqs. (32) and (28a), it follows that $z_{1/2}$ may be obtained as a solution to

$$\frac{1}{4(1+\lambda)^{1/2}} = \frac{[z_{1/2}(1-z_{1/2})]^{1/2}}{1+\lambda z_{1/2}} ,$$

which yields values

$$z_{1/2} = \frac{7\lambda + 8 \pm 4\sqrt{3}\,(1+\lambda)}{\lambda^2 + 16\lambda + 16} . \tag{38}$$

Using Eqs. (38) and (28b) one can evaluate the FWHM in t space as

$$\Delta t_{\mathrm{FWHM}} = T\left[\ln\left(\frac{7\lambda + 8 + 4\sqrt{3}\,(1+\lambda)}{7\lambda + 8 - 4\sqrt{3}\,(1+\lambda)}\right) - (1+\lambda)\ln\left(\frac{\lambda + 8 - 4\sqrt{3}}{\lambda + 8 + 4\sqrt{3}}\right)\right] . \tag{39}$$

TABLE I. Pulse characteristics.

	Amplitude[a] A_0	$t_{\max}/T$	Asymmetry A	FWHM	HAW	Comments
$\lambda=-1+\epsilon$ $(0<\epsilon\ll 1)$	$\frac{1}{2\sqrt{\epsilon}}$	$-\epsilon(1+\ln\epsilon)$	$-1+\frac{4\sqrt{\epsilon}}{\pi}$	$19.1\epsilon T$	$T\ln 2$	Most pulse area for $t<t_{\max}$
$\lambda=0$	$\frac{1}{2}$	0	0	$5.27T$	∞	Symmetric, hyperbolic-secant pulse
$\lambda\gg 1$	$\frac{1}{2\sqrt{\lambda}}$	$1-\ln\lambda$	$1-\frac{4}{\pi\sqrt{\lambda}}$	$19.1T$	$\lambda T\ln 2$	Most pulse area for $t>t_{\max}$

[a] In units of $(S/\pi T)$.

Half-area width (HAW). Another useful parameter is the HAW defined as

$$\Delta t_{\mathrm{HAW}}=|t_H-t_{\max}|\ ,$$

where t_H is the time defined such that half the pulse area lies between t_H and $t_{\max}$. Setting

$$\tfrac{1}{2}S=\pm\int_{t_H}^{t_{\max}}P(t)\,dt$$

and using Eqs. (28a), (21), (31a), (28b), and (31b), one may obtain

$$\Delta t_{\mathrm{HAW}}=T\left|\ln[g^2(1+g^2)^\lambda]-\ln\frac{(2+\lambda)^\lambda}{(1+\lambda)^{1+\lambda}}\right|\ , \qquad (40a)$$

where

$$g=\frac{1-(1+\lambda)^{1/2}}{1+(1+\lambda)^{1/2}}\ . \qquad (40b)$$

For symmetric pulses the HAW is infinite since half of the pulse area lies between $t=-\infty$ and $t=t_{\max}$. However, for very asymmetric pulses ($A\approx -1$ or 1), the HAW is a characteristic long-time scale pulse width.

The pulse properties are summarized in Table I for $\lambda=-1+\epsilon(0<\epsilon\ll 1)$, $\lambda=0$, and $\lambda\gg 1$. For $\lambda=-1+\epsilon$ or $\lambda\gg 1$ the pulses are very asymmetric, containing narrow central peaks and long tails extending out toward $t=-\infty$ and $t=+\infty$, respectively. The case $\lambda=0$ represents the symmetric hyperbolic secant pulse. Less extreme pulse asymmetries are represented in Fig. 1 where pulse shapes $P(t)(\pi T/S)$ are drawn for $\lambda=-0.8$, $\lambda=0$, and $\lambda=5$.

IV. NATURE OF THE SOLUTION

The general solution for the state amplitudes is given by Eqs. (26) along with Eqs. (20), (12c), (13), and (21); the class of pulse envelope functions $f(\tau)$ for which this solution is valid has been described in the previous section. Although Eqs. (26) could be used to determine the transient response to a pulse, we consider only the transition probability induced by the pulse. That is, we take as initial conditions

$$a_1(t=-\infty)=a_1(z=0)=1\ ,$$

$$a_2(t=-\infty)=a_2(z=0)=0\ , \qquad (41)$$

and calculate the probability

$$P_2=|a_2(t=\infty)|^2=|a_2(z=1)|^2 \qquad (42)$$

that the atom has been excited by the pulse. Setting $z=1$ on the rhs of Eq. (26b) and using Eq. (41), we find

$$P_2=\frac{|ab|}{|c|^2}|F(c-a,c-b,1+c;1)|^2$$

$$=\frac{|ab|}{|c|^2}\left|\frac{\Gamma(1+c)\Gamma(1-c+a+b)}{\Gamma(1+a)\Gamma(1+b)}\right|^2\ , \qquad (43)$$

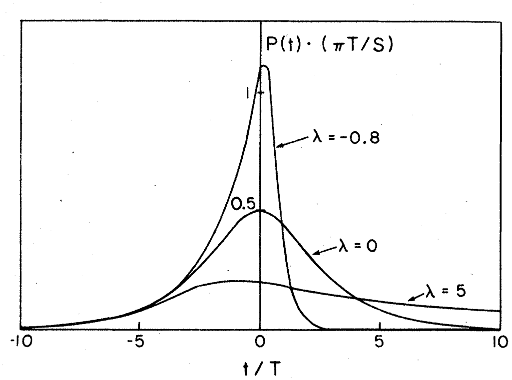

FIG. 1. Graphs of the pulse function $P(t)(\pi T/S)$ versus t/T for $\lambda=-0.8$, $\lambda=0$ (hyperbolic secant), and $\lambda=5$.

where Γ is the gamma function.[3] By substituting the values for a,b,c from Eqs. (20) into Eq. (43) and using some elementary properties of the gamma functions,[3] one may obtain the transition probability

$$P_2=[\sinh^2\delta+\sin^2(S^2-\delta^2)^{1/2}]\,\mathrm{sech}(\pi\alpha)\,\mathrm{sech}(\pi\alpha+2\delta)\,, \tag{44}$$

where S is the pulse area (2c) and

$$\delta=\pi\alpha\lambda/2 \quad (-\pi\alpha/2<\delta<\infty)\,. \tag{45}$$

As a function of S, P_2 increases until $S=|\delta|$ and then oscillates between $\mathrm{sech}(\pi\alpha)\,\mathrm{sech}(\pi\alpha+2\delta)$ $(\sinh^2\delta)$ and $\mathrm{sech}(\pi\alpha)\,\mathrm{sech}(\pi\alpha+2\delta)$ $(\cosh^2\delta)$.

Whereas the pulse was characterized by the parameters T, S, and λ, the transition probability is a function of the detuning parameter $\alpha=\omega T$, the pulse area S, and the quantity $\delta=\pi\alpha\lambda/2$ which reflects the pulse asymmetry through Eq. (35). We now examine the nature of the solution (44) for several specific cases in light of the pulse structure described in the previous section.

$\alpha=0$. For zero detuning, the solution (44) reduces to the well-known solution [see Eq. (5)]

$$P_2=\sin^2 S\,. \tag{46}$$

$\delta=0$. For α arbitrary and $\delta=0$, one must have $\lambda=0$. The pulse is the hyperbolic secant given in Eq. (36) and Eq. (44) becomes

$$P_2=\sin^2 S\,\mathrm{sech}^2\pi\alpha\,, \tag{47}$$

which is the Rosen-Zener solution.[2]

Both solutions (46) and (47) are of the form $P_2=|\mathfrak{F}(\alpha,S)\sin S/S|^2$, where $\mathfrak{F}(\alpha,S)$ is the Fourier transform of the pulse evaluated at frequency α/T. Rosen and Zener conjectured that this result will be valid for arbitrary smooth pulses. For asymmetric pulses, the general solution (44) clearly violates this conjecture. Moreover, even for symmetric smooth pulses, one can show that the conjecture is false by numerically integrating Eqs. (1). However, numerical calculations using Lorentzian and Gaussian pulses do seem to indicate that, for symmetric pulses, there is an oscillatory behavior of P_2 as a function of S, and there are values of the pulse area S for which $P_2=0$. In contrast to this result, the result for asymmetric pulses and nonzero detuning $\delta\neq 0$ always yields $P_2>0$ regardless of the value of S.

$\pi\alpha\ll 1$, $\delta\gtrsim 1$. This limit implies that

$$\lambda=2\delta/\pi\alpha\gg 1$$

which is an asymmetric pulse of amplitude $1/2\sqrt{\lambda}$, FWHM $19.1T$, and HAW $\lambda T\ln 2$ (Table I). The corresponding transition probability (44) is given by

$$P_2=[\sinh^2\delta+\sin^2(S^2-\delta^2)^{1/2}]\,\mathrm{sech}2\delta\,. \tag{48}$$

The solution is graphed as a function of S for $\alpha=0.001$ and two non-zero values of δ in Fig. 2 along with the corresponding $\delta=0$ solution [Eq. (47)] for the hyperbolic-secant pulse. One notes that P_2 oscillates as a function of S about its saturation value of $\frac{1}{2}$ and that the oscillation amplitude decreases with increasing δ (increasing λ). With increasing δ, it is the central peak region that is providing the major contribution to the transition probability since the pulse wing is becoming increasingly adiabatic [i.e., $\Delta t_{\mathrm{HAW}}(\alpha/T)>1$—see Table I]. The sharply asymmetric nature of the central peak *cannot* give rise to the zero-transition probability effect (i.e., $P_2=0$ for $S\neq 0$) that occurs with symmetric pulses. Even though the peak amplitude decreases as $\lambda^{-1/2}$, the transition probability from the central peak region still leads to saturation behavior for $S>\delta$.

$\pi\alpha\approx 1$, $\delta=(\pi\alpha/2)(-1+\epsilon)(0<\epsilon\pi\alpha\ll 1)$. This limit implies $\lambda=-1+\epsilon(\epsilon\ll 1)$ which is an asymmetric pulse of amplitude $1/2\sqrt{\epsilon}$, FWHM $19.1\epsilon T$, and HAW $T\ln 2$. The transition probability is given by

$$P=[\sinh^2(\pi\alpha/2)+\sin^2(S^2-\pi^2\alpha^2/4)^{1/2}]\,\mathrm{sech}\pi\alpha\,. \tag{49}$$

and is plotted in Fig. 3 for $\alpha=\frac{1}{3}$, $\epsilon=0.001$ along with the corresponding P_2 for the $\delta=0$ hyperbolic-

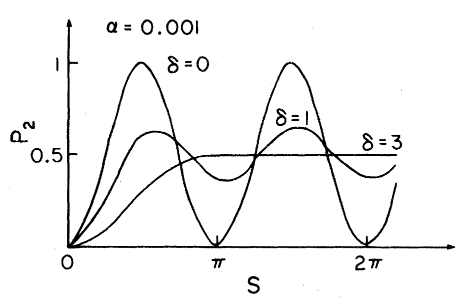

FIG. 2. Graph of the transition probability P_2 as a function of pulse area S for $\alpha=0.001$ and $\delta=0$ $(\lambda=0)$, $\delta=1$ $(\lambda=637)$, and $\delta=3$ $(\lambda=1910)$.

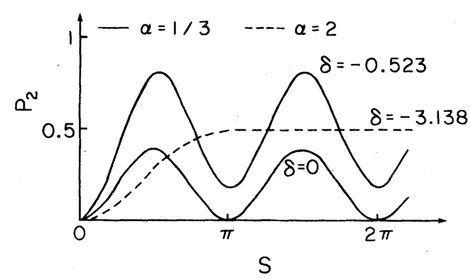

FIG. 3. Graph of P_2 versus S for $\alpha=\frac{1}{3}$, $\delta=0$ $(\lambda=0)$, $\alpha=\frac{1}{3}$, $\delta=-0.523$ $(\lambda=-1+0.001)$, and $\alpha=2$, $\delta=-3.138$ $(\lambda=-1+0.0011)$.

secant pulse. A graph of P_2 versus S for $\alpha = 2$, $\epsilon = 0.0011$ is also drawn (the corresponding hyperbolic-secant solution has amplitude 1.4×10^{-5}) showing its similarity to the $\delta = 3$ curve of Fig. 2.

These graphs are explained by the fact that the hyperbolic-secant pulse is "semiadiabatic" ($\alpha \approx 1$), and becomes increasingly adiabatic with increasing α (with a corresponding decrease of P_2). In contrast, the central peak region of the asymmetric pulse is always sudden with respect to $1/\alpha$. It is true that the long-tail region of the asymmetric pulse is also "semiadiabatic" $[\Delta t_{\mathrm{HAW}}(\alpha/T) \approx 1]$ and this tail gives rise to the oscillations in P_2. However, for $\alpha \gtrsim 2$ the central asymmetric region dominates the contribution to P_2 and a saturation behavior similar to the $\delta = 3$ curve of Fig. 1 results.

$\pi\alpha \approx 1$, $\delta \gg 1$. This limit corresponds to the asymmetric pulse with $\lambda \gg 1$, amplitude $1/2\sqrt{\lambda}$, FWHM $19.1T$, and HAW $\lambda T \ln 2$. The transition probability is given by

$$P_2 = \tfrac{1}{2}(S^2/\delta)\operatorname{sech}\pi\alpha\, e^{-\pi\alpha}, \quad S^2 \ll \delta$$
$$= \tfrac{1}{2}\operatorname{sech}\pi\alpha\, e^{-\pi\alpha}, \quad S^2 \gtrsim \delta^2 . \tag{50}$$

Only the central peak contributes to P_2 since the HAW wing is adiabatic for a detuning $\pi\alpha \approx 1$ [i.e., $\pi(\alpha/T)\cdot\Delta t_{\mathrm{HAW}} \gg 1$]. Thus, the probability is much less than that in the corresponding hyperbolic-secant case (47), except when the pulse area S is strong enough to have the central peak region of the $\lambda \gg 1$ pulse saturate P_2.

$\pi\alpha \gg 1$, $\delta = (\pi\alpha/2)(-1+\epsilon)$, $\epsilon\pi\alpha \approx 1$. This limit corresponds to the asymmetric pulse $\lambda = -1+\epsilon$ ($0 < \epsilon \ll 1$) having amplitude $1/2\sqrt{\epsilon}$, FWHM $19.1\epsilon T$, and HAW $T \ln 2$. The transition probability is given by

$$P_2 = \frac{S^2}{\pi\alpha} e^{-\pi\alpha\epsilon}\operatorname{sech}(\pi\alpha\epsilon), \quad S^2 \ll |\delta|$$
$$= \tfrac{1}{2} e^{-\pi\alpha\epsilon}\operatorname{sech}(\pi\alpha\epsilon), \quad S^2 \gtrsim \delta^2 . \tag{51}$$

The central peak of the pulse is "semisudden" ($\epsilon\pi\alpha \approx 1$) and gives rise to the major contribution to P_2. As in the previous case, for large enough S, the pulse is strong enough to lead to saturation behavior. The transition probability for this case is larger than that for the corresponding hyperbolic-secant pulse since the hyperbolic-secant pulse is adiabatic for a detuning $\pi\alpha \gg 1$.

$\pi\alpha \gg 1$, $2\delta + \pi\alpha \gg 1$, $|\delta| \gg 1$. This adiabatic limit can apply to a wide variety of pulses. The transition probability is given by

$$P_2 = (S^2/|\delta|)e^{-2\pi\alpha}e^{2(|\delta|-\delta)}, \quad S^2 \ll |\delta|$$
$$= e^{-2\pi\alpha}e^{2(|\delta|-\delta)}, \quad S^2 \gtrsim \delta^2 \quad (\lambda \neq 0) \tag{52}$$

and

$$P = 4\sin^2 S\, e^{-2\pi\alpha} \quad (\lambda = 0) . \tag{53}$$

The entire pulse is adiabatic for the conditions given, but the central portions of the asymmetric pulse can still provide the major contribution to the transition probability. The transition probability for the asymmetric pulse does not oscillate as a function of S in contrast to that for the $\lambda = 0$ (hyperbolic-secant pulse). (Actually there is an oscillatory term in P_2 even for the asymmetric case, but its amplitude relative to the background term is negligible.)

To summarize, we have found a new class of functions for which analytic solutions of the two-state problem may be obtained. These positive definite pulses vanish at $t = \pm\infty$. With the exception of the hyperbolic-secant pulse which is a member of this class of functions, the pulses are asymmetric. For the asymmetric pulses, the transition probability does not vanish for any pulse area $S > 0$ (provided $\alpha \neq 0$), a result that differs from the corresponding calculation for symmetric pulses.

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[1]The normalization (4) cannot be maintained for a constant pulse amplitude. In this work we shall consider only pulses which vanish at $t = \pm\infty$ for which Eq. (4) may be satisfied.

[2]N. Rosen and C. Zener, Phys. Rev. 40, 502 (1932).

[3]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (U. S. Government Printing Office, Washington, D. C., 1965).