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Stability criteria for non-Doppler lasers

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Under some conditions many types of laser oscillators are known to produce their output in the form of an infinite periodic pulse train, even when the pump rate and cavity losses are independent of time. For some lasers the underlying cause of the instability is well understood, while for others the source of pulsations remains unknown. Emphasis here is on a coherent pulsation effect which previously has only been analyzed in relation to Doppler-broadened gas lasers. The same effect is shown to apply also to lasers with non-Doppler line broadening and the governing stability criteria are derived.

I. INTRODUCTION

It is ordinarily expected that a laser with time-independent pumping and cavity loss rates will produce its output in the form of a cw beam of light. In fact this conclusion can be demonstrated analytically for a single-mode laser if one uses the standard pair of phenomenological rate equations governing the inversion and photon densities. These equations can be solved numerically, and any perturbations of the steady-state solution invariably damp out after a train of relaxation oscillations.[1,2] The damping rate and pulsation frequency depend on the details of the pump rates and lifetimes. The stability of these equations can also be demonstrated analytically. Against this background one then encounters the real-world result that a wide variety of "stable" lasers actually emit periodic pulse trains. One is thus obliged to consider more general models which might somehow account for the observed instabilities.

This study is primarily concerned with a recently reported model in which the rate equations are replaced by a more general set of semiclassical equations.[3,4] This set has been shown to provide the explanation for the spontaneous pulsations that have been observed in xenon lasers and it probably accounts for instabilities in other low pressure gas lasers as well. The purpose of this work is to extend the model to also include non-Doppler broadening mechanisms and the resulting stability criteria may be relevant to instabilities that have been observed in nongaseous laser media. Semiconductor lasers, in particular, represent important practical systems that are noted for their instabilities. Even single-mode GaAs lasers often produce strong periodic pulsations, and in some recent studies this phenomenon has been attributed to inherent saturable absorption in the semiconductor medium.[5-8] However, the existence of a suitable absorption mechanism has not yet been demonstrated experimentally.[9] While such pulsations are generally regarded as undesirable, they yield useful information about basic processes in the laser material. They might also provide a convenient light source for communications applications requiring steady or frequency-modulated pulse trains, and in any case it would be desirable to understand the physical principles underlying the instabilities.

In Sec. II the stability criteria are derived for one-directional non-Doppler ring lasers and the corresponding results for standing-wave lasers are obtained in Sec. III. Of special interest in Sec. III are new analytic formulas governing the light intensity of standing-wave lasers for arbitrary levels of saturation. Although the physics is quite different, the methods used here are similar to those employed in deriving stability criteria for Doppler-broadened lasers. Basically, the laser is first assumed to be operating in a single cw laser mode. After the intensity and frequency of this mode have been derived, the equations are reexamined to determine whether any infinitesimal sidebands of the saturating mode can also satisfy the oscillation phase condition and exhibit net gain. If any such sidebands exist, it is concluded that the original cw mode is unstable. The result of these calculations is that the semiclassical equations for a non-Doppler inhomogeneously broadened laser are indeed unstable for certain broad ranges of values of the various damping coefficients. Stability contours are presented for easy determination of the stability of specific laser systems.

II. ONE-DIRECTIONAL RING LASERS

The most general approach that is usually needed for studying the characteristics of laser oscillators involves a set of equations governing the ensemble-averaged density matrix coupled to Maxwell's wave equation for the electric field.[4,10,11] These equations are fully adequate for the pulsation phenomenon of interest here. The equations governing the ensemble-averaged density matrix for a four-level system with both Doppler broadening and a distribution of intrinsic resonance frequencies can be written

$$(\partial/\partial t+v\partial/\partial z)\rho_{ab}(v,\omega_a,z,t) = -(i\omega_a+\gamma)\rho_{ab}(v,\omega_a,z,t)-(i\mu/\hbar)E(z,t) \times[\rho_{aa}(v,\omega_a,z,t)-\rho_{bb}(v,\omega_a,z,t)], \quad (1)$$

$$(\partial/\partial t+v\partial/\partial z)\rho_{aa}(v,\omega_a,z,t) = \lambda_a(v,\omega_a,z,t)-\gamma_a\rho_{aa}(v,\omega_a,z,t) + [(i\mu/\hbar)E(z,t)\rho_{ba}(v,\omega_a,z,t)+\text{c.c.}], \quad (2)$$

$$(\partial/\partial t+v\partial/\partial z)\rho_{bb}(v,\omega_a,z,t) = \lambda_b(v,\omega_a,z,t)-\gamma_b\rho_{bb}(v,\omega_a,z,t) - [(i\mu/\hbar)E(z,t)\rho_{ba}(v,\omega_a,z,t)+\text{c.c.}], \quad (3)$$

$$\rho_{ba}(v,\omega_a,z,t)=\rho^*_{ab}(v,\omega_a,z,t), \quad (4)$$

where γ_a and γ_b represent the decay rates of the diagonal matrix elements, $\gamma=(\gamma_a+\gamma_b)/2+\gamma_{ph}$ is the decay rate for the off-diagonal elements with γ_{ph} the rate of phase interrrupting collisions, λ_a and λ_b are the pumping terms, and ω_a is the center frequency of the laser transition for members of an atomic or molecular class a. Maxwell's wave equation for the electric field of a linearly polarized wave in a laser medium can be written as

$$\frac{\partial^2E(z,t)}{\partial z^2}-\mu_0\sigma\frac{\partial E(z,t)}{\partial t}-\mu_0\epsilon_0\frac{\partial^2E(z,t)}{\partial t^2} = \mu_0\frac{\partial^2P(z,t)}{\partial t^2}. \quad (5)$$

The polarization driving this equation can be related back to the off-diagonal matrix elements by

$$P(z,t)=\int_0^\infty\int_{-\infty}^\infty \mu\rho_{ab}(v,\omega_a,z,t)dv\,d\omega_a+\text{c.c.} \quad (6)$$

Equations (1)–(6) are a complete set from which the time and space dependences of the electric field can be determined, subject to the boundary conditions at the resonator mirrors. In a previous stability analysis, only the possibility of Doppler broadening was considered, and the starting equations for that case can be derived from Eqs. (1)–(6) by imagining a very narrow range of center frequencies ω_a near ω_0 and defining new matrix elements and pump rates according to

$$\rho_{ab}(v,z,t)=\int_0^\infty \rho_{ab}(v,\omega_a,z,t)d\omega_a,$$

$$\lambda_a(v,z,t)=\int_0^\infty \lambda_a(v,\omega_a,z,t)d\omega_a,$$

etc. In the present study we are concerned with the alternative case of nongaseous laser media where the inhomogeneous broadening, if significant, will be due to intrinsic differences in the center frequencies of the nearly stationary atoms or molecules. The governing equations for this situation can be derived from Eqs. (1)–(4) by imagining a very narrow range of velocities centered on $v=0$ and defining new matrix elements and pump rates according to

$$\rho_{ab}(\omega_a,z,t)=\int_{-\infty}^\infty \rho_{ab}(v,\omega_a,z,t)dv,$$

$$\lambda_a(\omega_a,z,t)=\int_{-\infty}^\infty \lambda_a(v,\omega_a,z,t)dv,$$

etc. With these substitutions Eqs. (1)–(4) reduce to

$$(\partial/\partial t)\rho_{ab}(\omega_a,z,t)=-(i\omega_a+\gamma)\rho_{ab}(\omega_a,z,t) -(i\mu/\hbar)E(z,t)[\rho_{aa}(\omega_a,z,t) -\rho_{bb}(\omega_a,z,t)], \quad (7)$$

$$(\partial/\partial t)\rho_{aa}(\omega_a,z,t)=\lambda_a(\omega_a,z,t)-\gamma_a\rho_{aa}(\omega_a,z,t) +[(i\mu/\hbar)E(z,t)\rho_{ba}(\omega_a,z,t)+\text{c.c.}], \quad (8)$$

$$(\partial/\partial t)\rho_{bb}(\omega_a,z,t)=\lambda_b(\omega_a,z,t)-\gamma_b\rho_{bb}(\omega_a,z,t) -[(i\mu/\hbar)E(z,t)\rho_{ba}(\omega_a,z,t)+\text{c.c.}], \quad (9)$$

$$\rho_{ba}(\omega_a,z,t)=\rho^*_{ab}(\omega_a,z,t). \quad (10)$$

For a one-directional ring laser both the electric field and polarization are traveling waves. Thus it is helpful to factor the rapid time and space variations from Eqs. (7)–(10) by means of the substitutions

$$E(z,t)=\tfrac{1}{2}E'(t)\exp(ikz-i\omega t)+\text{c.c.}, \quad (11)$$

$$\rho_{ab}(\omega_a,z,t)=P'(\omega_a,t)\exp(ikz-i\omega t)/2\mu, \quad (12)$$

and with the standard rotating-wave approximation, one obtains the set

$$\frac{\partial P'(\omega_a,t)}{\partial t}=i(\omega-\omega_a)P'(\omega_a,t)-\gamma P'(\omega_a,t) -(i\mu^2/\hbar)E'(t)D(\omega_a,t), \quad (13)$$

$$\frac{\partial D(\omega_a,t)}{\partial t}=\lambda_a(\omega_a)-\lambda_b(\omega_a) -\tfrac{1}{2}(\gamma_a+\gamma_b)D(\omega_a,t)-\tfrac{1}{2}(\gamma_a-\gamma_b)M(\omega_a,t) +(1/2\hbar)[iE'(t)P'^*(\omega_a,t)+\text{c.c.}], \quad (14)$$

$$\frac{\partial M(\omega_a, t)}{\partial t} = \lambda_a(\omega_a) + \lambda_b(\omega_a) - \tfrac{1}{2}(\gamma_a + \gamma_b)M(\omega_a, t) - \tfrac{1}{2}(\gamma_a - \gamma_b)D(\omega_a, t)\,, \tag{15}$$

where $D(\omega_a, t) = \rho_{aa}(\omega_a, t) - \rho_{bb}(\omega_a, t)$ is the population difference and $M(\omega_a, t) = \rho_{aa}(\omega_a, t) + \rho_{bb}(\omega_a, t)$ is the population sum.

If the output of the laser is believed to be harmonic in time, it is reasonable to introduce the Fourier expansions

$$E'(t) = \sum_n E_n \exp(-in\Delta\omega t)\,, \tag{16}$$

$$P'(\omega_a, t) = \sum_n P_n(\omega_a) \exp(-in\Delta\omega t)\,, \tag{17}$$

$$D(\omega_a, t) = \sum_n D_n(\omega_a) \exp(-in\Delta\omega t)\,, \tag{18}$$

$$M(\omega_a, t) = \sum_n M_n(\omega_a) \exp(-in\Delta\omega t)\,, \tag{19}$$

where $\Delta\omega$ is the frequency difference between the harmonic components or alternatively the fundamental pulsation frequency. With these substitutions in Eqs. (13)–(15), the equations for the nth frequency harmonic take the form

$$0 = i(\omega + n\Delta\omega - \omega_a)P_n(\omega_a) - \gamma P_n(\omega_a) - \frac{i\mu^2}{\hbar}\sum_j E_{n-j}D_j(\omega_a)\,, \tag{20}$$

$$0 = [\lambda_a(\omega_a) - \lambda_b(\omega_a)]\,\delta_{n0} + in\Delta\omega D_n(\omega_a) - \tfrac{1}{2}(\gamma_a + \gamma_b)D_n(\omega_a) - \tfrac{1}{2}(\gamma_a - \gamma_b)M_n(\omega_a) + \frac{i}{2\hbar}\sum_j [E_{j+n}P_j^*(\omega_a) - E_{j-n}^*P_j(\omega_a)]\,, \tag{21}$$

$$0 = [\lambda_a(\omega_a) + \lambda_b(\omega_a)]\,\delta_{n0} + in\Delta\omega M_n(\omega_a) - \tfrac{1}{2}(\gamma_a + \gamma_b)M_n(\omega_a) - \tfrac{1}{2}(\gamma_a - \gamma_b)D_n(\omega_a)\,. \tag{22}$$

If these same substitutions are used in the wave equation, Eq. (5), and one employs the standard rotating-wave approximation, isolation of the nth harmonic, and integration over z, the result is

$$\left(\frac{i}{2t_c} + (\omega + n\Delta\omega - \Omega)\right)E_n = -\frac{\omega_0 l}{2\epsilon_0 L}\int_0^\infty P_n(\omega_a)\,d\omega_a\,, \tag{23}$$

where l is the length of the amplifying medium and L is the length of the cavity. The cavity lifetime can be related to the distributed losses, the intensity reflectivities, and the round trip time t_{rt} by $t_c = t_{rt}[2\alpha l - \ln(R_1R_2)]^{-1}$.

In principle, the limit cycles associated with the spontaneous pulsations are completely characterized by their harmonic components which are governed by Eqs. (20)–(23). For the present stability analysis the general solutions of this set are not required. It is assumed initially that only a single frequency component is oscillating strongly, and this saturating field may greatly distort the gain and dispersion profiles. It is next inquired whether, in the presence of this saturation-induced distortion, some sideband of the oscillating field can also satisfy the oscillation phase condition and have net gain. If such a sideband exists, the initial single-frequency solution is unstable and pulsations must occur. On the other hand, if no such sideband exists, the single-frequency solution must be stable.

From Eqs. (20)–(23) the fundamental frequency component is described by the equations

$$0 = i(\omega - \omega_a)P_0(\omega_a) - \gamma P_0(\omega_a) - (i\mu^2/\hbar)E_0D_0(\omega_a)\,, \tag{24}$$

$$0 = \lambda_a(\omega_a) - \lambda_b(\omega_a) - \tfrac{1}{2}(\gamma_a + \gamma_b)D_0(\omega_a) - \tfrac{1}{2}(\gamma_a - \gamma_b)M_0(\omega_a) + (i/2\hbar)[E_0P_0^*(\omega_a) - E_0^*P_0(\omega_a)]\,, \tag{25}$$

$$0 = \lambda_a(\omega_a) + \lambda_b(\omega_a) - \tfrac{1}{2}(\gamma_a + \gamma_b)M_0(\omega_a) - \tfrac{1}{2}(\gamma_a - \gamma_b)D_0(\omega_a)\,, \tag{26}$$

$$\left(\frac{i}{2t_c} + (\omega - \Omega)\right)E_0 = -\frac{\omega_0 l}{2\epsilon_0 L}\int_0^\infty P_0(\omega_a)\,d\omega_a\,. \tag{27}$$

Without loss of generality, the phase of the electric field can be set equal to zero (E_0 real), and the polarization amplitude can be separated into its real and imaginary parts by means of the substitution

$$P_0(\omega_a) = C_0(\omega_a) + iS_0(\omega_a)\,. \tag{28}$$

As a result of this substitution Eqs. (24)–(27) can be replaced by the real set

$$0 = (\omega - \omega_a)C_0(\omega_a) - \gamma S_0(\omega_a) - (\mu^2/\hbar)E_0D_0(\omega_a)\,, \tag{29}$$

$$0 = -(\omega - \omega_a)S_0(\omega_a) - \gamma C_0(\omega_a)\,, \tag{30}$$

$$0 = \lambda_a(\omega_a) - \lambda_b(\omega_a) - \tfrac{1}{2}(\gamma_a + \gamma_b)D_0(\omega_a) - \tfrac{1}{2}(\gamma_a - \gamma_b)M_0(\omega_a) + E_0S_0(\omega_a)/\hbar\,, \tag{31}$$

$$0 = \lambda_a(\omega_a) + \lambda_b(\omega_a) - \tfrac{1}{2}(\gamma_a + \gamma_b)M_0(\omega_a) - \tfrac{1}{2}(\gamma_a - \gamma_b)D_0(\omega_a)\,, \tag{32}$$

$$\frac{E_0}{2t_c} = -\frac{\omega_0 l}{2\epsilon_0 L}\int_0^\infty S_0(\omega_a)\,d\omega_a\,, \tag{33}$$

$$(\omega - \Omega)E_0 = -\frac{\omega_0 l}{2\epsilon_0 L}\int_0^\infty C_0(\omega_a)\,d\omega_a\,. \tag{34}$$

If $C_0(\omega_a)$ is eliminated from Eqs. (29) and (30), the result can be written as

$$S_0(\omega_a) = -\frac{\mu^2 E_0 D_0(\omega_a)/\gamma\hbar}{1+[(\omega-\omega_a)/\gamma]^2} . \tag{35}$$

Similarly, Eqs. (31) and (32) may be combined to obtain

$$D_0(\omega_a) = \frac{\gamma_a+\gamma_b}{2\hbar\gamma_a\gamma_b} E_0 S_0(\omega_a) + N(\omega_a) , \tag{36}$$

where the unsaturated population difference is

$$N(\omega_a) = \lambda_a(\omega_a)/\gamma_a - \lambda_b(\omega_a)/\gamma_b . \tag{37}$$

With Eqs. (35) and (36) the population difference and the out-of-phase polarization component can be written explicitly as

$$D_0(\omega_a) = \frac{N(\omega_a)}{1+sI\{1+[(\omega-\omega_a)/\gamma]^2\}^{-1}} , \tag{38}$$

$$S_0(\omega_a) = -\frac{\mu^2 E_0 N(\omega_a)/\gamma\hbar}{1+[(\omega-\omega_a)/\gamma]^2+sI} , \tag{39}$$

where the normalized intensity is

$$sI = \frac{\mu^2 E_0^2}{2\hbar^2}\frac{\gamma_a+\gamma_b}{\gamma\gamma_a\gamma_b} . \tag{40}$$

With Eq. (30) the in-phase component is

$$C_0(\omega_a) = \left(\frac{\omega-\omega_a}{\gamma}\right)\frac{\mu^2 E_0 N(\omega_a)/\gamma\hbar}{1+[(\omega-\omega_a)/\gamma]^2+sI} . \tag{41}$$

Substituting Eqs. (39) and (41) into Eqs. (33) and (34) yields two implicit equations for the laser intensity and frequency. These equations can be written as

$$\frac{1}{r} = \int_0^\infty \frac{N(\omega_a)d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2+sI} \Big/ \int_0^\infty \frac{N(\omega_a)d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2} , \tag{42}$$

$$\frac{2(\omega-\Omega)t_c}{r} = -\int_0^\infty \left(\frac{\omega-\omega_a}{\gamma}\right)\frac{N(\omega_a)d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2+sI} \Big/ \int_0^\infty \frac{N(\omega_a)d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2} , \tag{43}$$

where r is a threshold parameter defined by

$$r = \frac{\mu^2\omega_0 t_c l}{\epsilon_0\hbar\gamma L}\int_0^\infty \frac{N(\omega_a)d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2} . \tag{44}$$

It is clear from Eq. (42) that at threshold ($sI=0$) the value of this parameter is unity.

It remains now to be determined whether any infinitesimal sideband at frequency offset $\Delta\omega$ can also satisfy the phase condition and have a net gain. From Eq. (20) the first-order polarization must satisfy

$$0 = i(\omega+\Delta\omega-\omega_a)P_1(\omega_a) - \gamma P_1(\omega_a) - (i\mu^2/\hbar)E_1 D_0(\omega_a) , \tag{45}$$

where population fluctuations have been assumed to be negligible (see the Appendix). Since the phase of the sideband at a remote time in the past must be irrelevant, E_1 is arbitrarily chosen to be real. Then Eq. (45) can be separated into the two equations

$$0 = (\omega+\Delta\omega-\omega_a)C_1(\omega_a) - \gamma S_1(\omega_a) - (\mu^2/\hbar)E_1 D_0(\omega_a) , \tag{46}$$

$$0 = -(\omega+\Delta\omega-\omega_a)S_1(\omega_a) - \gamma C_1(\omega_a) , \tag{47}$$

where $C_1(\omega_a)$ and $S_1(\omega_a)$ are, respectively, the real and imaginary parts of the sideband polarization. These equations may be combined to yield

$$S_1(\omega_a) = \frac{-\mu^2 E_1 D_0(\omega_a)/\gamma\hbar}{1+[(\omega+\Delta\omega-\omega_a)/\gamma]^2} , \tag{48}$$

$$C_1(\omega_a) = \left(\frac{\omega+\Delta\omega-\omega_a}{\gamma}\right)\frac{\mu^2 E_1 D_0(\omega_a)/\gamma\hbar}{1+[(\omega+\Delta\omega-\omega_a)/\gamma]^2} . \tag{49}$$

But from Eq. (23) the oscillation condition is

$$\frac{E_1}{2t_c} = -\frac{\omega_0 l}{2\epsilon_0 L}\int_0^\infty S_1(\omega_a)d\omega_a , \tag{50}$$

$$(\omega+\Delta\omega-\Omega)E_1 = -\frac{\omega_0 l}{2\epsilon_0 L}\int_0^\infty C_1(\omega_a)d\omega_a . \tag{51}$$

Thus, with Eqs. (38), (44), (48), and (49) these conditions become

$$\frac{1}{r} = \int_0^\infty \frac{1+[(\omega-\omega_a)/\gamma]^2}{1+[(\omega+\Delta\omega-\omega_a)/\gamma]^2}\frac{N(\omega_a)d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2+sI} \Big/ \int_0^\infty \frac{N(\omega_a)d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2} , \tag{52}$$

$$\frac{2(\omega+\Delta\omega-\Omega)t_c}{r} = -\int_0^\infty \left(\frac{\omega+\Delta\omega-\omega_a}{\gamma}\right)\frac{1+[(\omega-\omega_a)/\gamma]^2}{1+[(\omega+\Delta\omega-\omega_a)/\gamma]^2}\frac{N(\omega_a)d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2+sI} \Big/ \int_0^\infty \frac{N(\omega_a)d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2} . \tag{53}$$

To test a laser oscillator for stability, one now must first solve Eqs. (42) and (43) to determine the intensity sI and frequency ω of the oscillating mode. These values are then substituted into Eq. (53) to determine the frequency shift $\Delta\omega$ of a sideband, if any exists, which also satisfies the oscillation phase condition. Finally, all of these values are substituted into Eq. (52) to determine whether the sideband will have gain. Such gain would be indicated by the right-hand side of Eq. (52) being greater than the left-hand side and would be proof of an instability.

The four equations which have just been discussed can also be greatly simplified or solved for many cases. As a first step it may be noticed from Eq. (43) that when the laser cavity is tuned near the atomic center frequency ($\Omega = \omega_0$), the actual laser frequency is also at the atomic resonance ($\omega = \omega_0$), provided that $N(\omega_a)$ is an even function of $\omega_0 - \omega_a$. Thus, Eqs. (42), (52), and (53) reduce to

$$\frac{1}{r} = \int_0^\infty \frac{N(\omega_a)d\omega_a}{1+[(\omega_0-\omega_a)/\gamma]^2 + sI} \bigg/ \int_0^\infty \frac{N(\omega_a)d\omega_a}{1+[(\omega_0-\omega_a)/\gamma]^2}, \tag{54}$$

$$\frac{1}{r} = \int_0^\infty \frac{1+[(\omega_0-\omega_a)/\gamma]^2}{1+[(\omega_0+\Delta\omega-\omega_a)/\gamma]^2} \frac{N(\omega_a)d\omega_a}{1+[(\omega_0-\omega_a)/\gamma]^2+sI} \bigg/ \int_0^\infty \frac{N(\omega_a)d\omega_a}{1+[(\omega_0-\omega_a)/\gamma]^2}, \tag{55}$$

$$\frac{2\Delta\omega t_c}{r} = -\int_0^\infty \left(\frac{\omega_0+\Delta\omega-\omega_a}{\gamma}\right) \frac{1+[(\omega_0-\omega_a)/\gamma]^2}{1+[(\omega_0+\Delta\omega-\omega_a)/\gamma]^2} \frac{N(\omega_a)d\omega_a}{1+[(\omega_0-\omega_a)/\gamma]^2+sI} \bigg/ \int_0^\infty \frac{N(\omega_a)d\omega_a}{1+[(\omega_0-\omega_a)/\gamma]^2}. \tag{56}$$

If the inhomogeneous broadening function $N(\omega_a)$ has a width which is large compared to the homogeneous linewidth $\Delta\omega_h = 2\gamma$, then $N(\omega_a)$ may be replaced by its line center value $N(\omega_0)$ and removed from the integrals. Also the lower integration limit can be extended to minus infinity with negligible error. Thus the integrals in Eq. (54) can be evaluated analytically, and the result is

$$sI = r^2 - 1. \tag{57}$$

With the substitutions $U = \Delta\omega/\gamma$ and $V = (\omega_a - \omega_0)/\gamma$, Eqs. (55) and (56) take the simpler forms

$$\frac{1}{r} = \frac{1}{\pi}\int_{-\infty}^\infty \frac{1+V^2}{1+(V-U)^2}\frac{dV}{r^2+V^2}, \tag{58}$$

$$\frac{2\gamma U t_c}{r} = \frac{1}{\pi}\int_{-\infty}^\infty \frac{(V-U)(1+V^2)}{1+(V-U)^2}\frac{dV}{r^2+V^2}. \tag{59}$$

The integrals in Eqs. (58) and (59) can be evaluated analytically. First it is helpful to simplify Eq. (59) by long division with the divisor $1+(V-U)^2$ and the result is

$$\frac{2\gamma U t_c}{r} = \frac{1}{\pi}\int_{-\infty}^\infty \frac{V+U}{r^2+V^2}dV + \frac{U}{\pi}\int_{-\infty}^\infty \frac{UV-(2+U^2)}{1+(V-U)^2}\frac{dV}{r^2+V^2}. \tag{60}$$

The integrals in Eqs. (58) and (60) are all known, and after some algebra these equations reduce to

$$1 = \frac{(r^2-1)^2+(r^3-r^2+3r+1)U^2+rU^4}{(r^2-1)^2+2(r^2+1)U^2+U^4}, \tag{61}$$

$$2\gamma t_c = \frac{(r^2-1)[(r-1)^2+U^2]}{(r^2-1)^2+2(r^2+1)U^2+U^4}. \tag{62}$$

Equation (62) is a quadratic in U^2 and the solutions are

$$U^2 = \frac{(r^2-1)-2(r^2+1)\delta \pm [(r^2-1)^2-8r(r^2-1)\delta+16r^2\delta^2]^{1/2}}{2\delta}, \tag{63}$$

where $\delta = 2\gamma t_c$. The factor in brackets is a perfect square and the physically interesting solutions ($U^2 > 0$) may be written as

$$\frac{\nu_p}{\Delta\nu_h} = \left(\frac{(r^2-1)-(r+1)^2\delta}{4\delta}\right)^{1/2}, \tag{64}$$

where the pulsation frequency is $\nu_p = \Delta\omega/2\pi$, and the homogeneous linewidth is $\Delta\nu_h = \gamma/\pi$.

Equation (64) is plotted in Fig. 1 for various values of the parameter δ. The importance of these results is that oscillating sidebands are possible which satisfy the phase condition and have the same number of wavelengths between the mirrors as the dominant oscillation mode. It can be shown from Eq. (61) that these sidebands always have net gain. Another feature of the solutions is the fact that pulsations can never occur for $\delta > 1$. It follows from Eq. (64) that the minimum value of r for which pulsations will occur is

$$r_{\min} = (1+\delta)/(1-\delta), \tag{65}$$

and this stability condition is displayed in Fig. 2. Using this plot, one can determine immediately whether or not a given laser will pulse spontaneously.

The results presented here can be understood physically in terms of a spectral hole burning model of gain saturation, and this can be demonstrated by considering some explicit solutions to Eqs. (55) and (56). If the threshold parameter r is eliminated from these equations using Eq. (54), they may be written as

$$1=\int_{-\infty}^{\infty}\frac{1+V^2}{1+(V-U)^2}\frac{N(V)dV}{1+V^2+sI}\bigg/\int_{-\infty}^{\infty}\frac{N(V)dV}{1+V^2+sI}\,, \tag{66}$$

$$-U\delta=\int_{-\infty}^{\infty}\frac{(U-V)(1+V^2)}{1+(V-U)^2}\frac{N(V)dV}{1+V^2+sI}\bigg/\int_{-\infty}^{\infty}\frac{N(V)dV}{1+V^2+sI}\,. \tag{67}$$

To be specific, it is assumed that the inhomogeneous frequency distribution of the active atoms or molecules is a Gaussian of full width at half maximum $\Delta\nu_i$, and it is now convenient to normalize the frequency with respect to the inhomogeneous width using the relation $x=2(\nu-\nu_0)(\ln 2)^{1/2}/\Delta\nu_i$ and the damping ratio $\epsilon=\Delta\nu_h(\ln 2)^{1/2}/\Delta\nu_i$. With these definitions Eqs. (66) and (67) can be written as

$$1=\int_{-\infty}^{\infty}\frac{1+V^2}{1+(V-x/\epsilon)^2}\frac{\exp(-\epsilon^2V^2)dV}{1+V^2+sI}\bigg/\int_{-\infty}^{\infty}\frac{\exp(-\epsilon^2V^2)dV}{1+V^2+sI}\,, \tag{68}$$

$$\frac{-x\delta}{\epsilon}=-\int_{-\infty}^{\infty}\frac{(V-x/\epsilon)(1+V^2)}{1+(V-x/\epsilon)^2}\frac{\exp(-\epsilon^2V^2)dV}{1+V^2+sI}\bigg/\int_{-\infty}^{\infty}\frac{\exp(-\epsilon^2V^2)dV}{1+V^2+sI}\,. \tag{69}$$

The right-hand sides of Eqs. (68) and (69) are plotted, respectively, in Figs. 3 and 4 for the intensities $sI=0$ and 10 with the damping ratio $\epsilon=0.1$. These curves show the gain and dispersion that would be felt by an infinitesimal field detuned from line center by the amount x. The $sI=0$ curves give the gain and dispersion when the line-center mode is exactly at threshold. With higher levels of intensity a hole is burned in the gain curve to the depth of unity (gain always equals loss for the oscillating mode), and a glitch is introduced in the dispersion curve. Also shown in Fig. 4 is a negatively sloping straight line corresponding to the left-hand side of Eq. (69) with the values $\epsilon=0.1$, $\delta=0.1$. The points at which this straight line intersects the dispersion curve yield the frequencies of any sidebands having the same number of wavelengths between the mirrors as the saturating mode. It is clear from the figure that such sidebands can exist, and from Fig. 3 it is apparent that the sidebands have net gain. Thus, the spontaneous pulsations can be interpreted as a consequence of spectral hole burning and the associated distortion of the dispersion properties of the laser medium.

The spontaneous pulsations discussed here may appropriately be referred to as superoscillations, because they represent a low-frequency oscillation (typically 10^6–10^9 Hz) superimposed on the ordinary optical frequency fluctuations of the electric field. These superoscillations also bear a close relationship to super-radiant (sometimes

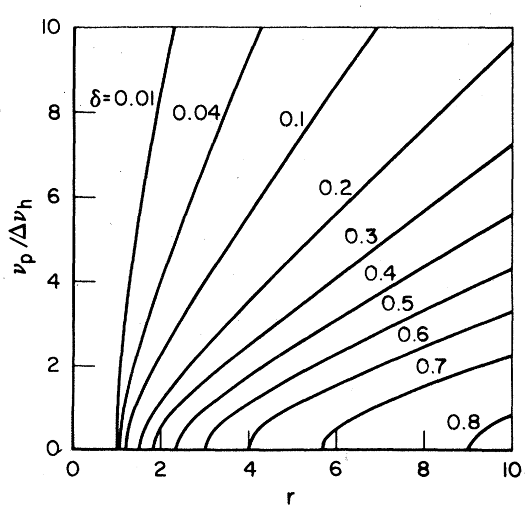

FIG. 1. Normalized pulsation frequency in a ring laser as a function of the threshold parameter r for various values of δ.

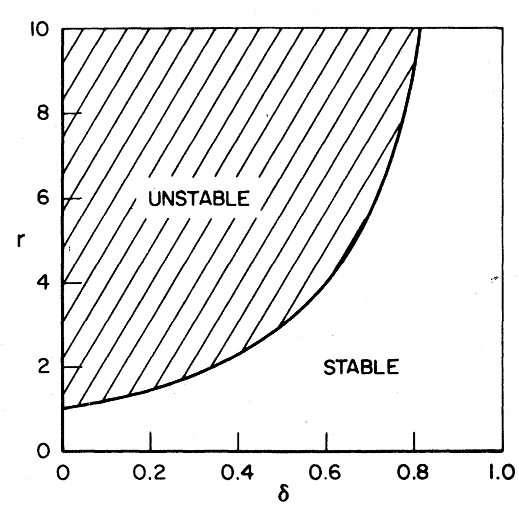

FIG. 2. Instability threshold in a ring laser in units of r as a function of δ. With δ greater than unity, no instability is possible.

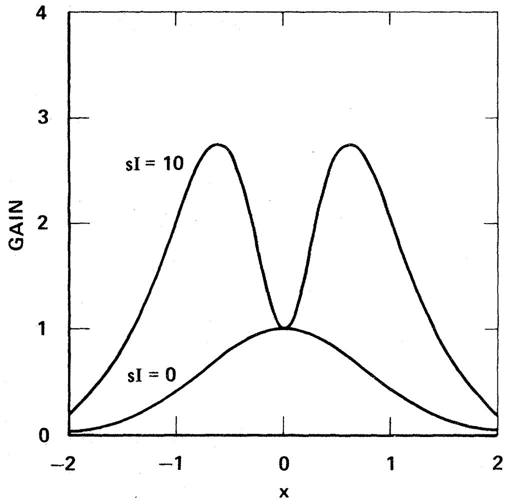

FIG. 3. Gain profiles [right-hand side of Eq. (68)] seen by a small amplitude signal propagating in a laser medium which is saturated by a line center signal of intensity sI. As the excitation level is increased the intensity increases and a hole is burned in the gain spectrum.

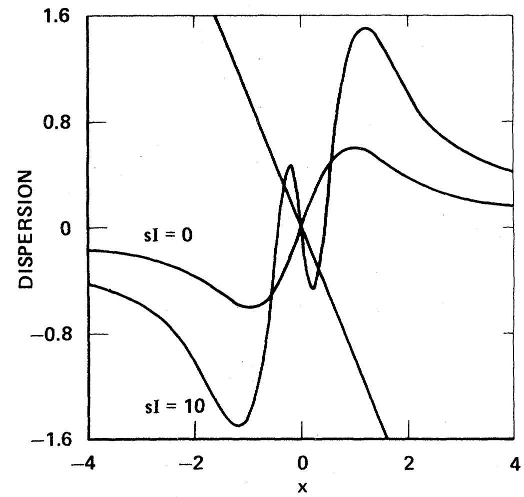

FIG. 4. Dispersion profiles [right-hand side of Eq. (69)] seen by a small amplitude signal propagating in laser medium having the gain profile shown in Fig. 3. The possible sideband frequencies having the same number of wavelengths as the saturating mode are found as the intersections of the dispersion curves and a straight line of slope $(-\delta/\epsilon)$.

called superfluorescent) pulsations. In a typical superfluorescence experiment, a system of atoms or molecules is raised to an excited state by means of a brief light pulse from a modulated laser. Following this excitation, the amplified fluorescence may be able to phase up all of the emitting dipoles if the gain is high enough and the coherence time long enough, and the result is a short radiation pulse. It should be noted, however, that there is no fundamental reason why cw excitation could not be employed in superfluorescence studies provided the initial field can be held to a sufficiently low level. An important result of this study is that an individual pulse in a superoscillation train may be equivalent to an ordinary superflourescent pulse. In both cases a weak radiation field in a laser amplifier builds up into a pulse which is short compared to the coherence time of the amplifying medium. This conclusion can be illustrated by displaying for comparison a superoscillation pulse obtained in xenon[3,12] and a superfluorescence pulse obtained in HF.[13] These results are shown in Fig. 5, and it is clear that except for the time scale the pulses are closely similar. It is also significant that both pulsations have been modeled by essentially the same Maxwell-Schrödinger semiclassical equations.[3,14] Increased feedback was found to reduce the superfluorescence pulse delay[13] and to increase the superoscillation frequency.[12] The only substantial difference between the experiments represented in Fig. 5 concerns the time duration of the pump.

It is now appropriate to inquire whether the spontaneous pulsations observed in semiconductor lasers might also be interpretable in terms of a semiclassical model, and in this regard a possible limitation of the more conventional saturable absorption rate-equation interpretations may be

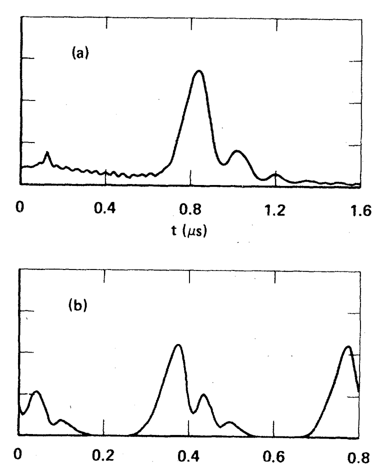

FIG. 5. Comparison of a super-radiant pulse in HF from Ref. 13, (a), with a superoscillation pulse in xenon from Ref. 3, (b). In the super-radiant case the optical excitation pulse is represented by the leftmost blip in part (a), while the laser of part (b) uses cw electron impact excitation.

noted. Rate equations are only applicable in light-matter interactions if all field and population quantities vary slowly compared to the coherence time. However, in spontaneously pulsing semiconductor lasers the field risetimes, photon lifetimes, etc., may be on the order of a picosecond (published data are often acknowledged to be limited by detection and display electronics). On the other hand, the coherence time in diode lasers is also on the order of a picosecond, so the rate-equation approximations should be used with caution. It will be interesting to see whether with sufficient detection resolution the output pulses from diode lasers having short and quasi-cw pumping will resemble the characteristic inhomogeneous semiclassical waveforms shown in Figs. 5(a) and 5(b), respectively, rather than the relatively simple pulse shapes predicted by the rate-equation models.

III. THE STANDING-WAVE LASER

In some practical lasers the electromagnetic field distribution is best described as a standing wave. For an exact treatment of the output characteristics including possible pulsation phenomena, one is obliged to take account of spatial hole burning. As before, the steady-state behavior is derived first, and then the possible existence of an infinitesimal sideband is tested. A useful aspect of these results is a set of simple new analytic formulas for the intensity and frequency characteristics of a steady-state laser. Previous studies of spatial hole burning have either been mainly numerical or have emphasized the operating regime near threshold.

Equations (1)–(6) are again the starting point for this stability calculation, but the initial substitutions involve the standing-wave field and polarization functions

$$E(z,t)=\tfrac{1}{2}\sin(kz)E'(t)\exp(-i\omega t)+\text{c.c.}\,, \tag{70}$$

$$\rho_{ab}(\omega_a,z,t)=P'(\omega_a,z,t)\exp(-i\omega t)/2\mu\,. \tag{71}$$

With these substitutions the density matrix equations take the form

$$(\partial/\partial t)P'(\omega_a,z,t)=i(\omega-\omega_a)P'(\omega_a,z,t)-\gamma P'(\omega_a,z,t)-(i\mu^2/\hbar)\sin(kz)E'(t)D(\omega_a,z,t)\,, \tag{72}$$

$$\begin{aligned}(\partial/\partial t)D(\omega_a,z,t)=\lambda_a(\omega_a)-\lambda_b(\omega_a)-\tfrac{1}{2}(\gamma_a+\gamma_b)D(\omega_a,z,t)-\tfrac{1}{2}(\gamma_a-\gamma_b)M(\omega_a,z,t)\\+[\sin(kz)/2\hbar][iE'(t)P'^*(\omega_a,z,t)+\text{c.c.}]\,,\end{aligned} \tag{73}$$

$$(\partial/\partial t)M(\omega_a,z,t)=\lambda_a(\omega_a)+\lambda_b(\omega_a)-\tfrac{1}{2}(\gamma_a+\gamma_b)M(\omega_a,z,t)-\tfrac{1}{2}(\gamma_a-\gamma_b)D(\omega_a,z,t)\,. \tag{74}$$

To investigate the possibility of stable limit cycles, these equations are expanded with harmonic expansions similar to those indicated in Eqs. (16)–(19). The results are the set of algebraic equations

$$0=i(\omega+n\Delta\omega-\omega_a)P_n(\omega_a,z)-\gamma P_n(\omega_a,z)-\frac{i\mu^2}{\hbar}\sin(kz)\sum_j E_{n-j}D_j(\omega_a,z)\,, \tag{75}$$

$$\begin{aligned}0=[\lambda_a(\omega_a)-\lambda_b(\omega_a)]\delta_{n0}+in\Delta\omega D_n(\omega_a,z)-\tfrac{1}{2}(\gamma_a+\gamma_b)D_n(\omega_a,z)-\tfrac{1}{2}(\gamma_a-\gamma_b)M_n(\omega_a,z)\\+\frac{i\sin(kz)}{2\hbar}\sum_j[E_{j+n}P_j^*(\omega_a,z)-E_{j-n}^*P_j(\omega_a,z)]\,,\end{aligned} \tag{76}$$

$$0=[\lambda_a(\omega_a)+\lambda_b(\omega_a)]\delta_{n0}+in\Delta\omega M_n(\omega_a,z)-\tfrac{1}{2}(\gamma_a+\gamma_b)M_n(\omega_a,z)-\tfrac{1}{2}(\gamma_a-\gamma_b)D_n(\omega_a,z)\,. \tag{77}$$

With the same substitutions in the wave equation, use of the rotating-wave approximation, isolation of the nth harmonic, multiplication by $\sin(kz)$, and integration over z, one obtains

$$\left(\frac{i}{2t_c}+(\omega+n\Delta\omega-\Omega)\right)E_n=-\frac{\omega_0}{\epsilon_0 L}\int_0^\infty\int_0^l\sin(kz)P_n(\omega_a,z)dz\,d\omega_a\,. \tag{78}$$

From Eqs. (75)–(78) the equations governing the fundamental $n=0$ frequency component can be written as

$$0=(\omega-\omega_a)C_0(\omega_a,z)-\gamma S_0(\omega_a,z)-(\mu^2/\hbar)\sin(kz)E_0D_0(\omega_a,z)\,, \tag{79}$$

$$0=-(\omega-\omega_a)S_0(\omega_a,z)-\gamma C_0(\omega_a,z)\,, \tag{80}$$

$$0=\lambda_a(\omega_a)-\lambda_b(\omega_a)-\tfrac{1}{2}(\gamma_a+\gamma_b)D_0(\omega_a,z)-\tfrac{1}{2}(\gamma_a-\gamma_b)M_0(\omega_a,z)+[\sin(kz)/\hbar]E_0S_0(\omega_a,z)\,, \tag{81}$$

$$0=\lambda_a(\omega_a)+\lambda_b(\omega_a)-\tfrac{1}{2}(\gamma_a+\gamma_b)M_0(\omega_a,z)-\tfrac{1}{2}(\gamma_a-\gamma_b)D_0(\omega_a,z)\,, \tag{82}$$

$$\frac{E_0}{2t_c}=-\frac{\omega_0}{\epsilon_0 L}\int_0^\infty\int_0^l\sin(kz)S_0(\omega_a,z)dz\,d\omega_a\,, \tag{83}$$

$$(\omega-\Omega)E_0=-\frac{\omega_0}{\epsilon_0 L}\int_0^\infty\int_0^l \sin(kz)C_0(\omega_a,z)dz\,d\omega_a\,, \tag{84}$$

where $C_0(\omega_a,z)$ and $S_0(\omega_a,z)$ are, respectively, the real and imaginary parts of the polarization. These equations may be combined in a manner similar to that used in Sec. II to obtain the set

$$\frac{1}{r}=\frac{2}{l}\int_0^\infty\int_0^l\frac{\sin^2(kz)N(\omega_a)dz\,d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2+4\sin^2(kz)sI}\Big/\int_0^\infty\frac{N(\omega_a)d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2}\,, \tag{85}$$

$$\frac{2(\omega-\Omega)t_c}{r}=-\frac{2}{l}\int_0^\infty\int_0^l\left(\frac{\omega-\omega_a}{\gamma}\right)\frac{\sin^2(kz)N(\omega_a)dz\,d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2+4\sin^2(kz)sI}\Big/\int_0^\infty\frac{N(\omega_a)d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2}\,, \tag{86}$$

where the normalized one-way intensity is

$$sI=\frac{\mu^2E_0^2}{8\hbar^2}\frac{\gamma_a+\gamma_b}{\gamma\gamma_a\gamma_b}\,, \tag{87}$$

and the threshold parameter is

$$r=\frac{t_c\omega_0\mu^2 l}{\epsilon_0\gamma\hbar L}\int_0^\infty\frac{N(\omega_a)d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2}\,. \tag{88}$$

Equations (85) and (86) are the governing formulas for the intensity and frequency characteristics of a laser oscillator including the effects of longitudinal spatial hole burning. The integrals can be evaluated analytically for the cases of greatest practical interest. We consider first the case of homogeneous line broadening. In the homogeneous limit the function $N(\omega_a)$ is very narrow compared to the Lorentizian terms and is centered at the frequency ω_0. Thus Eqs. (85) and (86) can be written as

$$\frac{1}{r}=\frac{2}{l}\int_0^l\frac{\sin^2(kz)dz}{1+4\sin^2(kz)sI\{1+[(\omega-\omega_0)/\gamma]^2\}^{-1}}\,, \tag{89}$$

$$\frac{2(\omega-\Omega)t_c}{r}=-\frac{2}{l}\left(\frac{\omega-\omega_0}{\gamma}\right)\times\int_0^l\frac{\sin^2(kz)dz}{1+4\sin^2(kz)sI\{1+[(\omega-\omega_0)/\gamma]^2\}^{-1}}\,. \tag{90}$$

With some simple manipulations these integrals may be put into a standard form and averaged over a wavelength. The results are[15]

$$\frac{1}{r}=2\left[1+\frac{4sI}{1+[(\omega-\omega_0)/\gamma]^2}+\left(1+\frac{4sI}{1+[(\omega-\omega_0)/\gamma]^2}\right)^{1/2}\right]^{-1}, \tag{91}$$

$$2(\omega-\Omega)t_c=(\omega_0-\omega)/\gamma\,. \tag{92}$$

Equations (91) and (92) can be readily solved for the intensity and frequency of the laser mode. Thus Eq. (91) is basically a quadratic equation, and it can be inverted to obtain

$$\frac{sI}{1+[(\omega-\omega_0)/\gamma]^2}=\frac{4r-1-(8r+1)^{1/2}}{8}\,. \tag{93}$$

Since the product $2\gamma t_c$ is usually large compared to unity, Eq. (92) implies a slight mode pulling toward gain center. The explicit solution is

$$\omega=(\Omega+\omega_0/2\gamma t_c)(1+1/2\gamma t_c)^{-1}\,, \tag{94}$$

which may be used to eliminate ω from Eq. (93). If the laser is tuned to line center ($\omega=\Omega=\omega_0$), Eq. (93) is simply

$$sI=\tfrac{1}{8}[4r-1-(8r+1)^{1/2}]\,. \tag{95}$$

Equation (95) is plotted in Fig. 6 and compared with the more conventional rate-equation result $sI=(r-1)/2$, where the factor of 2 accounts for saturation by the right and left traveling waves. It is evident from this comparison that for any given value of r, spatial hole burning has the effect of reducing the intensity by about 20 or 30 percent. It is reasonable that such a reduction should occur since the standing-wave fields cannot interact effectively with atoms that are situated near the field nodes. The largest discrepancy occurs near threshold ($sI\ll 1$) where Eq. (95) reduces to $sI=(r-1)/3$. Thus Eq. (95) provides a substantial improvement over the standard rate-equation intensity formula, but it is not significantly more difficult to apply.

The other limit of practical interest concerns lasers which are predominantly inhomogeneously broadened. The inhomogeneous limit also applies directly to the spontaneous pulsation phenomenon. In this case the function $N(\omega_a)$ is broad compared to the Lorentzians in Eq. (85) and to first order may be removed from the integrals leaving

$$\frac{1}{r}=\frac{2}{l}\int_0^\infty\int_0^l\frac{\sin^2(kz)dz\,d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2+4\sin^2(kz)sI}\Big/\int_0^\infty\frac{d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2}\,,$$
$$=\frac{2}{l}\int_0^l\frac{\sin^2(kz)dz}{[1+4sI\sin^2(kz)]^{1/2}}\,, \tag{96}$$

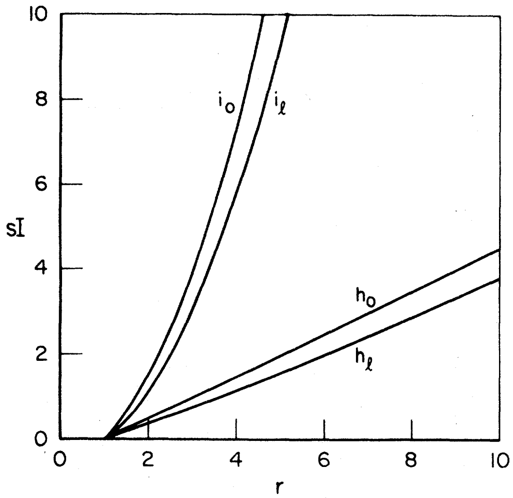

FIG. 6. Normalized internal one-way intensity sI versus the threshold parameter r for various types of lasers. The curve labeled h_0 is the conventional result $sI=\frac{1}{2}(r-1)$ for homogeneously broadened lasers neglecting longitudinal spatial hole burning, while h_l denotes the homogeneously broadened laser of Eq. (95) with spatial hole burning. Similarly i_0 represents the conventional inhomogeneously broadened laser governed by $sI=\frac{1}{2}(r^2-1)$, and i_l is from Eq. (98).

where the lower limit of the frequency integration has been extended to minus infinity. In this approximation the numerator frequency integration in Eq. (86) vanishes, since the integrand becomes an odd function of the frequency difference $(\omega - \omega_a)$. A more accurate evaluation of Eq. (86) would require detailed information about the function $N(\omega_a)$, and for most purposes mode pulling can simply be ignored.

The remaining integrand in Eq. (96) may be averaged and integrated using a tabulated integral, and the result is[16]

$$1/r=(2/\pi)B(\tfrac{3}{2},\tfrac{1}{2})F(\tfrac{1}{2},\tfrac{3}{2};2;-4sI)\,, \qquad (97)$$

where B is a beta function and F is a hypergeometric function. But the value of this beta function is[17] $\pi/2$, so Eq. (97) is simply

$$1/r=F(\tfrac{1}{2},\tfrac{3}{2};2;-4sI)\,. \qquad (98)$$

Equation (98) is plotted in Fig. 6 for comparison with the more familiar result $sI=(r^2-1)/2$, which is obtained for an inhomogeneously broadened laser neglecting longitudinal spatial hole burning. As in the case of homogeneous broadening, spatial hole burning has the effect of reducing the intensity. The largest percent reduction occurs near threshold, where Eq. (98) may be readily shown to have the useful lowest-order approximation $sI \simeq \frac{1}{3}(r^2-1)$.

As a final step in the stability analysis, it is necessary to determine the conditions under which the first-order sideband can satisfy the oscillation conditions. From Eqs. (75) and (78) the equations governing this sideband can be written as

$$0=(\omega+\Delta\omega-\omega_a)C_1(\omega_a,z)-\gamma S_1(\omega_a,z) -(\mu^2/\hbar)\sin(kz)E_1D_0(\omega_a,z)\,, \qquad (99)$$

$$0=-(\omega+\Delta\omega-\omega_a)S_1(\omega_a,z)-\gamma C_1(\omega_a,z)\,, \qquad (100)$$

$$\frac{E_1}{2t_c}=-\frac{\omega_0}{\epsilon_0 L}\int_0^\infty\int_0^l \sin(kz)S_1(\omega_a,z)dz\,d\omega_a\,, \qquad (101)$$

$$(\omega+\Delta\omega-\Omega)E_1=-\frac{\omega_0}{\epsilon_0 L}\int_0^\infty\int_0^l \sin(kz)C_1(\omega_a,z)dz\,d\omega_a\,. \qquad (102)$$

These equations may be combined to obtain the set

$$\frac{1}{r}=\frac{2}{l}\int_0^\infty\int_0^l \frac{\sin^2(kz)N(\omega_a)dz\,d\omega_a}{\{1+[(\omega+\Delta\omega-\omega_a)/\gamma]^2\}(1+4\sin^2(kz)sI\{1+[(\omega-\omega_a)/\gamma]^2\}^{-1})} \Bigg/ \int_0^\infty \frac{N(\omega_a)d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2}\,, \qquad (103)$$

$$\frac{2(\omega+\Delta\omega-\Omega)t_c}{r} =-\frac{2}{l}\int_0^\infty\int_0^l \frac{[(\omega+\Delta\omega-\omega_a)/\gamma]\sin^2(kz)N(\omega_a)dz\,d\omega_a}{\{1+[(\omega+\Delta\omega-\omega_a)/\gamma]^2\}(1+4\sin^2(kz)sI\{1+[(\omega-\omega_a)/\gamma]^2\}^{-1})} \Bigg/ \int_0^\infty \frac{N(\omega_a)d\omega_a}{1+[(\omega-\omega_a)/\gamma]^2}\,. \qquad (104)$$

Equations (85), (86), (103), and (104) now constitute a complete set with which one can test the stability of a longitudinal mode in a laser oscillator. First the intensity and frequency of the mode are determined using Eqs. (85) and (86). Then Eq. (104) is used to see whether in the presence of the saturating mode a sideband can also satisfy the oscillation phase condition. Finally, the gain of the sideband is tested using Eq. (103).

While some simplifications of the stability equations are possible, closed form solutions do not seem to exist except in the stable homogeneous limit. If mode pulling is neglected in the more interesting inhomogeneous limit, the frequency of the saturating mode is equal to the empty cavity frequency, and the intensity is given implicitly by Eq. (98). This result is also obtained rigorously if the laser is tuned near line center with a symmetric gain curve. In the same limit Eqs. (103) and (104) reduce to

$$\frac{1}{r}=\frac{2}{\pi l}\int_{-\infty}^{\infty}\int_{0}^{l}\frac{\sin^2(kz)dz\,dV}{[1+(V-U)^2][1+4\sin^2(kz)sI(1+V^2)^{-1}]}, \tag{105}$$

$$\frac{2\gamma t_c U}{r}=\frac{2}{\pi l}\int_{-\infty}^{\infty}\int_{0}^{l}\frac{(V-U)\sin^2 dz\,dV}{[1+(V-U)^2][1+4\sin^2(kz)sI(1+V^2)^{-1}]}, \tag{106}$$

where as before $U=\Delta\omega/\gamma$ and $V=(\omega_a-\omega)/\gamma$. These z integrations are similar to those in Eqs. (89) and (90) and the results are

$$\frac{1}{r}=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+(V-U)^2}\left[1+\frac{4sI}{1+V^2}+\left(1+\frac{4sI}{1+V^2}\right)^{1/2}\right]^{-1}dV, \tag{107}$$

$$\frac{2\gamma t_c U}{r}=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{V-U}{1+(V-U)^2}\left[1+\frac{4sI}{1+V^2}+\left(1+\frac{4sI}{1+V^2}\right)^{1/2}\right]^{-1}dV. \tag{108}$$

Equations (98), (107), and (108) can be evaluated numerically, and the resulting stability plot is given in Fig. 7. Perhaps the most notable feature of Fig. 7 is its close resemblance to Fig. 2, the stability plot for a ring laser. In both cases the stability boundary starts out at $r=1$ and approaches asymptotically to $\delta=1$. However, for any given value of δ the pulsation threshold r_{min} is slightly higher for the standing-wave laser than for the ring laser. Thus the spatial variations of the saturation in a standing-wave laser increase the laser stability with respect to coherent pulsations in much the same way that ordinary relaxation oscillations tend to be damped out by a radial field profile or longitudinal variations in the beam spot size.[18,19]

IV. DISCUSSION

A spontaneous coherence effect can cause a laser with cw pumping to produce its output in the form of an infinite periodic pulse train. In previous experiments and calculations this effect was shown to apply to Doppler-broadened gas lasers. In the present study it has been demonstrated that the same phenomenon must also be manifested by lasers subject to non-Doppler line broadening mechanisms, and both standing-wave lasers and one-directional ring lasers have been considered in the analysis. In short, the onset of the pulsations occurs close to the laser threshold for small values of a parameter $\delta=2\gamma t_c$. For larger values of δ, excess gain and saturation are required, and with δ greater than unity the pulsations cannot occur. The possible instability of specific laser systems can be readily assessed by reference to to Figs. 2 and 7.

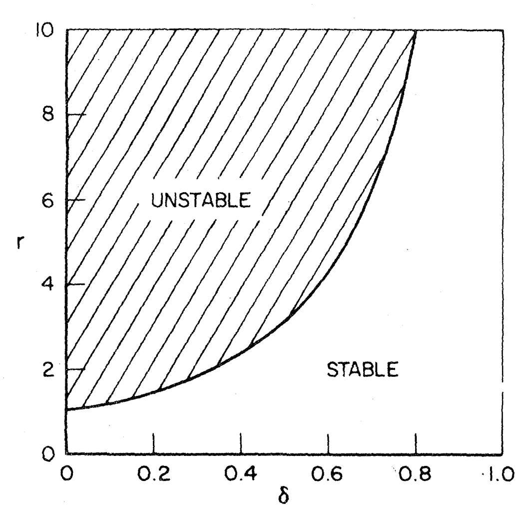

FIG. 7. Instability threshold in a standing-wave laser in units of r as a function of δ.

The source of the instability can be understood physically by frequency domain arguments involving spectral hole burning in the amplifying medium. In the presence of an intense oscillation mode the dispersion profile of an inhomogeneously broadened medium may be distorted to such an extent that sidebands of the saturating field can also satisfy the oscillation phase condition. Since these sidebands are displaced from the minimum of the hole in the gain profile, they experience net gain and pulsations result. In the time domain these "superoscillations" are the oscillator analog of super-radiance in laser amplifiers. An individual burst in the periodic output resembles a typical super-radiant burst, and the main experimental differences concern the level of optical feedback and the presence of continuous pumping.

Besides its intrinsic physical interest, the spontaneous pulsation instability may also have practical consequences. For nonlinear optical applications it is often desirable to have a laser's output condensed into short pulses, and such pulses are also commonly required in metrology and communications. One of the principal application areas for semiconductor lasers involves fiber optical communications, and for this purpose it is particularly important that the high-frequency laser behavior be well understood. The spontaneous pulsations in these lasers typically occur just in the frequency range (200–2000 MHz) that is of interest for optical communication.[7] A correct physical understanding of these pulsations would obviously increase the likelihood that they could be eliminated. Alternatively, suitable control of the pulsation phenomenon might lead to its direct employment in pulse modulation schemes.

ACKNOWLEDGMENT

The author is pleased to acknowledge the valuable assistance of Kendall C. Reyzer in checking all calculations.

APPENDIX: POPULATION FLUCTUATIONS

In developing the stability criteria used in this work and in a previous study[4] it has been assumed that the population difference is essentially constant in time and hence representable by a parameter D_0. This assumption would seem reasonable since the only saturating field is the constant E_0. However, in determining the gain of the pulsation sidebands, even infinitesimal population fluctuations may not be negligible. This appendix describes briefly the procedure that one could employ to study fluctuation effects.

From Eq. (23) the first sideband E_1 of the electric field is driven only by the polarization sideband P_1, and from Eq. (20) this polarization component is in principle related to all components of the pulsing population. These population components can in turn be expressed in terms of the fields by means of Eqs. (20)–(22). First, Eqs. (20) and (22) yield the polarization and population sum components:

$$P_n(\omega_a) = \frac{-(i\mu^2/\hbar)\sum_j E_{n-j}D_j(\omega_a)}{\gamma - i(\omega + n\Delta\omega - \omega_a)}\,, \tag{A1}$$

$$M_n(\omega_a) = \frac{[\lambda_a(\omega_a)+\lambda_b(\omega_a)]\delta_{n0} - D_n(\omega_a)(\gamma_a-\gamma_b)/2}{(\gamma_a+\gamma_b)/2 - in\Delta\omega}\,. \tag{A2}$$

These components may be substituted into Eq. (21) and the results written in the form

$$0 = \frac{2[\lambda_a(\omega_a)\gamma_b - \lambda_b(\omega_a)\gamma_a]}{\gamma_a+\gamma_b}\delta_{n0} + \left\{ in\Delta\omega - \frac{\gamma_a+\gamma_b}{2} + \frac{[(\gamma_a-\gamma_b)/2]^2}{(\gamma_a+\gamma_b)/2 - in\Delta\omega}\right\} D_n(\omega_a)$$
$$- \frac{\mu^2}{2\hbar^2}\sum_m\sum_j \left[\frac{E_{j+n}E^*_{j+m}D_m(\omega_a)}{i(\omega + j\Delta\omega - \omega_a)+\gamma} - \frac{E^*_{j-n}E_{j-m}D_m(\omega_a)}{i(\omega+j\Delta\omega-\omega_a)-\gamma}\right], \tag{A3}$$

where use has been made of the relationship $D^*_m = D_{-m}$, which is required for a real population difference.

Equation (A3) can be solved for any particular population component, and the result for the nth component is

$$D_n(\omega_a) = \frac{\dfrac{2[\lambda_a(\omega_a)\gamma_b - \lambda_b(\omega_a)\gamma_a]}{\gamma_a+\gamma_b}\delta_{n0} - \dfrac{\mu^2}{2\hbar^2}\sum_{m\neq n}\sum_j\left[\dfrac{E_{j+n}E^*_{j+m}}{\gamma + i(\omega+j\Delta\omega-\omega_a)} + \dfrac{E^*_{j-n}E_{j-m}}{\gamma - i(\omega+j\Delta\omega-\omega_a)}\right]D_m(\omega_a)}{\dfrac{\gamma_a+\gamma_b}{2} - in\Delta\omega - \dfrac{[(\gamma_a-\gamma_b)/2]^2}{(\gamma_a+\gamma_b)/2 - in\Delta\omega} + \dfrac{\mu^2}{2\hbar^2}\sum_j\left[\dfrac{|E_{j+n}|^2}{\gamma + i(\omega+j\Delta\omega-\omega_a)} + \dfrac{|E_{j-n}|^2}{\gamma - i(\omega+j\Delta\omega-\omega_a)}\right]}\,. \tag{A4}$$

For the present stability analysis the only saturating field is E_0, and D_0 is much larger than the other population components. Thus Eq. (A4) reduces to the two expressions

$$D_0 = \frac{\lambda_a(\omega_a)/\gamma_a - \lambda_b(\omega_a)/\gamma_b}{1 + sI\left[1 + \left(\dfrac{\omega-\omega_a}{\gamma}\right)^2\right]^{-1}} \tag{A5}$$

which is the same as Eq. (38), and

$$D_{n\neq 0}(\omega_a) = \frac{-sI[\alpha_0(\Delta\omega) + \alpha_n(\Delta\omega)]D_0(\omega_a)E_n/E_0}{\left[1 - 2in\Delta\omega\dfrac{\gamma_a+\gamma_b}{2\gamma_a\gamma_b} - \dfrac{(n\Delta\omega)^2}{\gamma_a\gamma_b}\right]\left[1 - \dfrac{2in\Delta\omega}{\gamma_a+\gamma_b}\right]^{-1} + \alpha_n(\Delta\omega)sI}\,. \tag{A6}$$

Here use has been made of the relationship $E^*_n = E_{-n}$, and the parameter α_n is defined by

$$\alpha_n(\Delta\omega) = \frac{\gamma/2}{\gamma + i(\omega - n\Delta\omega - \omega_a)} + \frac{\gamma/2}{\gamma - i(\omega + n\Delta\omega - \omega_a)}\,. \tag{A7}$$

The main interest is in the first sideband, and from Eq. (A1) the leading terms in the polarization driving this sideband are

$$P_1(\omega_a) = -\frac{(i\mu^2/\hbar)[E_1 D_0(\omega_a) + E_0 D_1(\omega_a)]}{\gamma - i(\omega + \Delta\omega - \omega_a)} \tag{A8}$$

With Eq. (A6) it is clear that the population pulsations lead to a correction of the first sideband polarization that is proportional to

$$\frac{E_0 D_1(\omega_a)}{E_1 D_0(\omega_a)} = \frac{-sI[\alpha_0(\Delta\omega) + \alpha_n(\Delta\omega)]}{\left[1 - 2i\Delta\omega\left(\frac{\gamma_a+\gamma_b}{2\gamma_a\gamma_b}\right) - \frac{(\Delta\omega)^2}{\gamma_a\gamma_b}\right]\left[1 - \frac{2i\Delta\omega}{\gamma_a+\gamma_b}\right]^{-1} + \alpha_n(\Delta\omega)sI}. \tag{A9}$$

This correction is clearly unimportant for small values of the saturating intensity, but if necessary the real and imaginary parts of Eq. (A8) could be substituted into Eqs. (50) and (51) to include the effects of population pulsations in the stability criteria.

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