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# What is left from stimulated electromagnetic shock radiation. A quantum approach

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Transition rates, angular distributions, and power spectra of stimulated-electromagnetic-shock radiation are calculated by means of relativistic quantum mechanics for a circularly polarized monochromatic plane-wave incident at an arbitrary angle to the direction of the electron beam. Usually the process can be divided into a high-frequency branch which is similar to ordinary Čerenkov radiation and a low-frequency branch which corresponds to high-intensity Compton scattering (HICS). No significant enhancement of Čerenkov radiation is found. The transition rates for the HICS branch, however, can be considerably larger than in vacuum.

## I. INTRODUCTION

The nomenclature "stimulated-electromagnetic-shock radiation (SESR)" has been introduced by Schneider and Spitzer[1] to refer to the radiation emitted by a relativistic electron moving in a dispersive nonabsorptive medium under the influence of a monochromatic electromagnetic plane wave. Schneider and Spitzer claimed a large enhancement of this type of radiation over ordinary Čerenkov radiation and a large frequency upshift compared to the incident radiation. Subsequent theoretical work on this effect by Šoln,[2] Zachary,[3,4], and Kroll[5] did not support all of the conclusions of Schneider and Spitzer and exhibited contradictory results: Whereas Zachary[3] still obtained a large enhancement due to SESR at low (microwave) frequencies, Kroll[5] denies any enhancement at all. The authors of Refs. 2–5 proceed by calculating the radiation from the classical current of the electron moving in the incident plane-wave field. Šoln[2] and Zachary[3,4] expand the current to first order in the intensity of the incident fields. Consequently they have no control over the occurence of higher harmonics. More seriously, this approximation turns out to be insufficient for high frequencies of the emitted radiation and to yield the above mentioned overestimate of the total power radiated by SESR. Kroll[5] starts from the exact current but this is only available in the case where the electron and the incident plane wave collide head on. Only Zachary[4] allows for an arbitrary geometry of all participating fields.

In the present investigation we employ relativistic quantum mechanics to recalculate SESR. Although quantum effects can be considered as small from the beginning, this approach gives a unified treatment of the process which is not at all more complicated than the classical procedure, in particular when an arbitrary geometry is allowed for. We calculate to lowest order the matrix element for emission of a photon by an electron under the conditions mentioned above. For the incident field we employ a circularly polarized monochromatic plane wave impinging at an arbitrary angle. The field is treated as a classical external field which is an excellent approximation for not too low intensities. The electron is described by the exact solutions of the Klein-Gordon equation in the presence of the external field. Hence we include all intensity-dependent frequency shifts.

The process called SESR has two well known limiting cases: Čerenkov radiation if there is no incident field and so called high-intensity Compton scattering (HICS) (Ref. 6) if there is no medium. The presence of both an incident field and a medium does more than simply to add the effects due to either external configuration. In principle a combined process occurs in complete analogy to the interplay between Čerenkov and synchroton radiation which has been referred to as synergic synchroton Čerenkov radiation by Schwinger, Tsai, and Erber.[7] In the present case we find that the effects depend crucially on the behavior of the refractive index as a function of frequency. We will assume the dielectric function to be approximately given by a sum of one or more damped Lorentzians. If the frequency of the incident field is far below the absorption frequencies of the medium we find that the frequencies of the emitted radiation can, in general, be split up into a HICS branch and one or more Čerenkov branches which are closely related to ordinary HICS and Čerenkov radiation, respectively. The corresponding transition rate for the Čerenkov branch does not deviate significantly from ordinary Čerenkov radiation, whereas the rate for the HICS branch can exceed ordinary HICS considerably under appropriate conditions. Since the total power radiated by the electron is dominated by the (comparatively high frequency) Čerenkov branch, it is not modified significantly by the SESR mechanism.

The explicit results agree apart from minor modifications with previous work[2-5] if the respective approximations are introduced. They are more general in as much as they exhibit (i) all intensity dependent effects; (ii) the incident field coming in at an arbitrary angle, except that it must not be too close to the Čerenkov angle; and (iii) quantum corrections. For low incident frequencies the latter are very small as expected. In this case the only vestige of quantum theory which might have some significance is the quantum recoil which accounts for a slight difference between the frequencies of emission and absorption.

In Sec. II the required solutions of the Klein-Gordon equation are presented. Since these are essentially given by Mathieu's functions we are forced to use appropriate approximations. In this process we have to exclude the immediate vicinity of the Čerenkov cone of the incident electron. The latter regime is highly nonlinear and would require different approximations for Mathieu's functions. This is not attempted in the present paper. In Sec. III we exploit kinematics in terms of conservation of effective momenta in order to obtain the relation between angle and frequency of the emitted radiation.

To solve for the frequency the dependence of the refractive index on the frequency must be known. Assuming an approximately Lorentzian shape the solutions are qualitatively discussed and the above-mentioned HICS and Čerenkov branches are found to evolve for low incident frequencies. In Sec. IV the general expression for the transition rate summed over polarizations is derived. In Sec. V angular distributions and power spectra of the emitted radiation are calculated for the HICS and the Čerenkov branch, separately. HICS type radiation can be considerably enhanced over HICS in vacuum. This effect can also be realized at large angles $\gtrsim \pi/2$ where the two branches do not interfere. On the other hand, Čerenkov-type SESR does not display any spectacular differences to ordinary Čerenkov radiation.

## II. WAVE FUNCTIONS

To describe the electron in the presence of a plane wave in a nonmagnetic transparent isotropic medium characterized by a real refractive index $n(\omega)$, we employ the solutions of the Klein-Gordon equation (we use natural units, $\hbar = c = 1$, and four-vector notation $a^\mu = (a^0, \vec{a})$ with scalar product $ab = a_0 b_0 - \vec{a}\vec{b}$)

$$[(i\partial^\mu - eA^\mu)^2 - m^2]\varphi(x) = 0\,, \tag{2.1}$$

where

$$A^\mu = \sum_{i=1}^{2} a e_i^\mu A_i(\xi) \tag{2.2}$$

is the vector potential of the plane wave with wave vector

$$k^\mu = (\omega, \vec{k})$$

and

$$e_i e_j = -\delta_{ij},\ k e_i = 0,\quad \vec{k}^2 = n^2\omega^2\,,$$
$$n = n(\omega)\,,\quad \xi = kx\,. \tag{2.3}$$

In using the Klein-Gordon instead of the Dirac equation we ignore the spin of the electron. This can be expected to be justified since in the two limiting cases of the present problem, Čerenkov radiation and high-intensity Compton scattering, spin is known to induce terms of relative order $\omega/E$ with $E$ the energy of the electron. This ratio is a small quantity under all reasonable conditions.

For a monochromatic plane wave, Eq. (2.1) reduces to a periodic differential equation. To be specific we choose circular polarization with the wave vector $\vec{k}$ in the $z$ direction and

$$A_1(\xi) = \cos\xi,\quad A_2(\xi) = -\sin\xi\,. \tag{2.4}$$

The solutions of Eq. (2.1) can then be expressed in terms of Mathieu's functions[8]

$$\varphi_p(x) = (2p_0 V\Delta)^{-1/2} \exp\left(-ipx + \frac{ipk}{k^2}\xi\right) \times me_\nu\left[\tfrac{1}{2}\left(\xi + \arctan\frac{p_2}{p_1}\right), h^2\right]\,, \tag{2.5}$$

where $p^2 = m^2$ and

$$h^2 = -\frac{4eap_T}{k^2}\,, p_T = (p_1^2 + p_2^2)^{1/2}\,. \tag{2.6}$$

The characteristic index $\nu$ is a function of

$$\lambda = \frac{4}{k^2}\left(\frac{(pk)^2}{k^2} + (ea)^2\right) \tag{2.7}$$

and $h^2$, and is given by the characteristic curves $\lambda = \lambda_\nu(h^2)$ (Ref. 9). The solutions (2.5) are normalized to one particle in the volume $V$ in the absence of the plane wave, if $\Delta = 1$ for $a = 0$. In general, $\Delta$ is fixed by the condition that the average current parallel to the four-dimensional direction of the incident field, i.e., $k^\mu j_\mu = \omega(j^0 - nj^3)$, be continuous.

Since Mathieu's functions are inconvenient to work with, we are depending on suitable approximations. For

$$\nu \gg 1\,,\quad h^2/\nu^2 \ll 1\,, \tag{2.8}$$

a useful approximation can be deduced from Ref. 10

$$me_\nu(z, h^2) = \sum_{r=-\infty}^{\infty} \exp i(\nu + 2r)z C_{2r}^\nu(h^2) = \exp i\nu\left(z - \frac{h^2}{2\lambda}\sin 2z\right)\left[1 + O\left(\frac{1}{\nu}, \frac{h^2}{\nu^2}\right)\right] \tag{2.9}$$

or

$$C_{2r}^{\nu}(h^2)=(-)^r J_r\left(\frac{h^2}{2\nu}\right)\left[1+O\left(\frac{1}{\nu},\frac{h^2}{\nu^2}\right)\right]. \qquad (2.10)$$

Within this approximation we derive [the current is given by (4.4)]

$$\Delta=1+\tfrac{1}{2}\left(\frac{ea}{m\gamma}\right)^2\frac{k^2}{(pk)^2}. \qquad (2.11)$$

Because this approximation is still valid for $h^2/\nu>1$ as long as $h^2/\nu^2\ll 1$, it is by far superior to a power series expansion of $C_{2r}^{\nu}(h^2)$ [cf. Eq. (2.16)]. If the second of conditions (2.8) applies we also have

$$\lambda_\nu(h^2)=\nu^2\left[1+O\left(\frac{h^4}{\nu^4}\right)\right].$$

The characteristic index $\nu$ can attain real or complex values.[9] In the latter case, $me_\nu(z,h^2)$ is exponentially decreasing for either positive or negative $z$. This indicates that the electron cannot penetrate into the plane-wave field but is trapped or totally reflected.[8,11] The regions in the $(\lambda,h^2)$ plane with complex $\nu$ are indicated in the so-called stability chart of Mathieu's functions.[9] They gather mainly in the region where $\lambda<2|h^2|$. We exclude these configurations from our considerations. This means that we have to exclude the immediate vicinity of the Čerenkov cone $C_i$ of the incident electron, which is given (apart from quantum contributions) by $pk=0$:

$$p=m\gamma(1,\vec{\beta}),\quad \vec{\beta}\vec{k}=\beta n\omega\cos\psi,$$

i.e.,

$$pk=m\gamma\omega(1-\beta n\cos\psi).$$

In view of (2.7) for $pk=0$ we have $\lambda<0$ since $k^2<0$. Hence $C_i$ is deeply inside the region with complex $\nu$.

We can, however, approach $C_i$ quite closely and remain still consistent with the conditions (2.8) as we are now going to show. In addition to (2.8) we assume

$$(ea)^2|k^2|/(pk)^2\ll 1, \qquad (2.12)$$

so that we may expand $\sqrt{\lambda}$ in powers of $(ea)^2$.

We then have

$$\frac{h^2}{\nu^2}\approx\frac{h^2}{\lambda}\approx-\frac{eap_Tk^2}{(pk)^2}\ll 1.$$

With $\cos\psi_C=(\beta n)^{-1}$, $\cos\psi=\cos(\psi_C+\delta)$ this is equivalent to $(n\beta>1)$,

$$\delta^2\gg\frac{ea}{m\gamma}\,\frac{(n^2-1)}{n(n^2\beta^2-1)^{1/2}}, \qquad (2.13)$$

whereas (2.12) is equivalent to

$$\delta^2\gg\left(\frac{ea}{m\gamma}\right)^2\frac{n^2-1}{n^2\beta^2-1}. \qquad (2.14)$$

For reasonable field strengths of the plane-wave field the right-hand sides of (2.13) and (2.14) are rather small: The dimensionless parameter $ea/m$ is related to the field strength of the plane-wave

$$\frac{ea}{m}=0.75\times 10^{-16}E(\text{V/cm})\,\frac{mc^2}{\omega\hbar}. \qquad (2.15)$$

Since the present theory makes sense only below the breakthrough field strength which may be of order $10^6$V/cm in typical materials, (2.15) is a small quantity even for microwave frequencies.

In what follows we shall take the conditions (2.8) for granted so that we can rely on the approximations (2.9)–(2.11). Restrictions corresponding to (2.8), (2.12), and $pk\neq 0$ apply, of course, also to the final momentum $p'$ of the electron after the emission of a SESR photon. Practically, however, they coincide with what we have already discussed, due to the relative smallness of the momentum of the emitted photon.

We note, finally, that the parameter

$$\frac{h^2}{\nu}\approx\frac{h^2}{\sqrt{\lambda}}\approx\frac{4eap_T}{pk}=4\frac{ea}{m}\frac{m}{\omega}\frac{\beta\sin\psi}{1-\beta n\cos\psi} \qquad (2.16)$$

is, in virtue of (2.15), not necessarily small in the context of our approximations.

## III. KINEMATICS

Before calculating in detail the cross section for the emission of radiation due to the incident plane-wave field we want to extract as much as possible from purely kinematical considerations. In fact, the frequencies of the emitted SESR radiation are determined by conservation of the effective four momenta.[12] The effective momentum $\tilde{p}$ of the electron in the presence of the stimulating plane wave can be read off from Eqs. (2.5) and (2.9)

$$\tilde{p}=p-\tfrac{1}{2}Nk,\quad N=\frac{2pk}{k^2}+\nu. \qquad (3.1)$$

Effective momenta are conserved up to the emission or absorption of an arbitrary number of quanta of the incident field, hence

$$\tilde{p}'=\tilde{p}-k'-sk, \qquad (3.2)$$

where $p'$ is the momentum of the electron after the emission of a SESR photon with momentum $k'=(\omega',\mathbf{k}')$, $\vec{k}'^2=n(\omega')^2\omega'^2=n'^2\omega'^2$. Calculating the frequency $\omega'$ of the emitted photon from (3.2) is made difficult by the dependence of $N$ on $p$. We are, however, consistent with the assumptions (2.8) and (2.12) if we let

$$\nu\approx\sqrt{\lambda}=-\frac{2pk}{k^2}-\frac{(ea)^2}{pk}\cdots, \qquad (3.3)$$

hence,

$$\tilde{p}=p+\frac{(ea)^2}{2pk}k,\quad N=-\frac{(ea)^2}{pk} \tag{3.4}$$

and

$$\tilde{p}^2=m^2+(ea)^2\,. \tag{3.5}$$

In view of (3.5) squaring Eq. (3.2) yields

$$\omega'^2(1-n'^2)+s^2\omega^2(1-n^2)-2\omega'\left(m\gamma(1-n'\beta\cos\varphi)+\frac{(ea)^2(1-nn'\cos\vartheta)}{2m\gamma(1-n\beta\cos\psi)}-s\omega(1-nn'\cos\vartheta)\right)$$

$$-2s\omega m\gamma(1-\beta n\cos\psi)-\frac{s(ea)^2\omega(1-n^2)}{m\gamma(1-\beta n\cos\psi)}=0\,, \tag{3.6}$$

where the angles $\varphi$ and $\vartheta$ between $\vec{k}'$ and $\vec{p}$, and between $\vec{k}$ and $\vec{k}'$, respectively, have been introduced

$$\vec{p}\vec{k}'=m\gamma\beta n'\omega'\cos\varphi,\quad \vec{k}\vec{k}'=n\omega n'\omega'\cos\vartheta\,.$$

For $s=0$, we have from (3.6)

$$\omega'=\frac{2m\gamma}{1-n'^2}\left(1-n'\beta\cos\varphi+\frac{(ea)^2(1-nn'\cos\vartheta)}{2(m\gamma)^2(1-n\beta\cos\psi)}\right) \tag{3.7}$$

or

$$\cos\varphi=\frac{1}{n'\beta}\left(1+\frac{(n'^2-1)\omega'}{2m\gamma}+\frac{(ea)^2(1-nn'\cos\vartheta)}{2(m\gamma)^2(1-n\beta\cos\psi)}\right) \tag{3.8}$$

These are the quantum-mechanical formulae for Čerenkov radiation modified by an intensity-dependent term which according to (2.12) is restricted to small values. In the dispersionless case $n(\omega')=\text{const}$, the emitted frequency is indeterminate in the classical limit $\omega'/m\to 0$ and emission occurs at a fixed angle.

Generally, dispersion cannot be neglected and the emitted frequency is only implicitly given by (3.7). The dielectric function may be considered to be a sum of one or more damped Lorentzians centered at frequencies $\omega_a$ in the UV or below. It is then clear that in the classical limit $\omega'$ cannot exceed the largest frequency $\omega_a$ since $n'<1$ for $\omega'>\omega_a$. One can easily convince oneself that this is also true if the quantum term is taken care of since $n'\to 1$ for $\omega'\gg\omega_a$. For $s\neq 0$, (3.6) is apart from the implicit dependence quadratic in $\omega'$. As we do not expect either the incident or the emitted frequencies to be comparable with $E=m\gamma$, we will neglect the first two terms in (3.6) which give rise to quantum corrections. We retain, however, the last term within the large parentheses of (3.6). This is also a very small quantum contribution but will turn out to yield the quantum recoil. We then have

$$\omega'=-s\omega\left(1-\beta n\cos\psi+\frac{(ea)^2(1-n^2)}{2(m\gamma)^2(1-\beta n\cos\psi)}\right)\times\left(1-\beta n'\cos\varphi+\frac{(ea)^2(1-nn'\cos\vartheta)}{2(m\gamma)^2(1-\beta n\cos\psi)}-\frac{s\omega}{m\gamma}(1-nn'\cos\vartheta)\right)^{-1}. \tag{3.9}$$

This is the usual result[2-5] modified by quantum-mechanical and intensity-dependent corrections which again according to (2.12) must be small. Note that even in the absence of dispersion Eq. (3.9) relates the frequency $\omega'$ and the angle $\varphi$ of the emitted radiation in contrast to classical Čerenkov radiation where a corresponding relation does not exist. For $n=n'=1$, (3.9) yields the *exact* frequency of HICS (Ref. 6), in spite of our approximations. The reason is that in this limit (3.4) and (3.5) are no longer approximations but exact.

Although we consider usually only the case of emission, Eq. (3.9) contains also the opposite case of absorption. If a solution $\omega'$ turns out to be negative, a photon with frequency $-\omega'$ can be absorbed [note that $n(-\omega')=n(\omega')$]. Letting $s\to -s$ we realize that the frequencies for emission and absorption differ only by the last term in the denominator of (3.9) which hence determines the quantum mechanical recoil.

For small dispersion, (3.9) can be considered as an explicit solution or can be solved by iteration. For $\omega\ll\omega_a$ this is even possible for quite large values of $\omega'$. Equation (3.9) then predicts a large frequency upshift near to the classical Čerenkov cone which is given by the zero of the right-hand side of (3.7), as well as higher harmonics with $\omega'_s\approx s\omega'_1$. This procedure is no longer possible when $\omega'$ approaches $\omega_a$. We will now discuss this behavior graphically in the simplest case of just one resonance frequency $\omega_a$ and find out that inevitably there are more solutions of (3.9) beyond those just discussed.

In Fig. 1 the refractive index $n'=n(\omega')$ as a function of $\omega'$ is depicted for a model with just one damped resonance at $\omega_a$. Figures 2(a)–2(c) exhibit the function $f(\omega')=(1-\beta n'\cos\varphi)^{-1}$ for $\cos\varphi<[\max(n')]^{-1}$, $\cos\varphi>[\max(n')]^{-1}$ but $\beta n(0)\cos\varphi<1$, $\beta n(0)\cos\varphi>1$, respectively. The solutions of (3.9) with the intensity dependent and quantum corrections for the time being neglected are given by the common points of $f(\omega')$ with the straight lines $g(\omega')=\omega'/[-s\omega(1-\beta n\cos\psi)]$ which are very steep for $\omega/\omega_a\ll 1$. In the case of Fig. 2(a), there is just one solution provided that $s(1-\beta n\cos\psi)<0$. It is of HICS type and exhibits a sequence of har-

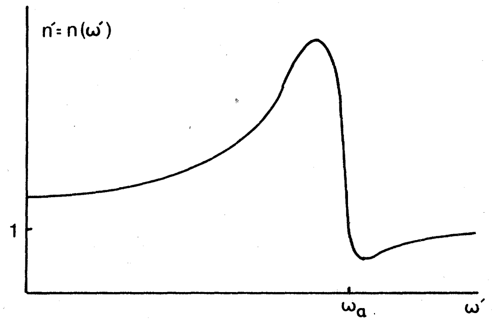


FIG. 1. Typical behavior of the refractive index $n(\omega')$ as a function of $\omega'$ for one resonance frequency $\omega_a$.

monics with $\omega_s' \approx s\omega_1'$. With $\cos\varphi$ increasing we approach the situation depicted in Fig. 2(b) (provided that $\beta$ is large enough). Now there exist two Čerenkov frequencies $\omega_{c1}$ and $\omega_{c2}$ given by $1 - \beta n(\omega_{ci})\cos\varphi = 0$. For $s(1-\beta n\cos\psi) > 0$, there are two solutions of (3.9), which due to the steepness of the straight lines $g(\omega')$ are very close to $\omega_{ci} (i=1,2)$. We shall refer to these solutions as Čerenkov-type. For $s(1-\beta n\cos\psi) < 0$, there is one further solution $\omega' \ll \omega_{ci}$ which is obviously of HICS-type. It displays reasonable harmonics with $\omega_s' \approx s\omega_1'$, whereas the Čerenkov-type solutions for different values of $s$ are closely grouped together and have nothing in common with what one usually expects from harmonics. This smearing of the Čerenkov radiation over a narrow range of frequencies or angles has been emphasized by Kroll.[5] If $\cos\varphi$ continues to increase, the HICS-type solution approaches larger values. Finally [for $\beta n(0)$ large enough, otherwise this situation does not occur] we will have $\omega_{c1} = 0$. At this stage what we up to now have referred to as the HICS-type solution has disappeared. An approximation to the solution with the lowest frequency can be obtained by expanding

$$n^2(\omega') \approx n^2(0) + [n^2(0) - 1](\omega'/\omega_a)^2 ,$$

where damping has been neglected. We then have

$$\omega' = \left( \frac{2s\omega\omega_a^2(1-\beta n\cos\psi)n^2(0)}{n^2(0)-1} \right)^{1/3} \qquad (3.10)$$

for $s > 0$ which is quite large. Obviously, labelling this solution as either HICS- or Čerenkov-type has become meaningless. With $\cos\varphi$ further increasing we end up with Fig. 2(c) where just one Čerenkov-type solution has been left and for $s(1-\beta n\cos\psi) > 0$ yet another one which again can be denoted as HICS-type. The HICS-type frequency as a function of $\cos\varphi$ becomes minimal at $\cos\varphi = \pm 1$ with values greater than zero. The corresponding discontinuities in the power spectrm have been estimated by Kroll[5] to be experimentally detectable.

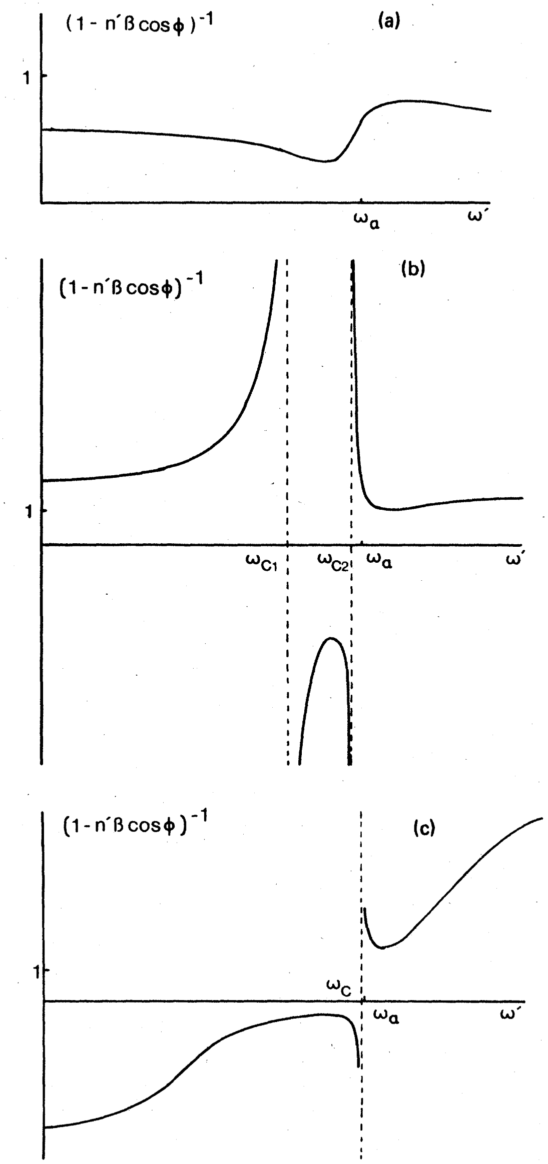


FIG. 2. The function $[1 - n(\omega')\beta\cos\varphi]^{-1}$ with $n(\omega')$ from Fig. 1 for (a) $\beta\cos\varphi = -0.5$; (b) $\beta\cos\varphi = 0.33$; (c) $\beta\cos\varphi = 0.8$. The common points with the straight lines $g(\omega') = \omega'/[-s\omega(1-\beta n\cos\psi)]$ yield the frequencies $\omega_s'$ of SESR. $\omega_c$ are the respective frequencies of Čerenkov radiation.

The quantitative impact of the intensity-dependent and quantum recoil corrections in (3.9) on the emitted frequencies $\omega'$ is small. The main effect comes from the intensity-dependent shift of the classical Čerenkov frequencies $\omega_c$ given by the zeroes of the right-hand side of (3.7) versus the zero-field values $\omega_c^0$ given by $1 - \beta n(\omega_c^0)\cos\varphi = 0$. The shift of the Čerenkov-type frequencies $\omega'$ versus $\omega_c$ is for $\omega/\omega_c \ll 1$,

$$\delta\omega = \omega' - \omega_c = (\beta \dot{n}_c' \cos\varphi)^{-1}\left(\frac{s\omega}{\omega_c}(1-\beta n\cos\psi) - \frac{s\omega}{m\gamma}(1 - nn_c\cos\vartheta)\right), \tag{3.11}$$

where $n_c = n(\omega_c)$, $\dot{n}_c' = dn(\omega')/d\omega' \,|\omega' = \omega_c$. Here the quantum term represents a very small correction which is even smaller than the quantum correction to the Čerenkov frequency $\omega_c$ which obtains from (3.7)

$$\omega_c(\text{qu. mech.}) - \omega_c = \frac{n_c^2 - 1}{\beta \dot{n}_c \cos\varphi}\,\frac{\omega_c}{2m\gamma}. \tag{3.12}$$

The above classification of the SESR radiation as either Čerenkov- or HICS-type is only feasible for $\omega \ll \omega_a$, and, even in this case, not in the vicinity of the angle where $\omega_{c1} = 0$ or only sufficiently far away from the corresponding frequency (3.10). Moreover, it relies on observing the doubly differential transition rate $d\Gamma/d\omega' d\Omega_{\vec{k}'}$, where the emitted frequencies at a particular angle are of interest. If just the power spectrum or the angular distribution are observed contributions of both HICS-type and Čerenkov-type SESR add in general. There are, however, two "pure" cases: For angles such that $\cos\varphi < [\beta \max(n')]^{-1}$ there is only HICS-type and for high frequencies $\omega' \lesssim \omega_a$ there is only Čerenkov-type radiation. Hence the total energy loss of an electron due to SESR is completely dominated by Čerenkov-type radiation.

For incident frequencies such that $\omega \approx \omega_a$ we encounter a completely different behavior. The straight lines $g(\omega')$ are then more or less flat. E.g., Fig. 2(a) exhibits that there may exist three HICS-type frequencies at large angles or from Fig. 2(b) one infers that there may be no Čerenkov-type frequency at all. Since, however, the parameter $ea/m$ which governs all SESR phenomena ($s \neq 0$) is very small for high frequencies, ordinary Čerenkov radiation ($s = 0$) will be largely dominant in this case.

## IV. CROSS SECTIONS

The matrix element for an electron with initial (final) momentum $p(p')$ to emit a photon with momentum $k' = (\omega', \vec{k}')$, $\vec{k}'^2 = n(\omega')^2\omega'^2 = n'^2\omega'^2$, is to first order given by

$$M = i\langle p' \,|\int d^4x \mathcal{L}_I(x)\,|p\rangle,$$

$$\mathcal{L}_I(x) = -ej_\mu(x) A^\mu_{k'}(x), \tag{4.1}$$

where

$$A^\mu_{k'}(x) = \frac{1}{n'}\left(\frac{2\pi}{\omega' V}\right)^{1/2} e'^\mu e^{ik'x} \tag{4.2}$$

is the vector potential of the emitted photon with polarization $e'$. We use the radiation gange, hence

$$e_0' = 0, \quad e'k' = 0. \tag{4.3}$$

$j_\mu(x)$ is the conserved Klein-Gordon current of the electron in the presence of the stimulating plane-wave field (2.2)–(2.4)

$$\langle p' \,|j_\mu(x)\,|p\rangle = i\varphi^*_{p'}(x)(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)\varphi_p(x) + 2eA_\mu(x)\varphi^*_{p'}(x)\varphi_p(x), \tag{4.4}$$

with the wave functions given by (2.5). Inserting the Fourier expansion (2.9) of Mathieu's functions in order to perform the $x$ integration we obtain

$$M = -\frac{ie(2\pi)^4}{n'\Delta}\left(\frac{2\pi}{\omega' V 2p_0 V 2p_0' V}\right)^{1/2} \sum_{r,s} \exp\frac{i}{2}\{(\nu + 2r)\alpha - [\nu' + 2(r-s)]\alpha'\}\,\delta(\tilde{p}' - \tilde{p} + k' + sk)$$
$$\times [C^\nu_{2r}(h^2) C^{\nu'}_{2(r-s)}(h'^2) e'(2\tilde{p} - k' - 2rk) + eae'(e_1 + ie_2) C^\nu_{2r}(h^2) C^{\nu'}_{2(r-s+1)}(h'^2) e^{-i\alpha'}$$
$$+ eae'(e_1 - ie_2) C^\nu_{2r}(h^2) C^{\nu'}_{2(r-s-1)}(h'^2) e^{i\alpha'}]. \tag{4.5}$$

Here $\tilde{p}(\tilde{p}')$ is the initial (final) effective momentum of the electron defined in (3.1). The δ function indicates conservation of effective momenta which has already been exploited in Sec. IV. Primed quantities $\nu'$, $h'$, $\alpha'$ refer to the final momentum of the electron and

$$\alpha = \arctan(p_2/p_1) \quad [\alpha' = \arctan(p_2'/p_1')]$$

is the azimutal angle of $\vec{p}(\vec{p}')$ relative to the $z$ axis given by the direction of $\vec{k}$.

Using the approximation (2.10) of the $C^\nu_{2r}$ in terms of Bessel functions the sum over $r$ in (4.5) can be done by means of the addition theorem for Bessel functions. The result is

$$M = -\frac{ie(2\pi)^4}{n'\Delta}\left(\frac{2\pi}{4\omega' p_0 p_0' V^3}\right)^{1/2} \sum_s \delta(\tilde{p}' - \tilde{p} + k' + sk) \exp i\left(\frac{\nu}{2}\alpha - \frac{\nu'}{2}\alpha' - s\chi + s\alpha'\right)$$
$$\times\Big(e'(2\tilde{p}' - k') J_s(R) + eae'(e_1 + ie_2) e^{i(\chi - \alpha')} J_{s-1}(R) + eae'(e_1 - ie_2) e^{-i(\chi-\alpha')} J_{s+1}(R)$$
$$+ \frac{h^2}{2\nu} e'k[e^{i(\chi+\alpha-\alpha')} J_{s-1}(R) + e^{-i(\chi+\alpha-\alpha')} J_{s+1}(R)]\Big), \tag{4.6}$$

where

$$R^2=\left(\frac{h^2}{2\nu}\right)^2+\left(\frac{h'^2}{2\nu'}\right)^2-\frac{h^2h'^2}{2\nu\nu'}\cos(\alpha-\alpha') \quad (4.7)$$

and

$$e^{2i\chi}=\left(\frac{h'^2}{2\nu'}-\frac{h^2}{2\nu}e^{-i(\alpha-\alpha')}\right)\Big/\left(\frac{h'^2}{2\nu'}-\frac{h^2}{2\nu}e^{i(\alpha-\alpha')}\right). \quad (4.8)$$

Next we sum $|M|^2$ over the polarizations of the emitted photon. A system of two mutually orthogonal polarization vectors $e'^{(i)}_\mu (i=1,2)$, satisfies due to (4.3)

$$\sum_{i=1}^{2} e'^{(i)}_k e'^{(i)}_l=\delta_{kl}-\frac{k'_k k'_l}{\vec{k}'^2}(k,l=1,2,3). \quad (4.9)$$

Since

$$M\sim j_\mu(k')e'^\mu,$$

where

$$j_\mu(k')=\int d^4x e^{ik'x}\langle p'|j_\mu(k)|p\rangle$$

obeys due to current conservation

$$k'_l j_l(k')=k'_0 j_0(k')$$

we are allowed to employ instead of (4.9)

$$\sum_{i=1}^{2} e'^{(i)}_k e'^{(i)}_l \to E_{kl}=\delta_{kl}-\frac{1}{n'^2}\delta_{k0}\delta_{l0}. \quad (4.10)$$

Since we had in mind to use (4.10) we have deliberately done without using $e'k'=0$ in (4.6): If terms proportional to $k'$ are dropped, the current $j_\mu(k')$ is no longer conserved and (4.10) may not be used. The advantage of (4.10) over (4.9) is that

$$k_k E_{kl}(e_1-ie_2)_l=(e_1-ie_2)_k E_{kl}(e_1-ie_2)_l=0,$$

and that it provides for a convenient decomposition into classical and quantum-mechanical terms, as we shall see below.

We then obtain

$$\sum_{pol}|M|^2=\frac{e^2(2\pi)^5T}{n'^2V^2\Delta}\sum_{s=-\infty}^{\infty}\frac{\mu_s}{4\omega'p_0p'_0}\delta(\tilde{p}'-\tilde{p}+k'+sk), \quad (4.11)$$

$$\mu_s=J_s(R)^2\left[A+\frac{2sea}{R}(Be^{-i(\chi-\alpha')}+B^*e^{i(\chi-\alpha')})+\frac{2h^2s}{\nu R}C\cos(\chi+\alpha-\alpha')+\frac{h^4s^2\omega^2}{\nu^2R^2}\left(n^2-\frac{1}{n'^2}\right)\cos2(\chi+\alpha-\alpha')\right]$$
$$+[J_{s+1}(R)^2+J_{s-1}(R)^2]\left[2(ea)^2+\frac{h^4\omega^2}{2\nu^2R^2}\left(n^2-\frac{1}{n'^2}\right)\sin^2(\chi+\alpha-\alpha')\right], \quad (4.12)$$

where

$$A=(2\tilde{p}-k')_k E_{kl}(2\tilde{p}-k')_l$$
$$=4\tilde{p}_k E_{kl}\tilde{p}_l+4\left\{pk'-\left(1-\frac{1}{n'^2}\right)p_0\omega' -\tfrac{1}{2}N\left[kk'-\left(1-\frac{1}{n'^2}\right)k_0\omega'\right]\right\}$$
$$+\left(n'^2-\frac{1}{n'^2}\right)\omega'^2, \quad (4.13)$$

$$\tilde{p}_k E_{kl}\tilde{p}_l\cong-[m^2+(ea)^2]+\left(1-\frac{1}{n'^2}\right)p_0(p_0-N\omega), \quad (4.14)$$

$$B=(2\tilde{p}-k')_k E_{kl}(e_1-ie_2)_l=2p_Te^{-i\alpha}-k'_Te^{-i\beta}, \quad (4.15)$$

$$C=(2\tilde{p}-k')_k E_{kl}k_l$$
$$=2\left[-pk+\left(1-\frac{1}{n'^2}\right)p_0\omega-\tfrac{1}{2}N\omega^2\left(n^2-\frac{1}{n'^2}\right)\right]$$
$$+kk'-\left(1-\frac{1}{n'^2}\right)\omega\omega'. \quad (4.16)$$

In obtaining (4.14) we have used (3.5). $\beta$ in (4.15) is the azimuthal angle of $k'$ relative to the $z$ axis defined by the direction of $\vec{k}$, it is related to $\psi$, $\vartheta$, $\varphi$ by

$$\cos\varphi=\cos\vartheta\cos\psi+\sin\vartheta\sin\psi\cos(\alpha-\beta). \quad (4.17)$$

The angles $\chi$ and $\alpha'$ are eliminated from (4.12) by

$$\sin\chi=\frac{h^2}{2\nu R}\sin(\alpha-\alpha'), \quad (4.18)$$

$$\sin(\alpha-\alpha')=\frac{k'_T}{p'_T}\sin(\beta-\alpha). \quad (4.19)$$

Equation (4.18) follows from the definition (4.8) of $\chi$, (4.19) is verified in the triangle formed by the two-dimensional vectors $\vec{p}_T=(p_1,p_2)$, $\vec{p}'_T$ and $k'_T(\vec{p}'_T-\vec{p}_T+\vec{k}'_T=0)$ in the plane perpendicular to $\vec{k}$. We then have

$$\cos(\chi+\alpha-\alpha')=\frac{h^2}{2R}\left(\frac{1}{\nu'}-\frac{1}{\nu}-\frac{k'_T}{\nu'p_T}\cos(\alpha-\beta)\right), \quad (4.20)$$

$$\cos2(\chi+\alpha-\alpha')=1-\frac{h^4k_T'^2\sin^2(\beta-\alpha)}{2\nu'^2R^2p_T^2}, \quad (4.21)$$

$$\cos(\chi+\beta-\alpha')=\frac{h^2}{2R}\left[\left(\frac{1}{\nu'}-\frac{1}{\nu}\right)\cos(\alpha-\beta)-\frac{k'_T}{\nu'p_T}\right]. \quad (4.22)$$

Inserting $A$, $B$, and $C$ into (4.12) and counting powers of $\hbar$ it turns out that the terms proportional to $k'$ in $A$, $B$, and $C$ just constitute the quantum corrections within the first square bracket of (4.12). Since these are of relative order $\omega'/p_0$ which is small up to very high frequencies, we shall neglect all these terms [which if needed can be inferred from (4.13), (4.15) and (4.16)]. Moreover, we let

$$N \cong -\frac{(ea)^2}{pk}, \quad \nu \cong -\frac{2pk}{k^2}, \quad \nu' \cong -\frac{2p'k}{k^2} \cong -\frac{2pk}{k^2},$$

except

$$\nu' - \nu \cong 2(kk' + sk^2)/k^2 .$$

This comes up to neglecting intensity-dependent corrections within the square brackets in (4.12) which are small in the framework of our approximations. Note, however, that for $p_T = 0$ the square brackets still contain the complete first-order intensity-dependent corrections. We are then left with

$$\sum_{pol} |M|^2 = \frac{e^2(2\pi)^5 T}{V^2 \Delta} \sum_s \frac{\mu_s^{cl}}{4n'^2 \omega' p_0 p_0'} \delta(\tilde{p}' - \tilde{p} + k' + sk), \tag{4.23}$$

$$\begin{aligned}\mu_s^{cl} = 4J_s(R)^2 \Bigg\{ &-[m^2 + (ea)^2] + \left(1 - \frac{1}{n'^2}\right) p_0 \left(p_0 + \frac{(ea)^2 \omega}{pk}\right) - \frac{2s(ea)^2 p_T p_0 \omega}{R^2 (pk)^2} \left(1 - \frac{1}{n'^2}\right) \left(k_T' \cos(\alpha - \beta) - \frac{p_T(kk' + sk^2)}{pk}\right) \\ &+ \left(\frac{seap_T \omega}{Rpk}\right)^2 \left(n^2 - \frac{1}{n'^2}\right) \left[1 - 2\left(\frac{eak_T' \sin(\beta - \alpha)}{Rpk}\right)^2\right]\Bigg\} \\ &+ 2(ea)^2 [J_{s+1}(R)^2 + J_{s-1}(R)^2] \left[1 + \left(n^2 - \frac{1}{n'^2}\right) \left(\frac{ea\omega p_T k_T' \sin(\beta - \alpha)}{R(pk)^2}\right)^2\right],\end{aligned} \tag{4.24}$$

where

$$\begin{aligned} R^2 &= \frac{4(ea)^2}{k^4} \left[p_T^2 \left(\frac{1}{\nu} - \frac{1}{\nu'}\right)^2 + \frac{k_T'^2}{\nu'^2} + \frac{2p_T k_T' \cos(\alpha - \beta)}{\nu'} \left(\frac{1}{\nu} - \frac{1}{\nu'}\right)\right] \\ &\cong \left(\frac{ea\omega\omega' m\gamma}{(pk)^2}\right)^2 r^2 . \end{aligned} \tag{4.25}$$

$$\begin{aligned} r^2 = {} & \beta^2 \sin^2\psi \left(1 - nn' \cos\vartheta + \frac{s\omega}{\omega'}(1 - n^2)\right)^2 + n'^2 \sin^2\vartheta (1 - \beta n \cos\psi)^2 \\ & - 2n' \sin\psi \sin\vartheta \left(1 - nn' \cos\vartheta + \frac{s\omega(1 - n^2)}{\omega'}\right)(1 - n\beta \cos\psi) \cos(\alpha - \beta) \end{aligned} \tag{4.26}$$

can be considered of order unity so that the order of magnitude of $R^2$ is given by the first factor in (4.25). Now $\mu_s^{cl}$ [Eq. (4.24)] consists exclusively of classical terms inasmuch as $R$ is a classical quantity. This is the case if the recoil corrections to $\omega'$ are neglected. The fact that $R$ is practically classical is surprising in view of its definition (4.7), since $h^2/\nu$ [eq. (2.16)] is not. Moreover, $h^2/\nu$ is generally very large whereas $R^2$ is not, except in some cases to be discussed below. The total transition rate per unit time from a specified initial state is given by

$$\begin{aligned} \Gamma &= \int \frac{V d^3p'}{(2\pi)^3} \frac{V d^3k'}{(2\pi)^3} \frac{1}{T} \sum_{pol} |M|^2 \\ &= \frac{e^2}{2\pi\Delta p_0} \sum_s \int d^4p' \delta(p'^2 - m^2) \Theta(p_0') \frac{d^3k'}{2\omega' n'^2} \\ &\qquad \times \delta(\tilde{p}' - \tilde{p} + k' + sk) \mu_s , \end{aligned} \tag{4.27}$$

with $\mu_s$ (or its classical approximation $\mu_s^{cl}$) defined in (4.12) [(4.24)]. We change from the integration variables $p_\mu$ to $\tilde{p}_\mu$, having in mind (3.4).

With the functional determinant

$$\left|\frac{d^4\tilde{p}}{d^4p}\right| = 1 - \frac{(ea)^2 k^2}{2(pk)^2} = \Delta^{-1}, \tag{4.28}$$

we have

$$\begin{aligned} \Gamma &= \frac{e^2}{2\pi p_0} \sum_s \int d^4\tilde{p}' \delta(\tilde{p}'^2 - [m^2 + (ea)^2]) \Theta(p_0') \frac{d^3k'}{2\omega' n'^2} \\ &\qquad \times \mu_s \delta(\tilde{p}' - \tilde{p} + k' + sk) \end{aligned} \tag{4.29}$$

$$= \frac{e^2}{4\pi p_0} \sum_s \int n'\omega' d\omega' d\Omega_{\vec{k}'} \delta[(\tilde{p} - k' - sk)^2 - m^2 - (ea)^2] \mu_s .$$

The sum over $s$ is in principle restricted by the condition $p_0' > 0$, but since $p_0' \gg \omega', \omega$ this does not have much practical significance. The argument of the $\delta$ function in (4.28) is the left-hand side of Eq. (3.6) and has been evaluated in different approximations in (3.7)–(3.9). If we denote this argument by $\phi$ the transition rate can be written as

$$\Gamma = \frac{e^2}{4\pi m\gamma}\sum_s \int n'\omega'\,d\omega'\,d\Omega_{\vec{k}'} \times \sum_i \frac{1}{|\partial\phi/\partial\omega'|}\delta(\omega'-\omega_i')\mu_s , \quad (4.30)$$

with $\omega_i'$ the solutions of $\phi = 0$ which have been discussed in Sec. III. We can also exhibit the angular dependence by solving for $\cos\varphi$,

$$\Gamma = \frac{e^2}{8\pi(m\gamma)^2\beta}\sum_s\int d\omega'\,d\Omega_{\vec{k}'}\,\delta(\cos\varphi - \cos\hat{\varphi})\mu_s , \quad (4.31)$$

where $\cos\hat{\varphi}$ is the unique solution of $\phi = 0$.

## V. POWER SPECTRA AND ANGULAR DISTRIBUTIONS OF SESR

In this section we shall consider the contributions from HICS-type and Čerenkov-type SESR separately although as we remarked in Sec. III, a clear distinction is not always possible. For the angular distributions we will need

$$\partial\phi/\partial\omega' = 2m\gamma\left[\frac{\omega'}{m\gamma}(1-n'^2-n'\omega'\dot{n}') - \left(1-n'\beta\cos\varphi + \frac{(ea)^2(1-nn'\cos\vartheta)}{2(m\gamma)^2(1-n\beta\cos\psi)} - \frac{s\omega}{m\gamma}(1-nn'\cos\vartheta)\right) + \omega'\dot{n}'\left(\beta\cos\varphi + \frac{(ea)^2 n\cos\vartheta}{2(m\gamma)^2(1-n\beta\cos\psi)} - \frac{s\omega}{m\gamma}n\cos\vartheta\right)\right] \quad (5.1)$$

from (3.6) with $\dot{n}' = dn/d\omega'$. To estimate the relative importance of the terms in (5.1) we assume

$$n^2(\omega') = 1 + \frac{n^2(0)-1}{1-(\omega'/\omega_a)^2}$$

so that

$$\omega'\dot{n}' = \frac{1}{n'}\,\frac{n^2(0)-1}{[1-(\omega'/\omega_a)^2]^{-2}}\left(\frac{\omega'}{\omega_a}\right)^2 , \quad (5.2)$$

hence for low frequencies $\omega'$ the terms proportional to $\omega'\dot{n}'$ are negligible.

### A. HICS-type SESR

#### *1. Angular distribution*

For low frequencies we drop the first parentheses and the last large parentheses in (5.1) such that in view of (3.9) and (2.11)

$$\partial\phi/\partial\omega' = \frac{s\omega}{\omega'}(1-\beta n\cos\psi)\Delta , \quad (5.3)$$

and (4.30) yields

$$\Gamma = \frac{1}{8\pi(m\gamma)^2\Delta}\sum_{s=0}\int d\Omega_{\vec{k}}\,\frac{n'^2\omega'^2}{|s\omega(1-\beta n\cos\psi)|}\mu_s \quad (5.4)$$

with $\omega_s'$ from (3.9).

In vacuum $n = n' = 1$, $k^2 = k'^2 = 0$, and

$$\mu_s = -4[m^2 + (ea)^2]J_s(R)^2 + 2(ea)^2[J_{s-1}(R)^2 + J_{s+1}(R)^2] = \mu_s^{\rm vac} \quad (5.5)$$

from (4.24). With some algebra one can convince oneself that the same expression follows also from the exact quantum-mechanical formulae (4.12)–(4.16). Since $\mu_s$ is a Lorentz invariant, $R$ must be also. Hence we can evaluate $R$ from (4.25) in the rest frame of the electron

$$R^2 = \left(\frac{eak_T'}{p'k}\right)^2 \quad (\vec{p} = 0) ,$$

and then introduce invariants by means of $pk = m\omega$, $pk' = m\omega'$, $kk' = \omega\omega'(1-\cos\vartheta)$ to obtain a manifestly invariant form of $R^2$. Noting in addition that $\tilde{p}k' = -sp'k$ we have

$$R^2 = \left(\frac{s\,ea}{(pk)(\tilde{p}k')}\right)^2 kk'[2(pk)(pk') - m^2kk'] . \quad (5.6)$$

The differential cross section per incoming laser photon, $\sigma = \Gamma/(j_{in}N)$, $j_{in} = 1/V$, $N = a^2\omega V/4\pi$, is then (here we let $s \to -s$)

$$\frac{d\sigma}{d\Omega_{\vec{k}'}} = \frac{e^4}{(m\gamma)^2}\sum_{s\geq 1}\frac{1}{s(1-\beta\cos\psi)}\left(\frac{\omega_s'}{\omega}\right)^2 \times\left\{J_{s+1}(R)^2 + J_{s-1}(R)^2 - 2\left[1+\left(\frac{m}{ea}\right)^2\right]J_s(R)^2\right\} . \quad (5.7)$$

With the emitted frequencies given by

$$\omega_s' = \frac{s\omega(1-\beta\cos\psi)}{1-\beta\cos\varphi + \left[\left(\frac{ea}{m\gamma}\right)^2\frac{1}{1-\beta\cos\psi} + \frac{2s\omega}{m\gamma}\right]\sin^2\frac{\vartheta}{2}} , \quad (5.8)$$

which is the *exact* cross section for HICS off spinless particles.[6] Note that the only quantum contribution enters via the recoil term in the denominator of (5.8).

In the medium we restrict ourselves for simplicity to the case $p_T = 0$ so that

$$\mu_s^l = 4J_s(R)^2\left[-[m^2+(ea)^2] + \left(1-\frac{1}{n'^2}\right)(m\gamma)^2\left(1+\frac{(ea)^2}{(m\gamma)^2(1\mp\beta n)}\right)\right] + 2(ea)^2[J_{s-1}(R)^2 + J_{s+1}(R)^2] \quad (5.9)$$

and

$$R=\frac{ea\omega_s' n' \sin\varphi}{m\gamma\omega(1\mp n\beta)} \tag{5.10}$$

for $\psi=0$ and $\psi=\pi$, respectively. A comparison of (5.9) with (5.5) indicates a considerable enhancement of HICS-type SESR over ordinary HICS if (i) $\gamma\gg 1$, $n'$ sufficiently large and $\mu_s$ is dominated by the first term in (5.9); or (ii) $\omega_s'$ exceeds the corresponding vacuum value to some amount. Expanding the Bessel functions for $R\ll 1$ we have for $s=\pm 1$,

$$\mu_{\pm1}^{cl}=(ea)^2\left[\left(\frac{n'\omega'\sin\varphi}{\omega(1\mp n\beta)}\right)^2\left(-\frac{1}{\gamma^2}+1-\frac{1}{n'^2}\right)+2\right] + O(ea)^4 . \tag{5.11}$$

E.g. for $n'=2$, $\varphi=\pi/2$ we have $\omega'=(1\mp n\beta)\omega$, and for $\gamma\gg 1$, $\mu_{-1}^{cl}/\mu_{-1}^{vac}=5/2$. Note that at $\varphi=\pi/2$ there are no additional contributions from Čerenkov-type SESR. The additional terms in (4.24) which contribute for $p_T\neq 0$ can be shown to be at least of relative order $\omega/\omega'$.

### 2. *Power spectrum*

The power spectrum is related to the transition rate by

$$\Gamma=\int d\omega' \frac{P(\omega')}{\omega'} , \tag{5.12}$$

hence from (4.31)

$$P(\omega')=\frac{e^2\omega'}{8\pi(m\gamma)^2\beta}\sum_{s\neq 0}\int d\Omega_{\vec{k}'}\mu_s\,\delta\left(\cos\varphi-\frac{1}{\beta n'} -\frac{s\omega}{\beta n'\omega'}(1-\beta n\cos\psi)\right) =\sum_{s\neq 0}P_s(\omega') . \tag{5.13}$$

The $\delta$ function restricts the range of frequencies for which $P_s(\omega')$ is different from zero. In the subluminal region $(1-\beta n'>0)$

$$-\frac{s\omega(1-\beta n\cos\psi)}{1+\beta n'}<\omega'<-\frac{s\omega(1-\beta n\cos\psi)}{1-\beta n'} , \tag{5.14}$$

so depending upon the sign of $1-\beta n\cos\psi$ only positive or negative values of $s$ contribute to the sum in (5.13). In the superluminal region $1-\beta n'<0$ the frequencies are only restricted from below

$$\omega'>\max\left(-\frac{s\omega(1-\beta n\cos\psi)}{1+\beta n'},\ \frac{s\omega(1-\beta n\cos\psi)}{\beta n'-1}\right). \tag{5.15}$$

All $P_s(\omega')$ contribute in a certain range of frequencies. Note that in (5.14) and (5.15) we treated $n'$ as almost constant which is only justified for low frequencies. Equations (5.14) and (5.15) give rise to discontinuities in the power spectrum which have been emphasized and estimated in some cases by Kroll.[5]

## B. Čerenkov-type SESR

We are first going to extract the power spectrum of ordinary quantum Čerenkov radiation from Eqs. (4.12)–(4.16). In the absence of an incident field $ea=0$ and hence $R=0$ so that only the term with $s=0$ survives.

$$\begin{aligned}\tfrac14\mu_0=\tfrac14 A&=-m^2+\left(1-\frac{1}{n'^2}\right)p_0(p_0-\omega') +pk'+\omega'^2\left(n'^2-\frac{1}{n'^2}\right)\\ &=\left(\vec{p}-\frac{\vec{k}}{2}\right)^2-\frac{1}{n'^2}\left(p_0-\frac{\omega'}{2}\right)^2=\vec{p}^2\sin^2\varphi .\end{aligned} \tag{5.16}$$

in the last line $\phi=(2p-k')k'=0$ has been used. From (4.31) and (5.12) we obtain the power spectrum

$$P(\omega')=e^2\beta\omega'\int d\cos\varphi\,\sin^2\varphi \times\delta\left[\cos\varphi-\frac{1}{\beta n'}\left(1+\frac{(n'^2-1)\omega'}{2m\gamma}\right)\right] . \tag{5.17}$$

This is the usual result derived from the Klein-Gordon equation. It differs from the Dirac equation result[13] by the absence of spin-induced second-order quantum corrections. The argument of the $\delta$ function in (5.17) given by pure kinematics is the same in both cases.

### 1. *Power spectrum*

In the presence of the incident field we have

$$P(\omega')=\frac{e^2\omega'}{8\pi(m\gamma)^2\beta}\int d\Omega_{\vec{k}}\sum_s \mu_s^{cl}\,\delta(\cos\varphi-\cos\hat{\varphi}) , \tag{5.18}$$

$$\cos\hat{\varphi}=\frac{1}{\beta n'}\left(1+\frac{(ea)^2(1-nn'\cos\vartheta)}{2(m\gamma)^2(1-\beta n\cos\psi)} +\frac{s\omega}{\omega'}(1-\beta n\cos\psi)\,\Delta\right) , \tag{5.19}$$

up to quantum corrections. Equation (5.18) is of interest mainly for large frequencies such that $\omega'\lesssim\omega_a$, hence $\omega/\omega'\sim 10^{-4}$. But then $R^2$ [Eq. (4.25)] is not necessarily small and an expansion in powers of $R^2$ is impossible.

Because $\omega/\omega' \ll 1$ it is tempting to drop the $s$-dependent term in (5.19). By means of the summation formulae[14]

$$\sum_{s=-\infty}^{\infty} J_s(R)^2=1\,, \quad \sum_{s=-\infty}^{\infty} sJ_s(R)^2=0\,, \quad \sum_{s=-\infty}^{\infty} s^2 J_s(R)^2=\tfrac{1}{2}R^2\,,$$

we then have

$$\sum_{s=-\infty}^{\infty} \mu_s^{cl}=4(m\gamma\beta)^2\left\{1-\frac{1}{(\beta n')^2}+\left(\frac{ea}{m\gamma}\right)^2\left[\frac{1}{\beta^2}\left(1-\frac{1}{n'^2}\right)\frac{1}{1-\beta n\cos\psi}+\frac{1}{2}\left(n^2-\frac{1}{n'^2}\right)\left(\frac{\sin\psi}{1-\beta n\cos\psi}\right)^2 + (1-n^2)\left(1-\frac{1}{n'^2}\right)\frac{\sin^2\psi}{(1-\beta n\cos\psi)^3}\right]\right\}\,, \tag{5.20}$$

which reproduces the power spectrum (5.17) of ordinary Čerenkov radiation apart from small intensity-dependent corrections. The latter, however, are not reliable: expanding the $\delta$ function in (5.18) in powers of $s$ and including these terms in the summation contributes further terms of the same order of magnitude as the corrections exhibited in (5.20).

Hence we shall now improve the summation procedure. We treat only the case $p_T=0$, where $R^2$ is given by (5.10) and depends in a simple way on the angle $\varphi$. Note that for $\psi=0,\pi$ we have $\vartheta=\varphi,\pi-\varphi$. We then have to calculate

$$I=\zeta^{-1}\sum_{s=-\infty}^{\infty}\int_{-1}^{1} d\cos\varphi\,\delta(\cos\varphi-\xi-s\eta)J_s(R)^2\,, \tag{5.21}$$

with

$$\zeta=1\pm\frac{n(ea)^2}{2\beta(m\gamma)^2(1\mp\beta n)}\,, \tag{5.22a}$$

$$\xi=\frac{1}{\beta n'\zeta}\left(1+\frac{(ea)^2}{2(m\gamma)^2(1\mp\beta n)}\right)\,, \tag{5.22b}$$

$$\eta=\frac{\omega}{\beta n'\omega'\zeta}(1\mp\beta n)\Delta\,, \tag{5.22c}$$

where we carefully retain all first-order intensity-dependent corrections.

Introducing the Fourier representation of the $\delta$ function the summation can be done by means of the addition theorem of Bessel functions,[14] hence

$$I=\zeta^{-1}\int_{-\infty}^{\infty}\frac{dt}{2\pi}\int_{-1}^{1} d\cos\varphi\, e^{-it(\cos\varphi-\xi)}J_0\left(2R\sin\frac{\eta t}{2}\right)\,.$$

Replacing due to (5.22c) the sine by its argument the integral over $t$ can be done[14] with the result

$$I=(\pi\zeta)^{-1}\int_{-1}^{1} d\cos\varphi[R^2\eta^2-(\cos\varphi-\xi)^2]^{-1/2}\times\Theta(R\eta-|\cos\varphi-\xi|)\,, \tag{5.23}$$

where

$$R\eta=\sigma\sin\varphi\,,\quad \sigma=\frac{ea}{m\gamma\beta}\,\frac{\Delta}{\zeta}\approx\frac{ea}{m\gamma\beta}\,.$$

The zeros of the argument of the square root are

$$\cos\varphi=(1+\sigma^2)^{-1}[\xi\pm\sigma(1-\xi^2+\sigma^2)^{1/2}]=x_\pm\,, \tag{5.24}$$

hence the actual limits of the integration in (5.23) are min $(1,x_+)$ and $x_->-1$. We then have

$$I=\zeta^{-1}(1+\sigma^2)^{-1/2}\begin{cases}1 & (x_+<1)\,,\\ \dfrac{1}{2}+\dfrac{1}{\pi}\arcsin\dfrac{(1-\xi+\sigma^2)m\gamma\beta}{(1-\xi^2+\sigma^2)^{1/2}ea} & (x_+>1)\,.\end{cases} \tag{5.25}$$

Because the argument of the arcsin is smaller than one, we have

$$I\le\zeta^{-1}(1+\sigma^2)^{-1/2}\approx 1-\frac{1}{2}\left(\frac{ea}{m\gamma\beta}\right)^2\frac{1}{1\mp\beta n}\,. \tag{5.26}$$

In order to evaluate (5.18) completely we need in addition the integrals $I_\pm$ which are obtained from $I$ replacing $s\eta$ by $(s\pm1)\eta$. Obviously this comes up to replacing $\xi$ by $\xi\pm\eta$ in (5.25).

The power spectrum for Čerenkov-type SESR is then

$$P(\omega')=\frac{e^2\omega'}{4(m\gamma)^2\beta}\left\{4\left[-m^2-(ea)^2+\left(1-\frac{1}{n'^2}\right)p_0\left(p_0+\frac{(ea)^2\omega}{pk}\right)\right]I+2(ea)^2(I_++I_-)\right\}\,,$$

or letting $I_++I_-\approx 2I$,

$$P(\omega')=e^2\omega'\beta\left[1-\frac{1}{(\beta n')^2}+\left(1-\frac{1}{n'^2}\right)\left(\frac{ea}{m\gamma\beta}\right)^2\frac{1}{1\mp\beta n}\right]I \tag{5.27}$$

or, for $\beta n'$ not too close to one and sufficiently small intensity such that $x_+<1$,

$$P(\omega')=e^2\omega'\beta\left\{\left(1-\frac{1}{(\beta n')^2}\right)\left[1+\frac{1}{2(1\mp\beta n)}\left(\frac{ea}{m\gamma\beta}\right)^2\right]-\frac{1}{1\mp\beta n}\left(\frac{ea}{m\gamma^2\beta n'}\right)^2\right\}\,, \tag{5.28}$$

which for $\psi=\pi$ exceeds insignificantly ordinary Čerenkov radiation. For $x_+>1$ the reverse may happen. Equation (5.27) deviates qualitatively from Kroll's results: If $1-(\beta n')^{-2}$ and $(ea/m\gamma\beta)^2$

are fixed there is no dependence on $\omega/\omega'$ left. Since the decisive conclusion, that the Čerenkov-type SESR power spectrum does not significantly deviate from ordinary Čerenkov radiation, agrees in both cases we will not pursue the discrepancy further.

### 2. *Angular distribution*

The angular distribution of ordinary Čerenkov radiation can be obtained from (4.30) and (5.1) or from (5.17) and reads

$$\frac{d\Gamma}{d\cos\varphi}=\sum_i \frac{e^2}{|\dot{n}_i'|}\tan^2\varphi\,, \qquad (5.29)$$

where the sum extends over the Čerenkov frequencies $\omega_i=\omega_i(\varphi)$ which occur at the angle $\varphi$ and $n_i'=dn/d\omega'\,|\,\omega'=\omega_i$. Note that (5.29) is contrary to appearance a rather implicit representation.

The corresponding angular distribution in the presence of the incident field is

$$\frac{d\Gamma}{d\Omega_{\vec{k}'}}=\frac{e^2}{4\pi m\gamma}\sum_{i,s} n_{is}'\omega_{is}'\mu_s^{cl}\,|\,(\partial\phi/\partial\omega')_{\omega'=\omega_{is}}|^{-1}\,, \qquad (5.30)$$

with $\omega_{is}(\varphi)$ denoting the frequencies of Čerenkov-type SESR. In Sec. III we had found that the frequencies $\omega_{is}'$ deviate only very slightly from the Čerenkov frequencies $\omega_i'$ given by (3.7) [cf. (3.11)], so that we may let $\omega_{is}'=\omega_i'$. Now all the Bessel-functions occurring in $\mu_s$ have the common argument $R(\omega_{is}')=R(\omega_i')$. Hence we can perform the sum over $s$ in (5.30) by means of (5.20). In view of (3.7) we have neglected quantum contributions ($\psi=0,\pi$)

$$\partial\phi/\partial\omega'=2m\gamma\beta\zeta\omega'\dot{n}'\cos\varphi\,.$$

The angular distribution is then to first order in $(ea/m\gamma)^2$

$$\frac{d\Gamma}{d\cos\varphi}=\sum_i\frac{e^2}{|\dot{n}_i'|}\left[\tan^2\varphi\left(1+\frac{(ea)^2(3\beta\mp2n)}{2\beta(m\gamma)^2(1\mp\beta n)}\right)+\left(\frac{ea}{m\gamma}\right)^2\frac{1}{1\mp n\beta}\left(1\mp\frac{n}{\beta}+\frac{1}{\gamma^2\beta^2\cos^2\varphi}\right)\right]. \qquad (5.31)$$

## C. Comparison with earlier work

The work of Šoln[2] and Zachary[3,4] is restricted to a linear approximation with respect to the intensity of the incident field. Whereas this is sufficient for HICS-type it is not for Čerenkov-type SESR. Consequently the origin of the strong enhancement of SESR over ordinary Čerenkov radiation claimed by Zachary[3] is that his linear approximation comes up to replacing $J_1(R)^2$ by $R^2/4$ for $R\gg1$. Within the framework of the classical linear approximation our results agree with Refs. 2 and 3 if the linear polarization which has been assumed there is averaged over. In Ref. 4 an essentially arbitrary geometry of the participating fields is allowed for as in the present paper. Of course, also in that paper the immediate vicinity of the Čerenkov cone must be excluded. We recognize a minor difference to Ref. 4 because there the terms proportional to $sk^2=s\omega^2(1-n^2)$ which are present in (4.24) and (4.26), are missing.

Relating our results to Kroll's[5] we have to take into account that the velocities $\beta$, $\beta_\perp$, and $\beta_z$ in Kroll's work (henceforth referred to by $\beta^K$, $\beta_\perp^K$, and $\beta_z^K$) denote the velocity *in presence* of the incident field whereas our $\beta$ is the velocity of the *free* electron. The connection is established by noting that the zero component $\tilde{p}_0$ of the effective momentum (3.1) is the energy of the electron inside the incident field, hence (for $\psi=\pi$)

$$\gamma^K=\gamma\left[1+\frac{1}{2(1+\beta n)}\left(\frac{ea}{m\gamma}\right)^2\right] \qquad (5.32)$$

and, because

$$\beta^{K2}=\beta_z^{K2}+\beta_\perp^{K2}\,,\quad \beta_\perp^{K2}=\left(\frac{ea}{m\gamma}\right)^2,$$
$$\beta_z^K=\beta-\frac{1}{2}\,\frac{n+\beta}{1+n\beta}\left(\frac{ea}{m\gamma}\right)^2 \qquad (5.33)$$

to first order in $(ea/m\gamma)^2$. With this in mind we notice that our Eq. (4.24) for $\psi=\pi$ and $n=n'$ agrees with Kroll's Eq. (3.11) up to one difference: instead of the term $(1-n'^{-2})(ea)^2p_0\omega/pk$ in (4.24) he has $(ea)^2/(1+\beta n)\gamma^2$. For $n'\beta=1$ both expressions are identical. We do not have an explanation for the discrepancy between Kroll's Table I and our Eq. (5.27).

We conclude with an argument in favor of our quantum approach: Kroll's procedure relies on the exact solution of the Lorentz force equation in the incident field which yields a simple helical motion only if the electron moves initially parallel or antiparallel to $\vec{k}$. On the other hand, the Klein-Gordon equation provides for a compact solution also for arbitrary angles $\psi$. Our calculational efforts would have been considerably reduced if we had restricted ourselves from the beginning to the case $\psi=0,\pi$.

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