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Intermediate- and strong-coupling approximations for scattering in a laser field

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A study is made of the range of applicability of various versions of the low-frequency approximation for electron-atom scattering in a laser field. A key physical parameter—the ratio of the electron-field interaction energy to the photon energy—is identified and a distinction is drawn between two regimes, corresponding to values of this ratio of order unity (intermediate coupling) and values large compared to unity (strong coupling). Separate derivations are given which are appropriate to one or the other of these regimes, with proper account taken of the possibility of resonances or composite bound-state poles in the field-free scattering amplitude. The strong-coupling case is characterized by the emission or absorption of large numbers of virtual soft photons in initial and final states. The analogy with the problem of spontaneous infrared radiation which this physical picture suggests is put into more precise form here through the derivation of sum rules for the total cross section and for the average energy transferred to the field. The strong-coupling approximation is closely related to the approximate treatment, due to Keldysh and others, of multiphoton ionization as a tunneling process. This relation is demonstrated explicitly here by recognizing that ionization may be thought of as the second half of an induced resonance reaction and applying, to the amplitude obtained in this way, some of the approximation techniques developed for the scattering problem.

I. INTRODUCTION

While the problem of electron-atom scattering in a laser field is greatly simplified in the low-frequency limit the remaining dynamical complexity is still considerable and a specification of the domains of validity of various versions of the low-frequency approximation requires some care. For the sake of numerical orientation let us consider a scattering experiment in which the power density is $I = 10^8$ W/cm^2, the initial kinetic energy of the electron is $K \equiv p^2/2\mu = 10$ eV, and the energy of the laser photon is $\hbar\omega = 0.1$ eV. (These parameters correspond closely with those in the experiment of Weingartshofer *et al.*[1]) A dimensionless parameter providing a measure of the field strength may be taken to be $\delta_1 = eA_c/cp$, where $A_c = (2\pi cI/\omega^2)^{1/2}$ is the magnitude of the classical vector potential (in the Coulomb gauge) corresponding to power density I and frequency ω. With the above choice of parameters we have $\delta_1 \approx 10^{-2}$, $\delta_2 \equiv (\hbar\omega/K) \approx 10^{-2}$, and $\delta_3 \equiv (p/\mu c) \approx 6 \times 10^{-3}$. With each of the δ_i small compared with unity, and with δ_1/δ_2 of order unity, one is well within the domain of validity of the low-frequency approximation discussed, in various versions, by several authors.[2-4]

A derivation of the low-frequency approximation, in a form appropriate to the above-mentioned experimental regime, can be carried out in a number of ways. The essential element, however, is the use of exact solutions of the Schrödinger equation for the electron in the field to represent the initial and final states. Perturbative solutions are inappropriate for this purpose even when, as in the example just considered, the electron-field interaction energy is two orders of magnitude smaller than the kinetic energy of the projectile. Actually, it is the ratio of the interaction energy to the *photon* energy (or, equivalently, δ_1/δ_2) which gives a proper measure of the effective coupling strength in asymptotic states. This is just a consequence of the near-degeneracy of the electron-field states—for fixed electron energy the levels are separated by $\hbar\omega$. Of course the field also acts in intermediate states, and affects the target atom in initial and final states, but in the absence of near degeneracies the relevant coupling strength is of order δ_1 and ordinary perturbation theory suffices.[5,6] The low-frequency approximation corresponds to dropping terms of second and higher order in this modified expansion. The derivation of the first order correction term is very much simplified by the introduction of a gauge transformation which enables one to express the perturbation in terms of the electric-dipole interaction.[6] The merit of this transformation lies in the fact that the interaction in the electric-dipole form is effectively weaker than that in the Coulomb gauge by a factor of order $\hbar\omega/\Delta E$, where ΔE represents an average excitation energy of the atomic system.[7] Indeed one is able to reproduce the low-frequency approximation in the new gauge by treating the electric-dipole interaction in *lowest* order (see Sec. III B).

Versions of the low-frequency approximation which remain valid in the presence of resonances in the field-free scattering amplitude have been introduced by Krüger and Jung[4] and by Mittleman[8] for potential scattering. The resonant case is reconsidered in Sec. III B, in the context of electron scattering from an atom. The result obtained includes a first-order correction term and

involves only the on-shell scattering amplitude. the on-shell scattering amplitude.

Suppose now, in the example discussed above, we allow the power density to increase from 10^8 to 10^{12} W/cm^2, with the other parameters held fixed. Since in this case δ_1 is of order unity the validity of the low-frequency approximation discussed above is called into question and the derivation must be reexamined.[9] Putting the problem in somewhat more general terms we may consider an approach to the static limit in which the frequency is decreased, with the electric field strength held fixed; δ_1 will then increase like ω^{-1}. If the electric field is sufficiently weak the electric-dipole interaction may be treated in perturbation theory. (In this case the gauge transformation mentioned above will play a crucial role since an expansion in powers of δ_1 will be inappropriate.) The low-frequency approximation corresponds once again to treating the electric-dipole interaction in lowest order but rather than taking δ_1 to be a small parameter in the analysis of the resultant expression we assume that δ_1/δ_2 is a *large* parameter and apply stationary phase arguments. We are distinguishing, then, between an intermediate-coupling regime, in which δ_1/δ_2 is of order unity, and a strong-coupling regime where δ_1/δ_2 is large compared to unity. The latter case was first analyzed by Kroll and Watson for potential scattering.[10] In the more general electron-atom scattering problem studied here we find that, in the absence of resonances, the low-frequency approximation for the transition amplitude may be expressed in the same form in the intermediate- and strong-coupling regimes [see Eq. (3.29) below], although the derivation differs in detail for the two cases. It is then a simple matter to verify that the sum rule for the total cross section, derived earlier for the intermediate-coupling case,[11] is valid in the strong-coupling regime as well. A sum rule of this type was first derived for the relativistic scattering problem by Brown and Goble[12] who based their argument on general field-theoretic considerations. These authors have provided a clear discussion of the connection between soft-photon approximations and the classical limit; this connection has also been pointed out by Kroll and Watson. The physical picture which emerges from these studies is very similar to that developed by Bloch and Nordsieck[13] in their analysis of *spontaneous* infrared radiation in the scattering of charged particles. To emphasize this similarity we have summarized results obtained here, in Sec. IIIC, by formulating what might be called a strong-field version of the Bloch-Nordsieck theorem.

In approaching the static limit one takes into account the effect of a large number of soft photons acting coherently. This effect can lead to a transfer of energy of the order $N\hbar\omega$, with $N \gg 1$, as it does in ionization through the tunneling mechanism. The relationship between scattering and multiphoton ionization in the low-frequency limit is developed in Sec. IIID. The basis of the discussion is the recognition that ionization may be viewed as the second half of a resonance transition induced by the field. In this way a representation of the ionization amplitude is obtained which can serve as the basis for a modified perturbation theory of the type discussed above for scattering. The leading term in the expansion gives rise to the tunneling approximation studied earlier.[14-16] It might be noted that the problem of developing a consistent and practical theoretical description of final-state electron-field interaction effects in multiphoton ionization has taken on added interest now that these effects are being studied experimentally.[17]

II. PRELIMINARIES: SCATTERING IN A LASER FIELD

A formulation of the problem of electron-atom scattering in the presence of a laser field, developed previously,[3,6] is now briefly reviewed to provide the background for the discussion to follow. In the initial and final states of the scattering process the projectile electron and target atom interact with the laser field but not with each other. These asymptotic states satisfy

$$(H - E_\beta)\,|\,\Phi_\beta\rangle = V_\beta\,|\,\Phi_\beta\rangle\,, \tag{2.1}$$

where β is a channel index representing the set of observables which defines the state[18] and V_β is the net projectile-target interaction in channel β. In the occupation-number representation of the field the Hamiltonian is expressed as

$$H = \sum_j \left(\vec{p}_j - \frac{e}{c}\vec{A}(\vec{r}_j)\right)^2/2\mu + H_F + V\,. \tag{2.2}$$

Here V is the sum of the interparticle Coulomb potentials. For a plane-wave laser field of propagation vector $\vec{k}$, frequency ω, and complex polarization $\vec{\lambda}$ the field Hamiltonian is $H_F = \hbar\omega a^\dagger a$ and

$$\vec{A}(\vec{r}) = \left(\frac{2\pi\hbar c^2}{\omega L^3}\right)^{1/2}(a\vec{\lambda}e^{i\vec{k}\cdot\vec{r}} + a^\dagger\vec{\lambda}^* e^{-i\vec{k}\cdot\vec{r}}) \tag{2.3}$$

is the vector potential appropriate to a quantization volume L^3. The transition amplitude may then be put in the form

$$T_{\beta'\beta} = \langle\Phi_{\beta'}|\,[V_\beta + V_{\beta'}(E - H)^{-1}V_\beta]\,|\,\Phi_\beta\rangle\,, \tag{2.4}$$

where E differs from the total energy $E_\beta = E_{\beta'}$

by the addition of an infinitesimal positive imaginary part.

In treatments of the interaction of atomic systems with a classical radiation field one commonly makes use of a particular gauge transformation to facilitate the introduction of the electric-dipole approximation for the atom-field interaction.[19] In the intense-field limit the classical and quantum treatments of the field are equivalent and the role of the gauge transformation is played, in the quantum treatment, by the unitary transformation e^{g}, with

$$g = (ie/\hbar c)\sum_j \vec{A}(\vec{r}_j)\cdot\vec{r}_j . \tag{2.5}$$

The effect of the transformation is readily determined using the relations

$$\vec{p}_j - \frac{e}{c}\vec{A}(\vec{r}_j) = e^{g}\left(\vec{p}_j + \frac{e}{c}\hat{k}\vec{r}_j\cdot\vec{E}(\vec{r}_j)\right)e^{-g} \tag{2.6}$$

and (ignoring photon depletion effects[6])

$$H_F = e^{g}\left(H_F - e\sum_j \vec{E}(\vec{r}_j)\cdot\vec{r}_j\right)e^{-g} ; \tag{2.7}$$

here

$$\vec{E}(\vec{r}) \equiv i\left(\frac{\omega}{c}\right)\left(\frac{2\pi\hbar c^2}{\omega L^3}\right)^{1/2}(a\vec{\lambda}e^{i\vec{k}\cdot\vec{r}} - a^{\dagger}\vec{\lambda}^{*}e^{-i\vec{k}\cdot\vec{r}}) \tag{2.8}$$

represents the electric field. One finds that the Hamiltonian can be written as

$$H = e^{g}\overline{H}e^{-g} , \tag{2.9}$$

with the transformed Hamiltonian given by

$$\overline{H} = \sum_j \left(\vec{p}_j + \frac{e}{c}\hat{k}\vec{r}_j\cdot\vec{E}(\vec{r}_j)\right)^2/2\mu + H_F + V - e\sum_j \vec{E}(\vec{r}_j)\cdot\vec{r}_j . \tag{2.10}$$

We may anticipate that the electric-dipole approximation is appropriate not only for the description of the interaction with the field in intermediate states of the scattering but for the target-field interaction in initial and final states as well. This suggests that we introduce the transformed state $|\overline{\Phi}_\beta\rangle = e^{-g_T}|\Phi_\beta\rangle$, where

$$g_T = (ie/\hbar c)\sum_j^{(T)} \vec{A}(\vec{r}_j)\cdot\vec{r}_j , \tag{2.11}$$

the sum running over coordinates of the target electrons only. One easily verifies that $|\overline{\Phi}_\beta\rangle$ satisfies an equation of the form (2.1) but with the target-field interaction

$$H_T' = \sum_j^{(T)}\left(-\frac{e}{\mu c}\vec{p}_j\cdot\vec{A}(\vec{r}_j) + \frac{e^2}{2\mu c^2}A^2(\vec{r}_j)\right) \tag{2.12}$$

replaced by

$$\overline{H}_T' = \sum_j^{(T)}\left(\frac{e}{\mu c}\vec{p}_j\cdot\hat{k}\vec{r}_j\cdot\vec{E}(\vec{r}_j) + \frac{e^2}{2\mu c^2}(\vec{r}_j\cdot\vec{E}(\vec{r}_j))^2 - e\vec{r}_j\cdot\vec{E}(\vec{r}_j)\right) . \tag{2.13}$$

In the nonrelativistic limit considered here the first two terms in this expression will be small compared with the dipole interaction represented by the third term. Furthermore, the dipole interaction will itself be small compared with H_T' in the low-frequency limit. The matrix elements of $\overline{H}_T'$ and H_T' are, in order of magnitude, in the ratio $\hbar\omega/\Delta\epsilon$, where $\Delta\epsilon$ represents some average excitation energy of the target. A similar analysis can be made of the interaction of the field with the electron-atom system in intermediate states.

To apply the transformation to the transition amplitude itself we write $(E-H)^{-1} = e^{g}(E-\overline{H})^{-1}e^{-g}$ in Eq. (2.4). Then, with $g = g_e + g_T$ for the initial state and $g = g_e' + g_T'$ for the final state (the primes account for the possibility of electron exchange) we have the alternative form

$$T_{\beta',\beta} = \langle e^{-g_e'}\overline{\Phi}_{\beta'}|[V_\beta + V_{\beta'}\overline{G}(E)V_\beta]|e^{-g_e}\overline{\Phi}_\beta\rangle , \tag{2.14}$$

where

$$\overline{G}(E) = (E-\overline{H})^{-1} . \tag{2.15}$$

III. GENERALIZED LOW-FREQUENCY APPROXIMATION

A. Lowest order in modified perturbation theory

We begin by examining the expression (2.14) for the transition amplitude in the approximation in which the interaction with the laser field, as it appears in the transformed Hamiltonian $\overline{H}$, is neglected. This interaction appears not only in the intermediate-state propagator $\overline{G}(E)$ but also in the Schrödinger equation for the transformed asymptotic states $|\overline{\Phi}_\beta\rangle$. As a first step we outline the procedure for constructing these states in this approximation.

In the time interval before the field is switched on the initial state is of the form $|\vec{p}\rangle|n\rangle|\chi_i\rangle$. Here $\vec{p}$ and n specify the electron momentum and photon number, respectively, and $|\chi_i\rangle$ is the isolated target state of energy ϵ_i. As the projectile-field interaction is switched on adiabatically the unperturbed state $|\vec{p}\rangle|n\rangle$ evolves into the state $|\psi_{n\vec{p}}\rangle$ satisfying

$$\left(\frac{(\vec{p}-e\vec{A}/c)^2}{2\mu} + \hbar\omega a^{\dagger}a\right)|\psi_{n\vec{p}}\rangle = E_{n\vec{p}}|\psi_{n\vec{p}}\rangle . \tag{3.1}$$

Since the transformed target-field interaction $\overline{H}_T'$ is ignored in this approximation we have

$$|\overline{\Phi}_\beta\rangle \approx |\psi_{n\vec{p}}\rangle|\chi_i\rangle . \tag{3.2}$$

The construction of the state $|\psi_{n\vec{p}}\rangle$ has been described previously[20] and for ease of reference we summarize the result here. We have the representation

$$|\psi_{n\vec{p}}\rangle=\sum_{l=-\infty}^{\infty}\gamma_l(\vec{p})|n+l\rangle|\vec{p}-l\hbar\vec{k}\rangle, \tag{3.3}$$

where the expansion coefficients are of the form

$$\gamma_l(\vec{p})=\int_0^{2\pi}\frac{d\phi}{2\pi}e^{il\phi}e^{iS_{\vec{p}}(\phi)}. \tag{3.4}$$

Here we have defined

$$S_{\vec{p}}(\phi)=\rho_{\vec{p}}\sin(\phi+\theta)+\alpha\sin2\phi\ . \tag{3.5a}$$

$\rho_{\vec{p}}$ and θ are real parameters satisfying

$$\rho_{\vec{p}}e^{i\theta}=\frac{2}{\hbar\omega-\hbar\vec{k}\cdot\vec{p}/\mu}\left(\frac{2\pi\hbar e^2n}{\mu^2\omega L^3}\right)^{1/2}\vec{p}\cdot\vec{\lambda}. \tag{3.5b}$$

We also have

$$\alpha=\frac{\Delta(\vec{\lambda}\cdot\vec{\lambda})}{2(\hbar\omega-\hbar\vec{k}\cdot\vec{p}/\mu)}, \tag{3.5c}$$

with

$$\Delta=\left(\frac{e^2}{2\mu c^2}\right)\left(\frac{4\pi n\hbar c^2}{\omega L^3}\right). \tag{3.5d}$$

The energy eigenvalue in Eq. (3.1) is determined to be $E_{n\vec{p}}=p^2/2\mu+n\hbar\omega+\Delta$.

To calculate $e^{-i\xi_e}|\psi_{n\vec{p}}\rangle$ we expand the exponential as

$$e^{-i\xi_e}=1-i\left(\frac{e}{\hbar c}\right)\vec{A}(\vec{r}_e)\cdot\vec{r}_e+\cdots, \tag{3.6}$$

where $\vec{r}_e$ is the position operator for the incident electron. With the aid of the representation (3.3) we find

$$e^{-i\xi_e}|\psi_{n\vec{p}}\rangle=\sum_l\int_0^{2\pi}\frac{d\phi}{2\pi}e^{il\phi}e^{iS_{\vec{p}}(\phi)}\left(1-i\frac{e}{\hbar c}\vec{A}_c(\phi)\cdot\vec{r}_e+\cdots\right)|n+l\rangle|\vec{p}-l\hbar\vec{k}\rangle, \tag{3.7}$$

where

$$\vec{A}_c(\phi)=\left(\frac{2\pi cI}{\omega^2}\right)^{1/2}(\vec{\lambda}e^{i\phi}+\vec{\lambda}^*e^{-i\phi}) \tag{3.8}$$

may be interpreted as the classical vector potential corresponding to a phase ϕ and a flux $I=(n\hbar\omega)c/L^3$. In calculating the higher-order terms in the expansion (they must be retained in an analysis of the strong-coupling regime) we ignore the commutator $[a,a^\dagger]$. This allows us to recombine terms in exponential form. Recognizing $\vec{r}_e$ as the generator of momentum translations we obtain

$$e^{-i\xi_e}|\psi_{n\vec{p}}\rangle=\sum_l\int_0^{2\pi}\frac{d\phi}{2\pi}e^{il\phi}e^{iS_{\vec{p}}(\phi)}\times|n+l\rangle|\vec{p}-\frac{e}{c}\vec{A}_c(\phi)-l\hbar\vec{k}\rangle, \tag{3.9}$$

in obvious correspondence to the result one would obtain in a semiclassical treatment.[21]

The approximation (3.2) along with the result shown in Eq. (3.9) provides us with a representation of the state vectors $e^{-i\xi_e}|\overline{\Phi}_\beta\rangle$ and $e^{-i\xi_e'}|\overline{\Phi}_{\beta'}\rangle$ which appear in Eq. (2.14). To proceed we apply the basic approximation to the resolvent $\overline{G}(E)$ in Eq. (2.14). Suppose we express the transformed Hamiltonian $\overline{H}$ as

$$\overline{H}=H_0+H_F+\overline{H}', \tag{3.10}$$

where $H_0=\sum_j p_j^2/2\mu+V$ is the Hamiltonian of the isolated electron-atom system. In the approximation in which $\overline{H}'$ is neglected $\overline{G}(E)$ may be replaced by $(E-H_0-H_F)^{-1}$. This allows us to express the scattering amplitude in the presence of the field in terms of the field-free electron-atom scattering amplitude

$$t(e;\vec{q}',\vec{q})=\langle\vec{q}'|\langle\chi_{i'}||V_\beta+V_{\beta'}(e-H_0)^{-1}V_\beta]|\chi_i\rangle\vec{q}\rangle. \tag{3.11}$$

(Channel labels on t are omitted for simplicity.) We readily find

$$T_{\beta'\beta}\cong\sum_l\int_0^{2\pi}\frac{d\phi'}{2\pi}\int_0^{2\pi}\frac{d\phi}{2\pi}\exp\{-i[l+n-n')\phi'+S_{\vec{p}'}(\phi')]\}e^{i[l\phi+S_{\vec{p}}(\phi)]}t(E-(n+l)\hbar\omega;\vec{q}',\vec{q}), \tag{3.12}$$

with

$$\vec{q}=\vec{p}-\frac{e}{c}\vec{A}_c(\phi)-l\hbar\vec{k} \tag{3.13a}$$

and

$$\vec{q}'=\vec{p}'-\frac{e}{c}\vec{A}_c(\phi')-(l+n-n')\hbar\vec{k}. \tag{3.13b}$$

The t amplitude in Eq. (3.12) is off the energy shell since the scalar variables

$$\xi = E - (n+l)\hbar\omega - q^2/2\mu - \epsilon_i \tag{3.14a}$$

and

$$\xi' = E - (n+l)\hbar\omega - q'^2/2\mu - \epsilon_{i'} \tag{3.14b}$$

are nonzero. The energy conservation relations $E - n\hbar\omega = p^2/2\mu + \epsilon_i + \Delta$ and $E - n'\hbar\omega = p'^2/2\mu + \epsilon_{i'} + \Delta$ can be used along with Eqs. (3.13) to reexpress the off-shell parameters in the form

$$\xi = -(\hbar\omega - \vec{p}\cdot\hbar\vec{k}/\mu)(l + S'_{\vec{p}}(\phi)), \tag{3.15a}$$

$$\xi' = -(\hbar\omega - \vec{p}'\cdot\hbar\vec{k}/\mu)((n+l-n') + S'_{\vec{p}}(\phi')), \tag{3.15b}$$

where

$$S'_{\vec{p}}(\phi) \equiv \frac{dS_{\vec{p}}}{d\phi} = \rho_{\vec{p}}\cos(\phi+\theta) + 2\alpha\cos 2\phi. \tag{3.16}$$

This last relation is equivalent to

$$\hbar\omega S'_{\vec{p}}(\phi) = -\frac{e}{\mu c}\frac{\vec{p}\cdot\vec{A}_c}{1-\vec{p}\cdot\hat{k}/\mu c} + \frac{e^2A_c^2/2\mu c^2 - \Delta}{1-\vec{p}\cdot\hat{k}/\mu c}, \tag{3.17}$$

a version which is helpful in verifying Eqs. (3.15). To continue with the analysis it is convenient at this point to distinguish between the intermediate-coupling and strong-coupling regimes discussed in Sec. I.

B. Intermediate coupling

The function $S'_{\vec{p}}(\phi)$ in Eq. (3.16) is of the order of the ratio of the electron-field interaction energy to the photon energy (δ_1/δ_2 in the notation of Sec. I). Let us suppose that this ratio is of order unity. We may then argue that contributions to the sum in Eq. (3.12) corresponding to values of $|l|$ which are large compared to unity will be negligible due to rapid oscillations of the integrand as ϕ ranges between 0 and 2π. The argument would fail if the phase were stationary for some large l value but this possibility is excluded by the assumption that $S'_{\vec{p}}$ is at most of order unity. Then ξ and ξ' in Eqs. (3.15) may be treated as first-order parameters and t in Eq. (3.12) may be expanded in a Taylor series about $\xi = \xi' = 0$. The leading term is the on-shell amplitude $t(E-(n+l)\hbar\omega, \tau(\phi,\phi'))$, which is expressed as a function of the energy and the momentum-transfer-squared variable

$$\tau(\phi,\phi') = \left(\vec{p}' + n'\hbar\vec{k} - \vec{p} - n\hbar\vec{k} - \frac{e}{c}(\vec{A}_c(\phi') - \vec{A}_c(\phi))\right)^2. \tag{3.18}$$

The first-order correction terms vanish. To see this note that in the term proportional to $(l + S'_{\vec{p}})(\partial t/\partial\xi)$ we may write

$$l e^{il\phi} e^{iS_{\vec{p}}(\phi)} = -i\left(\frac{d}{d\phi} e^{il\phi}\right) e^{iS_{\vec{p}}(\phi)}$$

and integrate by parts. Since the surface term vanishes, and since the contribution coming from the derivative of t with respect to ϕ is of higher order and may be dropped, we have effectively replaced l by $-S'_{\vec{p}}$ to first order. Thus the coefficient of $\partial t/\partial\xi$ vanishes, and in a similar way one sees that the first-order correction term proportional to $\partial t/\partial\xi'$ makes no contribution. We are assuming that t is a smooth function of the scalar variables ξ, ξ', and τ but no assumption has been made concerning the dependence of t on the energy variable. In particular, resonances may be present in the field-free scattering amplitude.

The low-frequency approximation is of the form (3.12) with t replaced by the on-shell amplitude $t(E-(n+l)\hbar\omega, \tau(\phi,\phi'))$. To simplify the double integral in Eq. (3.12) we expand the τ variable (to first order in the small parameter δ_1) as

$$\tau(\phi,\phi') = \tau_0 - 2(\vec{p}' - \vec{p})\cdot\frac{e}{c}(\vec{A}_c(\phi') - \vec{A}_c(\phi))$$

with $\tau_0 = (\vec{p}' + n'\hbar\vec{k} - \vec{p} - n\hbar\vec{k})^2$. (We have used the transversality property $\vec{k}\cdot\vec{A}_c = 0$.) The t amplitude may be expanded in a Taylor series about $\tau = \tau_0$. Keeping only the first two terms we have

$$T_{\beta'\beta} = T^{(1)}_{\beta'\beta} + T^{(2)}_{\beta'\beta} \tag{3.19a}$$

with

$$T^{(1)}_{\beta'\beta} = \sum_l \gamma^*_{l+m-m'}(\vec{p}')\gamma_l(\vec{p})\, t(E-(n+l)\hbar\omega, \tau_0). \tag{3.19b}$$

The first-order correction term is

$$T^{(2)}_{\beta'\beta} = 2\frac{e}{c}\left(\frac{2\pi cI}{\omega^2}\right)^{1/2} \times(\vec{p}' - \vec{p})\cdot\sum_l [\vec{\lambda}\gamma^*_{l+m-m'}(\vec{p}')\gamma_{l+1}(\vec{p})\Delta_l^{(+)} + \vec{\lambda}^*\gamma^*_{l+m-m'}(\vec{p}')\gamma_{l-1}(\vec{p})\Delta_l^{(-)}]. \tag{3.19c}$$

Here we have used Eq. (3.8) and the abbreviation

$$\Delta_l^{(\pm)} = \frac{\partial}{\partial\tau}[t(E-(n+l)\hbar\omega, \tau) - t(E-(n+l\pm1)\hbar\omega, \tau)]\big|_{\tau=\tau_0}. \tag{3.20}$$

Note that the correction term $T^{(2)}_{\beta'\beta}$ need be retained, to the order considered here, only if the t amplitude contains resonances. In the nonresonant case $\Delta_l^{(\pm)}$ will be of first order so that $T^{(2)}_{\beta'\beta}$ will actually be of second order. It should also be pointed out that for the particular case of linear polarization of the field ($\vec{\lambda}$ is real in this case),

the general result, resonances included, simplifies considerably. We find that the two terms (3.19b) and (3.19c) can be combined and the result expressed in the form

$$T_{\beta'\beta} \cong \sum_l \gamma^*_{l+n-n'}(\vec{p}')\gamma_l(\vec{p})\, t\left(p_l^2/2\mu + \epsilon_i, (\vec{p}_l' - \vec{p}_l)^2\right). \tag{3.21}$$

The t amplitude appearing here corresponds to an on-shell scattering process in which the initial electron momentum is

$$\vec{p}_l = \vec{p} - l\hbar\vec{k} - \frac{\mu l}{\vec{p}\cdot\vec{\lambda}}(\hbar\omega - \vec{p}\cdot\hbar\vec{k}/\mu)\vec{\lambda} \tag{3.22a}$$

and the final momentum is

$$\vec{p}_l' = \vec{p}' - (n+l-n')\hbar\vec{k} - \frac{\mu(n+l-n')}{\vec{p}'\cdot\vec{\lambda}}(\hbar\omega - \vec{p}'\cdot\hbar\vec{k}/\mu)\vec{\lambda}. \tag{3.22b}$$

The on-shell property is expressed by the relations (correct to first order)

$$E - (n+l)\hbar\omega = p_l^2/2\mu + \epsilon_i = p_l'^2/2\mu + \epsilon_{i'}. \tag{3.23}$$

The derivation of Eq. (3.21) is based on the recursion relation satisfied by the γ coefficients. That relation follows directly from the representation (3.4).[20] To the accuracy required in the analysis of the first-order correction term we may neglect the term proportional to α in Eq. (3.5a); we then have $\gamma_l(\vec{p}) \cong J_{-l}(\rho_{\vec{p}})$ (having set $\theta = 0$ since $\vec{\lambda}$ is real) and the recursion relation reduces to

$$\rho_{\vec{p}}[\gamma_{l+1}(\vec{p}) + \gamma_{l-1}(\vec{p})] = -2l\gamma_l(\vec{p}). \tag{3.24}$$

After making use of this relation in Eq. (3.19c) we find that the correction term may be put in the form

$$T^{(2)}_{\beta'\beta} = \sum_l \gamma^*_{l+n-n'}(\vec{p}')\gamma_l(\vec{p})[(\vec{p}_l' - \vec{p}_l)^2 - \tau_0] \times \frac{\partial}{\partial\tau} t\left(E - (n+l)\hbar\omega, \tau\right)\Big|_{\tau=\tau_0}. \tag{3.25}$$

The result of combining this with the leading term (3.19b) is to introduce a shift in the momenta as indicated by Eqs. (3.21) and (3.22). Equation (3.21) extends the result of Krüger and Jung[4] through the inclusion of these first-order momentum shifts, as well as the first-order correction obtained by replacing J_{-l} by γ_l. There still remains the problem of estimating higher-order corrections.[8] As a result of the near singularity associated with the resonance the term which is nominally of second order may be comparable in magnitude to the first-order correction calculated here.

In the absence of resonances the low-frequency approximation takes the form shown in Eq. (3.19b). Since by assumption $t(e, \tau_0)$ is a slowly varying function of energy for e in the neighborhood of $E - n\hbar\omega$ we may expand to first order and write

$$t\left(E - (n+l)\hbar\omega, \tau_0\right) \cong t - l\hbar\omega\frac{\partial t}{\partial e}, \tag{3.26}$$

with t and its derivative evaluated at the energy $E - n\hbar\omega$. We observe that

$$-l\gamma_l(\vec{p}) = \int_0^{2\pi}\frac{d\phi}{2\pi} e^{il\phi} e^{iS_{\vec{p}}(\phi)} S'_{\vec{p}}(\phi), \tag{3.27}$$

as may be verified by an integration by parts. The sum in Eq. (3.19b) may now be performed using the relation

$$\frac{1}{2\pi}\sum_l e^{il(\phi-\phi')} = \delta(\phi - \phi'). \tag{3.28}$$

After combining the two terms in the Taylor series we obtain the result

$$T_{\beta'\beta} \cong \int_0^{2\pi}\frac{d\phi}{2\pi} e^{i(n'-n)\phi} e^{i[S_{\vec{p}}(\phi) - S_{\vec{p}'}(\phi)]} t\left(e_{\vec{p}}(\phi), \tau_0\right). \tag{3.29}$$

Here $e_{\vec{p}}(\phi) = E - n\hbar\omega + \hbar\omega S'_{\vec{p}}$, which may be expressed, using Eq. (3.17), as

$$e_{\vec{p}}(\phi) = p^2/2\mu + \Delta + \epsilon_i - \frac{e}{\mu c}\frac{\vec{p}\cdot\vec{A}_c}{1 - \vec{p}\cdot\hat{k}/\mu c} + \frac{e^2A_c^2/2\mu c^2 - \Delta}{1 - \vec{p}\cdot\hat{k}/\mu c}. \tag{3.30}$$

A more suggestive version is $e_{\vec{p}}(\phi) = P^2(\phi)/2\mu + \epsilon_i$, with

$$\vec{P}(\phi) = \vec{p} - \frac{e}{c}\vec{A}_c(\phi) - \hat{k}\frac{1}{c}\left(\vec{p}\cdot\frac{e}{\mu c}\vec{A}_c(\phi) - \frac{e^2}{2\mu c^2}A_c^2(\phi) + \Delta\right). \tag{3.31}$$

Here $\vec{P}(\phi)$ is the momentum (expressed as a function of the phase of the field rather than the time, and corresponding to an average momentum $\vec{p}$) obtained from a solution of the classical nonrelativistic equation of motion for the charged particle within a radiation field which is represented by the vector potential $\vec{A}_c(\phi)$.[22] This result is one of a number of indications of the close connection which exists between soft-photon approximations and the classical limit. A systematic investigation of this connection, within the context of relativistic field theory, was given some time ago by Brown and Goble.[12] These authors did not express their soft-photon approximation in a form analogous to that shown in Eqs. (3.19b) and (3.29).

However, as observed subsequently,[23] their results could readily be rewritten in those forms. A nonrelativistic version of the Brown-Goble sum rule for the total cross section was derived earlier for the case of intermediate coupling.[11] This result will be rederived below in a manner which makes it clear that it holds in both the intermediate- and strong-coupling regimes. The sum rule provides another expression of the connection with the classical limit.

It is a simple matter to establish the equivalence, to first order, between the expression shown in Eq. (3.29) and the approximation previously derived for the nonresonant case in intermediate coupling.[3]

C. Strong coupling

We return to the analysis of the off-shell contributions to the t amplitude in Eq. (3.12). With $S'_{\vec{p}}(\phi)$ now taken to be large compared with unity it will no longer be appropriate to treat the parameters ξ and ξ', given by Eqs. (3.15), as first-order quantities. One may, however, argue that the dominant contribution to the integrals in Eq. (3.12) will come from values of ϕ and ϕ' for which the phase factors are stationary. It is assumed here that t varies much more slowly than the exponential factors as ϕ and ϕ' are varied. Now the stationary phase conditions $l+S'_{\vec{p}}(\phi)=0$ and $n+l-n'+S'_{\vec{p}'}(\phi')=0$ correspond precisely to the conditions $\xi=\xi'=0$. According to Eqs. (3.14) these conditions place t on the energy shell; the on-shell amplitude which now appears in Eq. (3.12) is expressed as $t(E-(n+l)\hbar\omega,\tau(\phi,\phi'))$, with $\tau(\phi,\phi')$ given by Eq. (3.18).

Recall that the strong-coupling limit is characterized by the conditions $\delta_2 \ll \delta_1$. There will still be a range of experimental conditions for which $\delta_1 \ll 1$. If δ_1 treated as a small parameter the analysis given above, which led to Eqs. (3.19) for the transition amplitude, may be taken over directly. With this case accounted for we suppose in the following that δ_1 is not small; we include, in particular, the limit in which the frequency is decreased with the electric field strength held fixed. With δ_1 of order unity or greater the approach based on a Taylor-series expansion in the τ variable is no longer useful. To make progress in simplifying Eq. (3.12) we shall assume, in the remainder of this subsection, that t is a smooth function of the energy. (Resonance effects may be treated along the lines described in Sec. IIID.) We now observe that with l replaced by $-S'_{\vec{p}}(\phi)$, valid at the point of stationary phase, the energy variable $E-(n+l)\hbar\omega$ becomes

$$E-n\hbar\omega+\hbar\omega S'_{\vec{p}}(\phi)=P^2(\phi)/2\mu+\epsilon_i\,.$$

Then t in Eq. (3.12) may be replaced by $t(P^2(\phi)/2\mu+\epsilon_i,\tau(\phi,\phi'))$. Having removed the l dependence from t we may now perform the sum over l in Eq. (3.12). This leads immediately to the form (3.29) for the transition amplitude. We have shown, then, that in the absence of resonances Eq. (3.29) provides a low-frequency approximation valid in both the intermediate- and strong-coupling regimes.

If the number of photons emitted or absorbed in the scattering process is not observed the relevant cross section is given by

$$d\sigma=(2\pi)^4\left(\frac{\mu}{p}\right)\hbar^2\,dQ\,, \tag{3.32}$$

where

$$dQ=\sum_{n'}\int d^3p'\,\delta\left(\frac{p'^2}{2\mu}+\epsilon_{i'}-\frac{p^2}{2\mu}-\epsilon_i+(n'-n)\hbar\omega\right)\times|T_{\beta'\beta}|^2\,. \tag{3.33}$$

We adopt the low-frequency approximation (3.29) for the amplitude $T_{\beta'\beta}$. The dominant contribution to the integral in Eq. (3.29) will come from values of ϕ which satisfy the stationary phase condition

$$n'-n=S'_{\vec{p}'}(\phi)-S'_{\vec{p}}(\phi)\,, \tag{3.34}$$

and if, for a given n', this condition cannot be satisfied for $0\leq\phi\leq 2\pi$ the contribution to the sum in Eq. (3.33) will be negligible.[24] This permits us to replace the momentum-transfer variable $\tau_0=[\vec{p}'-\vec{p}+(n'-n)\hbar\vec{k}]^2$ by

$$\tau(\phi)=(\vec{P}'(\phi)-\vec{P}(\phi))^2\,, \tag{3.35}$$

where $\vec{P}'(\phi)$ is the classical final-state momentum determined by replacing $\vec{p}$ in Eq. (3.31) with $\vec{p}'$. [In checking this recall that terms of order $(p/\mu c)^2$ are to be ignored.] The approximation for $T_{\beta'\beta}$ to be inserted in Eq. (3.33) is now taken to be

$$T_{\beta'\beta}\cong\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(n'-n)\phi}e^{i[S_{\vec{p}}(\phi)-S_{\vec{p}'}(\phi)]}\times t(e_{\vec{p}}(\phi),\tau(\phi))\,. \tag{3.36}$$

Using the representation

$$\delta(x)=\int_{-\infty}^{\infty}\frac{ds}{2\pi}e^{ixs} \tag{3.37}$$

in Eq. (3.33) we find all of the n' dependence to be in the exponential. The sum over n' may then be carried out using Eq. (3.28), with the result[25]

$$dQ=\int_0^{2\pi}\frac{d\phi}{2\pi}\int d^3p'\,\delta(e_{\vec{p}'}(\phi)-e_{\vec{p}}(\phi))\,|t(e_{\vec{p}}(\phi),\tau(\phi))|^2\,. \tag{3.38}$$

The average energy transferred to the field, call it $\langle(n'-n)\hbar\omega\rangle$, can be defined by the relation

$$dR = \langle (n' - n)\hbar\omega \rangle dQ \tag{3.39}$$

with

$$dR \equiv \sum_{n'} \int d^3p' (n' - n)\hbar\omega \delta\left(\frac{p'^2}{2\mu} + \epsilon_{i'} - \frac{p^2}{2\mu} - \epsilon_i + (n' - n)\hbar\omega\right) \times |T_{\beta'\beta}|^2 . \tag{3.40}$$

The sum in Eq. (3.40) is readily evaluated using the approximation (3.29) (along with an integration by parts) to give

$$dR = \int_0^{2\pi} \frac{d\phi}{2\pi} \int d^3p' \left(\frac{p^2}{2\mu} + \epsilon_i - \frac{p'^2}{2\mu} - \epsilon_{i'}\right) \times \delta(e_{\vec{p}}(\phi) - e_{\vec{p}'}(\phi)) |t(e_{\vec{p}}(\phi), \tau(\phi))|^2 . \tag{3.41}$$

The average energy transferred to the field is then seen to be precisely that which would be obtained by calculating the average energy lost by the electron-atom system assuming the electron to follow classical trajectories before and after the collision, with the collision taking place instantaneously, and without influence from the field. The result depends on the phase of the field at the instant of collision and the energy loss is averaged over the phase in addition to being weighted by the collision probability.[26] [The phase-space integral can be taken over $\vec{p}'$ rather than the momentum $\vec{P}'(\phi)$ which the electron has immediately after the collision since the Jacobian of the transformation from $\vec{P}'(\phi)$ to $\vec{p}'$ is unity.] A similar classical interpretation can be given for the cross-section sum rule (3.38). It is worth remarking on the similarity between these sum rules and the Bloch-Nordsieck theorem for scattering in a spontaneously generated infrared radiation field.[13] The present results may be summarized by stating a strong-field version of the Bloch-Nordsieck theorem of the following form.

(i) The probability for scattering accompanied by a finite change in the photon number of the field vanishes in the static limit. This follows from the fact that the amplitude given in Eq. (3.29) decreases in magnitude as the frequency is lowered due to the increasingly rapid oscillations of the integrand.[27]

(ii) The total probability for scattering, summed over final states of the field, is determined by the field-free cross section as shown in Eq. (3.38).

(iii) While the average number of photons emitted or absorbed grows without bound as the static limit is approached, the average energy transferred to the field approaches a finite value which is determined, according to Eq. (3.41), from a classical treatment of the motion of the electron in the field.

To see the relation between Eq. (3.29) and the Kroll-Watson version of the low-frequency approximation[10] we specialize to the case of linear polarization and ignore electron recoil effects of order $p/\mu c$. The stationary phase condition (3.34) then becomes

$$(n - n')\hbar\omega = -\frac{e}{\mu c} (\vec{p} - \vec{p}') \cdot \vec{A}_c .$$

With $\epsilon_i = 0$ (appropriate to potential scattering) and with $\vec{A}_c(\phi) = \vec{a}\cos\phi$ the energy variable becomes, at the point of stationary phase,

$$\frac{P^2(\phi)}{2\mu} = \frac{1}{2\mu}\left(\vec{p} + \frac{\mu(n' - n)\hbar\omega\hat{a}}{\hat{a} \cdot (\vec{p}' - \vec{p})}\right)^2 . \tag{3.42}$$

Neglect of electron recoil allows us to replace $\tau(\phi)$ by $(\vec{p}' - \vec{p})^2$. With t now independent of ϕ it may be removed from under the integral sign in Eq. (3.29); the result is identical to Eq. (5.18) of Ref. 10.

D. Induced resonance and ionization

Suppose that in the absence of the field the electron-atom system has a single composite bound state satisfying

$$H_0 |B\rangle = \mathcal{E} |B\rangle . \tag{3.43}$$

The condition for a field-induced resonance is (ignoring level shifts)

$$\mathcal{E} + n_R\hbar\omega \cong p^2/2\mu + \epsilon_i + n\hbar\omega . \tag{3.44}$$

For simplicity let us assume that the width of the composite state in the presence of the field is small enough compared with $\hbar\omega$ that the resonance condition is *not* satisfied for $n_R \to n_R \pm \hbar\omega$. The unperturbed state $|a\rangle \equiv |B\rangle |n_R\rangle$ may be subtracted out in the standard way with the aid of the projection operators $P = |a\rangle\langle a|$ and $Q = 1 - P$. (More generally, if the composite state is degenerate, P should be taken to be the projection operator onto the subspace of degenerate particle-field states.) To exhibit the resonance contribution to the scattering amplitude defined in Eq. (2.14) we introduce the resolvent identity

$$\overline{G} = \overline{G}^Q + (1 + \overline{G}^Q \overline{H}') |a\rangle (E - E_a)^{-1} \langle a| (1 + \overline{H}' \overline{G}^Q) . \tag{3.45}$$

Here we have defined

$$\overline{G}^Q = [Q(E - \overline{H})Q]^{-1} \tag{3.46}$$

and

$$E_a = \mathcal{E} + n_R\hbar\omega + \langle a| [\overline{H}' + \overline{H}' \overline{G}^Q \overline{H}'] |a\rangle , \tag{3.47}$$

and have made use of the relation $Q\overline{H}P = Q\overline{H}'P$.

The decomposition (3.45) leads to a representation of the transition amplitude as the sum

$$T_{\beta'\beta} = T^P_{\beta'\beta} + T^Q_{\beta'\beta} . \tag{3.48}$$

The nonresonant component, from which the state $|a\rangle$ has been subtracted out, is

$$T^Q_{\beta'\beta} = \langle e^{-i\vec{g}'\cdot\vec{e}}\overline{\Phi}_{\beta'} \left| [V_\beta + V_{\beta'}\overline{G}^Q V_\beta] \right| e^{-i\vec{g}\cdot\vec{e}}\overline{\Phi}_\beta \rangle . \tag{3.49}$$

The resonant component is

$$T^P_{\beta'\beta} = T_{\beta' a}(E - E_a)^{-1} T_{a\beta} \tag{3.50}$$

with

$$T_{\beta' a} = \langle e^{-i\vec{g}'\cdot\vec{e}}\overline{\Phi}_{\beta'} \left| (V_{\beta'} + V_{\beta'}\overline{G}^Q \overline{H}') \right| a \rangle , \tag{3.51}$$

and

$$T_{a\beta} = \langle a \left| (V_\beta + \overline{H}'\overline{G}^Q V_\beta) \right| e^{-i\vec{g}\cdot\vec{e}}\overline{\Phi}_\beta \rangle . \tag{3.52}$$

The residue at the pole in the resonant term (3.50) may be related to the ionization amplitude, as shown in Ref. 28. If for simplicity we neglect the energy-dependence of the level shift in Eq. (3.47), thereby ignoring a (second-order) correction to the residue, we may identify the ionization amplitude as $T_{\beta' a}$, defined in Eq. (3.51). The ressolvent which appears in the second term in Eq. (3.51) may be expanded in powers of the interaction $\overline{H}'$. The resultant perturbation series provides what may be called the multiphoton ionization contribution, in which the electron gains sufficient energy to ionize by successive absorption of photons in intermediate states; in the form given here it is modified to allow for final-state interactions as well. As the frequency is lowered the relative strength of the final-state interactions is increased. With the effectiveness of the intermediate-state absorption mechanism diminished the first term

$$T^{(1)}_{\beta' a} = \langle e^{-i\vec{g}'\cdot\vec{e}}\overline{\Phi}_{\beta'} \left| V_{\beta'} \right| a \rangle , \tag{3.53}$$

which is of zeroth order in $\overline{H}'$ and represents the pure tunneling effect, may be expected to dominate. The representation (3.51) provides a convenient starting point for a study of the relative importance of the tunneling and multiphoton ionization mechanisms.

For the sake of completeness we now examine the ionization amplitude (3.53) in order to establish explicitly that this corresponds to the tunneling approximation formulated and studied in some detail in Refs. 14–16. Thus, we adopt the approximation (3.2) for the final state, with $|\psi_{n\vec{p}}\rangle$ given by Eq. (3.3). Defining the vertex function

$$F_{\beta'}(\vec{p}) = \langle \vec{p} \left| \langle \chi_{i'} \right| V_{\beta'} \left| B \right\rangle , \tag{3.54}$$

we arrive at the approximation

$$T^{(1)}_{\beta' a} \cong \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-i(n_R - n')\phi} e^{-i S_{\vec{p}'}(\phi)} F_{\beta'}(\vec{q}(\phi)) , \tag{3.55}$$

with

$$\vec{q}(\phi) = \vec{p}' - \frac{e}{c}\vec{A}_c(\phi) - (n_R - n')\hbar\vec{k} . \tag{3.56}$$

Now $n_R - n'$ represents the number of photons absorbed by the electron in the ionization process. This number becomes very large compared to unity in the low-frequency limit since the energy absorbed must at least be equal to the electron affinity $\epsilon_{i'} - \mathcal{E}$. With $n_R - n' \gg 1$ in Eq. (3.55) the integral may be evaluated using the method of steepest descent. The stationary phase condition is

$$n_R - n' + \frac{dS_{\vec{p}'}}{d\phi} = 0 . \tag{3.57}$$

With the (complex) value of ϕ determined by this condition we readily verify that

$$q^2(\phi)/2\mu = \mathcal{E} - \epsilon_{i'} . \tag{3.58}$$

As is well known,[29] the function $F_{\beta'}(\vec{q})$, with $\vec{q}$ satisfying Eq. (3.58), is determined from a knowledge of the asymptotic normalization of the coordinate-space representation of the composite bound state $|B\rangle$. This normalization factor can be related to the residue of the bound-state pole in the physical field-free scattering amplitude. Further analysis of the amplitude (3.55), leading to a low-frequency approximation for the total ionization probability, can be found in Refs. 14–16.

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