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No-go theorem for the superradiant phase transition without dipole approximation

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A diamagnetic inequality is used to extend beyond the dipole approximation an earlier result concerning the nonexistence of the superradiant phase transition.

Since the paper of Hepp and Lieb,[1] considerable effort has been devoted to the study of the impact of coupling to radiation on the equilibrium thermodynamic properties of atomic systems. It has been proved recently[2] that the superradiant phase transition found in Ref. 1 for the Dicke Hamiltonian cannot occur in a general system of atoms, if only charges (not spin magnetic moments) of the particles interact with a finite number of transverse radiation modes in the dipole approximation. The result was shown to hold without the dipole approximation if the Coulomb potential binding the electrons in the atoms is replaced by a harmonic potential.[3] This last result is a strong indication that an extended version of the no-go theorem without restriction to the dipole interaction and for arbitrary binding potentials, is also true. It is the purpose of this note to demonstrate that.

Our Hamiltonian H for the system is a sum of two terms: the atomic Hamiltonian

$$H_A = \sum_{j=1}^{N} h_j(a_i) = \sum_{j=1}^{N} \left[\frac{1}{2m} \left(\vec{p}_j - \frac{e}{c} \vec{A}(\vec{x}_j + \vec{R}_j) \right)^2 + V(\vec{x}_j) \right], \quad (1)$$

and the Hamiltonian of the photon field,

$$H_F = \sum_{i=1}^{m} \hbar\omega_i a_i^+ a_i . \quad (2)$$

Here $\vec{p}_j$ denotes the canonical momentum of the electron in the jth atom, $\vec{x}_j$ the electron's position with respect to the nucleus, and $\vec{R}_j$ the mean position of the jth atom, which will be regarded as a fixed parameter.

To simplify the discussion, and to keep it on a rigorous level we have the following:

(1) We have excluded here the direct interaction between different atoms. Such interactions, of course, can lead to a variety of different phase transitions and a large part of standard statistical mechanics is devoted to them. In an earlier publication,[2] however, we showed that they play no role in the study of the equilibrium thermodynamics of atomic systems interacting with transverse radiation in the dipole approximation.

(2) The potential $V(\vec{x})$ is assumed to go to infinity for $|\vec{x}| \to \infty$ at least as rapidly as, say, $(\ln|\vec{x}|)^{1+\epsilon}$. It is the same as the assumption that it has only a discrete spectrum without a finite limiting point. This is a purely technical assumption, which guarantees the finiteness of all the atomic traces encountered in our derivation. An alternative would be the truncation of the Coulomb potential with a subsequent elaborate proof of the existence of the limit for the intensive thermodynamic quantities when the truncation is removed. As the superradiant phase transition first occurs at $T=0$, physically the detailed properties of highly excited states should not matter anyway.

The transverse electromagnetic field is taken to be composed of (a fixed number) m modes, characterized by photon energies $h\omega_i$, wave vectors $\vec{k}_i$, and polarization vectors $\vec{\epsilon}_i$. Plane-wave decomposition of the vector potential has the form

$$\vec{A}(\vec{r}) = (2\pi\hbar c^2\rho)^{1/2} \sum_{i=1}^{m} \frac{1}{(\omega_i)^{1/2}} \left(\frac{\epsilon_i a_i e^{i\vec{k}\cdot\vec{r}}}{\sqrt{N}} + \text{H.c.} \right), \quad (3)$$

where the density $\rho = N/V$ has been introduced with the usual identification of the sample volume with the quantization volume. The simple argument outlined below *does not* require any approximation of the plane-wave exponents in (3), neither in the operator $\vec{x}$ nor in the parameter $\vec{R}$. That means that we *do not* make a dipole approximation in either global or local senses.

All equilibrium thermodynamic properties of the system can be derived from the partition function $Z(N,T)$ as follows:

$$Z(N,T)=\mathrm{Tr}_F\,\mathrm{Tr}_A(e^{-\beta H})\,, \tag{4}$$

requiring summation over both field and atomic variables. This task is simplified greatly by the observation of Wang and Hioe[4] that the leading term of Z in the $N\to\infty$ limit comes from that part of the $e^{-\beta H}$ operator that is normally ordered with respect to the field operators. The validity of this statement has been justified by Hepp and Lieb[5] for the case of an atomic system described by means of a finite dimensional Hilbert space—of which the notion of the "two-level atom" is the most common example. Their proof holds also for the case studied here, as finiteness of single atom *traces* and not finite dimensionality of the atomic space is required there.[6] It is just to secure the rigor of this statement that we have chosen the assumption concerning the behavior of the binding potential at large distances. Therefore we shall explore once again the c-number substitution for the field variables $a_i\to\alpha_i$, $a_i^{+}\to\alpha_i^{*}$.

The partition function takes the following form:

$$Z(N,T)\to\int\frac{d^2\alpha_1}{\pi}\cdots\frac{d^2\alpha_m}{\pi}\exp\left(-\beta\hbar\sum_{i=1}^{m}\omega_i|\alpha_i|^2\right)\times\prod_{j=1}^{N}\mathrm{Tr}_A(e^{-\beta h_j(\alpha_i)})\,. \tag{5}$$

The traces over atomic variables are finite for the considered class of potentials.

At this point we are dealing with noninteracting atoms subjected to the external magnetic field described by the amplitudes α_i. For this case several so-called diamagnetic inequalities have been recently discovered. They show that the spectrum of the atomic Hamiltonian goes up in mean when the magnetic field is turned on.[7-9] The inequality applicable here is due to E. Nelson, as cited in Ref. 8. We now have

$$|\langle\vec{x}|\exp\{-\beta[1/2m(\vec{p}-e/c\vec{A})^2+V]\}|\vec{y}\rangle| \leq|\langle\vec{x}|\exp[-\beta(\vec{p}^2/2m+V)]|\vec{y}\rangle|\,. \tag{6}$$

It follows from the path-integral representation[10] for the matrix element of

$$\langle\vec{x}|\exp\{-\beta[1/2m(\vec{p}-e/c\vec{A})^2+V]\}|\vec{y}\rangle=\int_{\vec{x}(0)=\vec{x};\vec{x}(\beta)=\vec{y}}\exp\left(i\frac{e}{c}\int_0^{\beta}\vec{A}(\vec{x})\dot{\vec{x}}\,ds\right)\exp\left(-\int_0^{\beta}[\tfrac{1}{2}m\dot{\vec{x}}^2+V(\vec{x})]ds\right)\prod_{0\leq s\leq\beta}d\vec{x}(s) \tag{7}$$

via estimation of the first exponent (which is just a phase factor) under the integral by unity.

Nelson's inequality, when taken for diagonal elements, implies the relation

$$\mathrm{Tr}_A[\exp-\beta h(\alpha_i)]\leq\mathrm{Tr}_A[\exp-\beta h(\alpha_i=0)]\,. \tag{8}$$

As we argued in Ref. 2, those traces are exactly equal in the dipole approximation.

But even in the present general case, due to (8), the maximum of the integrand of (5) is reached for $\alpha_i=0$. For the potentials considered here the intensive thermodynamic functions in the $N\to\infty$ limit are given by the saddle-point values of the α integrals. In particular, the mean photon number per atom must always vanish in the thermodynamic limit and the free energy per atom does not depend on the coupling to radiation. If there is any impact of the coupling to radiation on the equilibrium thermodynamic properties of the atomic systems, it can show up only in more realistic models, which account for spin magnetic moment, statistics, and infinitely many modes of electromagnetic field.

The study of the infinitely many mode case is both important and difficult. It is important since even a single atomic transition has its natural width; therefore, when approaching the thermodynamic limit an indefinitely growing number of field modes enters the resonance. It is difficult since no proof of the existence of the thermodynamic limit exists in the literature. Also, the c-number substitution cannot be exact in this case. Even a noninteracting system leads to a contribution of Planck's distribution of radiation to thermodynamic functions. With thermodynamics unknown, it is nevertheless certain that a superradiant phase transition cannot reemerge. It is not difficult to show that the ground-state energy of the atomic system described by the Hamiltonian [Eq. (1)], interacting with an arbitrary (and perhaps infinite) number of modes, is bounded from below by the sum of ground-state energies of atoms noninteracting with radiation. We may recall that the superradiant phase transition is caused by the appearance of a density-dependent ground state, whose energy indefinitely decreases to $-\infty$ with the increase of density.

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