Quantum mechanics of a constrained particle

R. C. T. da Costa

Departamento de Física e Ciência dos Materiais, Instituto de Física e Química de São Carlos, Universidade de São Paulo, São Carlos 13560, S. Paulo, Brasil

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The motion of a particle rigidly bounded to a surface is discussed, considering the Schrödinger equation of a free particle constrained to move, by the action of an external potential, in an infinitely thin sheet of the ordinary three-dimensional space. Contrary to what seems to be the general belief expressed in the literature, this limiting process gives a perfectly well-defined result, provided that we take some simple precautions in the definition of the potentials and wave functions. It can then be shown that the wave function splits into two parts: the normal part, which contains the infinite energies required by the uncertainty principle, and a tangent part which contains "surface potentials" depending both on the Gaussian and mean curvatures. An immediate consequence of these results is the existence of different quantum mechanical properties for two isometric surfaces, as can be seen from the bound state which appears along the edge of a folded (but not stretched) plane. The fact that this surface potential is not a bending invariant (cannot be expressed as a function of the components of the metric tensor and their derivatives) is also interesting from the more general point of view of the quantum mechanics in curved spaces, since it can never be obtained from the classical Lagrangian of an *a priori* constrained particle without substantial modifications in the usual quantization procedures. Similar calculations are also presented for the case of a particle bounded to a curve. The properties of the constraining spatial potential, necessary to a meaningful limiting process, are discussed in some detail, and, as expected, the resulting Schrödinger equation contains a "linear potential" which is a function of the curvature.

I. INTRODUCTION

The motion of a particle in a one- or two-dimensional domain of our Cartesian three-dimensional space is a well-known problem in classical mechanics. It is usually treated in two different ways. In the first, or Newtonian approach, the particle is first thought of as moving freely (that is, unconstrained) in the three-dimensional space, but subjected to spatial forces which maintain, at all instants of time, its velocity oriented along a preselected range of directions (the tangent plane of a surface or tangent line of a curve). In the second, or Lagrangian approach, the constraint is introduced from the beginning through the well-known generalized coordinates, and the calculations proceed without any necessary mention to the space in which our surface (or curve) is supposed to be embedded. For the purely spatial constraints considered here these two treatments yield the same equations of motion, the choice between one of them being, in general, a matter of convenience.

In quantum mechanics, however, the situation is much less clear. If we select the first approach to begin with we have the advantage of a ready-made Schrödinger equation but will have to deal with an unfamiliar situation in which the constraint can only be thought of as a kind of limiting process. In fact, due to the uncertainty relations we must have (besides the quantum analog of the classical forces to bend the momentum of the particle) infinite "squeezing forces," to contain the transversal spreading of the wave packet, which will be present even in the case of a particle moving along a perfectly flat surface. On the other hand, if we choose the second approach we can forget for the moment all properties related to the external space, but will still have to work out an adequate quantization procedure for the *a priori* curved motion.[1] The aim of this paper is to see how far we can go following the first of these two approaches. As mentioned above (and discussed in more detail below) we believe this idea to be unjustly neglected. The main point involved here is the choice of the spatial forces which simulate the mechanical constraint in a certain suitable limit. To better develop our reasoning let us consider a surface constraint: As is well known, in classical mechanics the constraint forces can only be uniquely determined if we assume them to be nondissipative (or frictionless); that is, they have the direction of the normal in all points of the surface. Well, since in quantum mechanics we can no more predict the position of the particle with pointlike accuracy it is perfectly natural to consider only constraint forces which are orthogonal to our surface in all points of the space where the particle can possibly be found (a similar procedure will be later developed for curves). This idea can be readily put in practice considering a potential which is constant over the surface but increases sharply for every small displacement in the normal direction, in such a way as to provide a normal "reaction" in a thin neighborhood of the surface in question. (Weaker requirements can possibly be found, but the one presented here is perfectly adequate for the ends we have in mind.) The constraint will then be considered as the limit of an infinitely strong attractive potential which maintains the particle permanently attached to a pre-established surface. In order to have the limit independent of the type of attractive potential we must have some kind of separation of the Schrödinger equation in which the surface part of the wave function obeys a special equation which does not contain the transverse variable appearing in the constraining potential. This is in fact what happens, as we shall now proceed to show.

II. PARTICLE BOUNDED TO A SURFACE

Let us consider a particle of mass m permanently attached to the surface S of parametric equations $\vec{r}=\vec{r}(q_1,q_2)$, where $\vec{r}$ is the vector position of an arbitrary surface point P. The portion of the space in an immediate neighborhood of S can be parametrized as (Fig. 1)

$$\vec{R}(q_1,q_2,q_3)=\vec{r}(q_1,q_2)+q_3\hat{N}(q_1,q_2)\ , \tag{1}$$

where $\hat{N}(q_1,q_2)$ is the value taken at P by a continuous unit normal to S. The absolute value of the coordinate q_3 gives, for points where (1) is nonsingular, the distance between the surface S and the point Q of coordinates (q_1,q_2,q_3). According to the ideas presented in the Introduction we shall now consider the spatial potential $V=V_\lambda(q_3)$, where λ is a "squeezing parameter" which measures the strength of the potential:

$$\lim_{\lambda\to\infty}V_\lambda(q_3)=\begin{cases}0, & q_3=0\ ,\\ \infty, & q_3\neq 0\ .\end{cases}$$

(If a specific example is required to guide our intuition, we can imagine the harmonic binding $V_\lambda(q_3)=\frac{1}{2}m\lambda^2q_3^2$, with λ eventually going to infinity, which gives $\langle q_3^2\rangle\approx\hbar/m\lambda$).

Before going to the Schrödinger equation it is worthwhile to briefly review the mathematical properties of the coordinate system (1). Let us

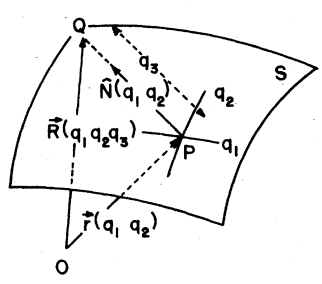

FIG. 1. Curvilinear coordinate system based on the surface S of parametric equation $\vec{r}=\vec{r}(q_1,q_2)$.

call $g_{ij}=(\partial\vec{r}/\partial q_i)(\partial\vec{r}/\partial q_j)$, $i,j=1,2$, the covariant components of the metric tensor of our surface S, $g=\det(g_{ij})$ and $h_{ij}=h_{ji}$, the coefficients of the second fundamental form.[2] Since the derivatives of the normal $\hat{N}(q_1,q_2)$ lie in the tangent plane we have

$$\frac{\partial\hat{N}}{\partial q_i}=\sum_{j=1}^{2}\alpha_{ij}\frac{\partial\vec{r}}{\partial q_j}\ , \tag{2}$$

with

$$\alpha_{11}=\frac{1}{g}(g_{12}h_{21}-g_{22}h_{11}),\quad \alpha_{12}=\frac{1}{g}(h_{11}g_{21}-h_{21}g_{11})\ ,$$
$$\alpha_{21}=\frac{1}{g}(h_{22}g_{12}-h_{12}g_{22}),\quad \alpha_{22}=\frac{1}{g}(h_{21}g_{12}-h_{22}g_{11}) \tag{3}$$

(Weingarten equations). From (1) and (2) we obtain

$$\frac{\partial\vec{R}}{\partial q_i}=\sum_{j=1}^{2}(\delta_{ij}+\alpha_{ij}q_3)\frac{\partial\vec{r}}{\partial q_j},\quad i,j=1,2$$
$$\frac{\partial\vec{R}}{\partial q_3}=\hat{N}(q_1,q_2)\ . \tag{4}$$

In our three-dimensional neighborhood of S the covariant components of the metric tensor are given by

$$G_{ij}=G_{ji}=\frac{\partial\vec{R}}{\partial q_i}\frac{\partial\vec{R}}{\partial q_j},\quad i,j=1,2,3\ .$$

Using (4) and denoting the transposed matrix by the superscript T, we have

$$G_{ij}=g_{ij}+[\alpha g+(\alpha g)^T]_{ij}q_3+(\alpha g\alpha^T)_{ij}q_3^2\ ,$$
$$G_{i3}=G_{3i}=0,\quad i=1,2;\ G_{33}=1\ . \tag{5}$$

We can now turn our attention to the Schrödinger equation. Writing the Laplacian in the curvilinear coordinates (q_1,q_2,q_3) we obtain

$$-\frac{\hbar^2}{2m}\sum_{i,j=1}^{3}\frac{1}{\sqrt{G}}\frac{\partial}{\partial q_i}\left(\sqrt{G}(G^{-1})_{ij}\frac{\partial\psi}{\partial q_j}\right)+V_\lambda(q_3)\psi=i\hbar\frac{\partial\psi}{\partial t}\ , \tag{6}$$

where $G=\det(G_{ij})$. Due to the structure of the G_{ij}'s given in (5) we can break up the Laplacian into two parts: the surface part, denoted by $\mathfrak{D}(q_1,q_2,q_3)$, given by the terms $i,j=1,2$, and the normal part, defined by $i=j=3$. We can then write

$$-\frac{\hbar^2}{2m}\mathfrak{D}(q_1,q_2,q_3)\psi-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial q_3^2}+\frac{\partial}{\partial q_3}(\ln\sqrt{G})\frac{\partial\psi}{\partial q_3}\right)+V_\lambda(q_3)\psi=i\hbar\frac{\partial\psi}{\partial t}\ . \tag{7}$$

Since we are hoping for the existence of a surface wave function, depending only on the variables q_1 and q_2, we are naturally led to the introduction of a new wave function χ from which, in the event of a separation $\chi(q_1, q_2, q_3) = \chi_t(q_1, q_2)\chi_n(q_3)$ we will be able to define the surface density probability $|\chi_t(q_1 q_2)|^2 \int |\chi_n(q_3)|^2 dq_3$. The adequate transformation $\psi \to \chi$ can be readily inferred from the volume dV expressed in terms of the curvilinear coordinates q_1, q_2, q_3. Really, using (4) we have

$$dV = f(q_1, q_2, q_3) dS\, dq_3 , \tag{8}$$

where $dS = \sqrt{g}\, dq_1 dq_2$ (the area element of the surface) and

$$f(q_1, q_2, q_3) = 1 + \mathrm{Tr}(\alpha_{ij}) q_3 + \det(\alpha_{ij}) q_3^2 . \tag{9}$$

Expression (8) now gives the desired result:

$$\chi(q_1, q_2, q_3) = [f(q_1, q_2, q_3)]^{1/2} \psi(q_1, q_2, q_3) . \tag{10}$$

Introducing this substitution into (7) we are left with

$$\sqrt{f}\left[-\frac{\hbar^2}{2m}\mathfrak{D}\left(\frac{\chi}{\sqrt{f}}\right)\right] - \frac{\hbar^2}{2m}\left\{\frac{\partial^2 \chi}{\partial q_3^2} + \frac{1}{4f^2}\left[\left(\frac{\partial f}{\partial q_3}\right)^2 - 2f\frac{\partial^2 f}{\partial q_3^2}\right]\chi\right\} + V_\lambda(q_3)\chi = i\hbar\frac{\partial \chi}{\partial t} . \tag{11}$$

We are now ready to take into account the effect of the potential $V_\lambda(q_3)$. Since in the limit when $\lambda \to \infty$ the wave function "sees" two steep potential barriers on both sides of the surface, its value will be significantly different from zero only for a very small range of values of q_3 around $q_3 = 0$. In this case we can safely take $q_3 \to 0$ in all coefficients of Eq. (11) [except of course in the term containing $V_\lambda(q_3)$]. The result from (5) and (9) is

$$-\frac{\hbar^2}{2m}\sum_{i,j=1}^{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q_i}\left(\sqrt{g}(\bar{g}^1)_{ij}\frac{\partial \chi}{\partial q_j}\right) - \frac{\hbar^2}{2m}\left([\tfrac{1}{2}\mathrm{Tr}(\alpha_{ij})]^2 - \det(\alpha_{ij})\right)\chi - \frac{\hbar^2}{2m}\frac{\partial^2 \chi}{\partial q_3^2} + V_\lambda(q_3)\chi = i\hbar\frac{\partial \chi}{\partial t} . \tag{12}$$

Equation (12) can now be easily separated by setting $\chi = \chi_t(q_1, q_2, t) \times \chi_n(q_3, t)$, where the subscripts t and n stand for "tangent" and "normal," respectively. The usual procedure yields the following equations:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \chi_n}{\partial q_3^2} + V_\lambda(q_3)\chi_n = i\hbar\frac{\partial \chi_n}{\partial t} , \tag{13}$$

$$-\frac{\hbar^2}{2m}\sum_{i,j=1}^{2}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q_i}\left(\sqrt{g}(\bar{g}^1)_{ij}\frac{\partial \chi_t}{\partial q_j}\right) - \frac{\hbar^2}{2m}\left([\tfrac{1}{2}\mathrm{Tr}(\alpha_{ij})]^2 - \det(\alpha_{ij})\right)\chi_t = i\hbar\frac{\partial \chi_t}{\partial t} . \tag{14}$$

Expression (13) is just the one-dimensional Schrődinger equation for a particle bounded by the transverse potential $V_\lambda(q_3)$, and can be ignored in all future calculations. Expression (14), however, is much more interesting, due to the presence of the surface potential $V_S(q_1, q_2) = -(\hbar^2/2m)\{\frac{1}{2}\mathrm{Tr}(\alpha_{ij})]^2 - \det(\alpha_{ij})\}$.

Using (3) this term can be written in a more useful form,

$$V_S(q_1, q_2) = -\frac{\hbar^2}{2m}(M^2 - K) = -\frac{\hbar^2}{8m}(k_1 - k_2)^2 , \tag{15}$$

where k_1 and k_2 are the principal curvatures of the surface S, and

$$M = \tfrac{1}{2}(k_1 + k_2) = \frac{1}{2g}(g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12}) \quad \text{(mean curvature)} , \tag{16}$$

$$K = k_1 k_2 = \frac{1}{g}\det(h_{ij}) \quad \text{(Gaussian curvature)} . \tag{17}$$

The dependence of V_S on q is especially remarkable due to the presence of the mean curvature M, since it cannot be obtained from the g_{ij}'s and their derivatives alone (contrary to what happens with K). This result has an important consequence: $V_S(q_1, q_2)$ will not be the same for two isometric surfaces (for which correspondent points can be found with the same g_{ij}'s). This is in striking contrast with the results of classical mechanics where the Lagrangian of the free surface motion, $\mathfrak{L}(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{1}{2}m(ds/dt)^2 = \frac{1}{2}m\sum_{i,j=1}^{2} g_{ij}(q_1 q_2)\dot{q}_i\dot{q}_j$, depends only on the metric properties of the surface. Strange as it may appear at first sight, this is not an unexpected result, since, independent of how small the range of value assumed for q_3, the wave function always "moves" in a three-dimensional portion of the space, so that the particle is "aware" of the external properties of the limit surface S. In order to illustrate the properties of $V_S(q_1, q_2)$ let us consider, for example, a bookcover shaped surface obtained by bending a plane around the surface of a cylinder of radius a (Fig. 2). Selecting as parameters the arc s of the cross section C and the Cartesian coordinate

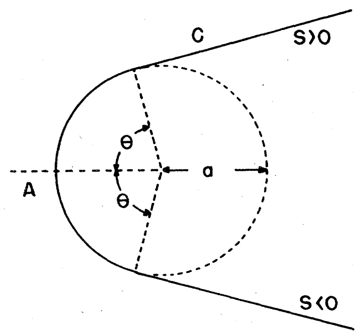

FIG. 2. Cross section C of the "bookcover" surface: A plane bent around the surface of a cylinder of radius a. The middlepoint A was chosen for the origin of the arc s.

z perpendicular to the plane of the figure, we have from Eqs. (14)–(17)

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\chi_t}{\partial s^2}+\frac{\partial^2\chi_t}{\partial z^2}\right)-\frac{\hbar^2}{8m}k(s)^2\chi_t=i\hbar\frac{\partial\chi_t}{\partial t}\,, \quad (18)$$

where $k(s)$ is the curvature of C at the point of arc s. If we consider a solution $\chi_t(s,t)$, independent of z, we obtain a one-dimensional Schrödinger equation in the presence of the square well potential,

$$V(s)=\begin{cases}V_0=-\dfrac{\hbar^2}{8ma^2}, & |s|<a\theta,\\ 0, & |s|>a\theta,\end{cases} \quad (19)$$

which (since $[(2m/\hbar^2)|V_0|(a\theta)^2]^{1/2}=\theta/2<\pi/4$) has only one bound state of energy E_0, $V_0<E_0<0$. In the limit $a\to 0$, $\theta=$constant, which corresponds to an infinitely sharp bend in our plane, the transmission coefficient of (19) goes to zero. The two sheets, $s<0$ and $s>0$, of Fig. 2 are then effectively disconnected, in strong contrast with the usual solutions where the term $V(s)$ is absent.

One last remark must be made on previous results stating that the limit obtained here does not actually exist. Our opinion is that those calculations, although mathematically unimpeachable, are badly conceived from the physical point of view since they involve potentials with nonzero tangent forces in every neighborhood of the surface S. To use the procedure described by Cheng[3] we imagine our particle squeezed between two impenetrable surfaces, our own surface S and another surface S', and let the distance $d(q_1,q_2)$, between them, go steadily to zero. If we take $d=\epsilon f(q_1 q_2)$, $\epsilon\to 0$, then, to quote Cheng's own words, "The Schrödinger equation would acquire a term proportional to $[\epsilon f(q_1,q_2)]^{-2}$ which varies wildly over the q's." This, in fact, could already be expected from the beginning, since this kind of potential prevents the separation of the wave function χ in tangent and normal parts, as given by (13) and (14). It goes also without saying that the fact that the forces tend to be normal to S in the limit $\epsilon\to 0$ does not imply the vanishing of the tangential components since the forces themselves go to infinity, precluding any direct comparison with the classical situation. If we now take $d(q_1,q_2)=\epsilon[1+\epsilon f(q_1,q_2)]$, then, again according to Cheng, "everything depends on the higher order terms $O(\epsilon^2)$." Here, however, we cannot forget that although the terms of order ϵ^2 may be a small perturbation for the total potential, they may still be important when compared with the energies involved in the surface motion.

III. PARTICLE BOUNDED TO A CURVE

Let us consider a pointlike particle of mass m, rigidly bounded to a curve C of arc q_1, parametric equation $\vec{r}=\vec{r}(q_1)$, and tangent normal and binormal denoted respectively by $\hat{t}(q_1)$, $\hat{n}(q_1)$, and $\hat{b}(q_1)$. Following the same reasoning of Sec. II we shall now introduce a curvilinear coordinate system based on the curve C (Fig. 3):

$$\vec{R}(q_1,q_2,q_3)=\vec{r}(q_1)+q_2\hat{n}_2(q_1)+q_3\hat{n}_3(q_1)\,, \quad (20)$$

$$\begin{aligned}\hat{n}_2&=\cos\theta(q_1)\hat{n}(q_1)-\sin\theta(q_1)\hat{b}(q_1)\,,\\ \hat{n}_3&=\sin\theta(q_1)\hat{n}(q_1)+\cos\theta(q_1)\hat{b}(q_1)\,,\end{aligned} \quad (21)$$

with

$$\frac{d\theta}{dq_1}=\tau(q_1)\,, \quad (22)$$

where $\tau(q_1)$ is the torsion of C. (For the sake of simplicity we have introduced a Cartesian coordinate system for each normal plane of C.)

From (20), (21), and (22) we get

$$\frac{d\vec{R}}{dq_1}=[1-k(q_1)f(q_1,q_2,q_3)]\hat{t}(q_1)\,, \quad (23)$$

$$\frac{d\vec{R}}{dq_j}=\hat{n}_j(q_1)\,, \quad (24)$$

where

$$f(q_1,q_2,q_3)=\cos\theta(q_1)q_2+\sin\theta(q_1)q_3\,, \quad (25)$$

and $k(q_1)=|d\hat{t}/dq_1|$ is the curvature of C at the point of arc q_1. Since our coordinate system is orthogonal, $(\partial\vec{R}/\partial q_i)(\partial\vec{R}/\partial q_j)=h_i^2\delta_{ij}$, we can write the classical force $\vec{F}$ due a potential $V(q_1,q_2,q_3)$ as

$$\vec{F}=-\operatorname{grad}V=-\sum_{j=1}^{3}\left(\frac{1}{h_j^2}\frac{\partial V}{\partial q_j}\right)\frac{\partial\vec{R}}{\partial q_j}\,. \quad (26)$$

Proceeding as in the case of the surface constraint, we shall select, from (26), a binding potential $V_\lambda(q_2, q_3)$ independent of q_1, in order to always maintain the force $-\operatorname{grad} V_\lambda$ in the normal planes of C. The Schrödinger equation is then written as

$$-\frac{\hbar^2}{2m}\left[\frac{1}{1-kf}\frac{\partial}{\partial q_1}\left(\frac{1}{1-kf}\frac{\partial\psi}{\partial q_1}\right)+\sum_{j=2}^{3}\left(\frac{\partial^2\psi}{\partial q_j^2}+\frac{\partial}{\partial q_j}\ln(1-kf)\frac{\partial\psi}{\partial q_j}\right)\right]+V_\lambda(q_2,q_3)\psi=i\hbar\frac{\partial\psi}{\partial t}\,. \quad (27)$$

The volume element is given by $dV=(1-kf)\times dq_1dq_2dq_3$, which suggests the introduction of the new wave function $\chi(q_1,q_2,q_3)=(1-kf)^{1/2}\psi\times(q_1,q_2,q_3)$. Equation (27) is then transformed into

$$-\frac{\hbar^2}{2m}\frac{1}{(1-kf)^{1/2}}\frac{\partial}{\partial q_1}\left(\frac{1}{1-kf}\frac{\partial}{\partial q_1}\frac{\chi}{(1-kf)^{1/2}}\right)-\frac{\hbar^2}{8m}\frac{k^2}{(1-kf)^2}\chi-\frac{\hbar^2}{2m}\left(\frac{\partial^2\chi}{\partial q_2^2}+\frac{\partial^2\chi}{\partial q_3^2}\right)+V_\lambda(q_2,q_3)\chi=i\hbar\frac{\partial\chi}{\partial t}\,. \quad (28)$$

Assuming for V_λ the expected properties of a constraining potential:

$$\lim_{\lambda\to\infty}V_\lambda(q_2,q_3)=\begin{cases}0, & q_2^2+q_3^2=0\,,\\ \infty, & q_2^2+q_3^2\neq 0\,,\end{cases} \quad (29)$$

we can directly take $f\to 0$ in (28), obtaining

$$-\frac{\hbar^2}{2m}\frac{\partial^2\chi}{\partial q_1^2}-\frac{\hbar^2}{8m}k(q_1)^2\chi-\frac{\hbar^2}{2m}\left(\frac{\partial^2\chi}{\partial q_2^2}+\frac{\partial^2\chi}{\partial q_3^2}\right)+V_\lambda(q_2,q_3)\chi=i\hbar\frac{\partial\chi}{\partial t}\,. \quad (30)$$

Equation (30) is now readily separated by setting $\chi=\chi_t(q_1,t)\times\chi_n(q_2,q_3,t)$. The result is

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\chi_n}{\partial q_2^2}+\frac{\partial^2\chi_n}{\partial q_3^2}\right)+V_\lambda(q_2,q_3)\chi_n=i\hbar\frac{\partial\chi_n}{\partial t}\,, \quad (31)$$

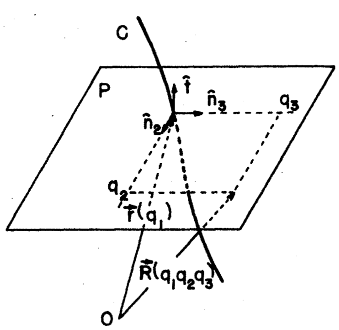

FIG. 3. Curvilinear coordinate system based on the curve C of parametric equation $\vec{r}=\vec{r}(q_1)$. Cartesian coordinates q_2 and q_3 were used for the normal plane P.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\chi_t}{\partial q_1^2}-\frac{\hbar^2}{8m}k(q_1)^2\chi_t=i\hbar\frac{\partial\chi_t}{\partial t}\,. \quad (32)$$

Here equation (32) has the same property of equation (14): although all curves are isometric each one has, depending on the curvature, its own distinct quantum mechanics. It must also be noted that Eq. (32) does not depend on the detailed behavior of the potential $V_\lambda(q_2, q_3)$ (its equipotentials around the curve C can be circles, elipses, rectangles, ect.), provided that once it is defined in *one* normal plane it is known in all points of the space by giving the same potential to all "parallel" curves with the same values of q_2 and q_3 (Combescure transforms to the mathematically minded). In a certain sense it can be said that in $V_\lambda(q_2q_3)$ we have introduced a generalization of the ordinary two-dimensional potential (obtained when C is a straight line).

One last remark must still be made about the possibility of binding a particle to a curve in two successive steps: First using a surface constraint of the type employed in Sec. II and, after that, assuming an extra surface potential to reduce the motion to a curve. It is not difficult to see that the result obtained in this way will, in general, depend on the intermediate surface selected in the process. The reason is that the normals to this intermediate surface are not necessarily contained in a normal plane of the curve. This means that the potential responsible for the surface constraint can give rise to forces with nonvanishing tangential components in a neighborhood of the curve, contrary to the definition of $V_\lambda(q_2, q_3)$. It can also be shown that the same result (29) can be obtained if the chosen surface belongs to the following family:

$$\vec{R}(q_1,S)=\vec{r}(q_1)+q_2(s)\hat{n}_2(q_1)+q_3(s)\hat{n}_3(q_1)\,, \quad (33)$$

where $q_1(s)$, $q_2(s)$ gives the intersection of the surface with the normal planes of C. Notice that, since q_2 and q_3 do not depend on q_1, the surface is completely determined from the knowledge of its intersection with one of the normal planes.

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