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Quantum kinetic theory of time-correlation functions

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A quantum kinetic theory of time-correlation functions is described in terms of a formally exact closure of the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy. The theory closely parallels existing treatments of the corresponding classical problem, so that direct comparisons between approximations for classical and quantum systems may be made. As an illustration, the formalism is applied to conditions of low density, but arbitrary degeneracy, and the resulting kinetic equation is shown to reduce to the linearized form of the Uehling-Uhlenbeck equation with the cross section appropriately modified to account for degeneracy. Also, classical approximations suitable for strongly coupled fluids are generalized to the quantum case. The results are applied to evaluation of the electrical conductivity for a two-component plasma in the following paper.

I. INTRODUCTION

There have been many experimental, theoretical, and computer studies of time-correlation functions for classical fluids in the past decade.[1] In particular, the description of time-correlation functions in terms of linear kinetic theory[2] has led to a much deeper understanding of the complexity of many-body dynamics, including such effects as anomalous short-time behavior,[3] persistent correlations at long time,[4] and nonanalytic density dependence of the collision rate.[5] Presently, classical kinetic theory provides the clearest microscopic basis for the recently discovered mode-coupling phenomena responsible for anomalous transport[6] and divergence of Burnett coefficients.[7] The development of classical kinetic theory has progressed from several complementary points of view,[8-15] and provides the basis for realistic optimism that a molecular description of the dynamics of dense gases[16] and liquids[17] may be developed in the near future. For example, a recent application of kinetic theory to the classical one-component plasma[18] yields transport and other dynamic properties of the plasma in substantial agreement with molecular-dynamics results, even for very large plasma parameters (strong coupling). The objective here is to demonstrate that much of the formal development of classical kinetic theory extends straightforwardly to the case of a normal quantum fluid. The reason for such an extension is to provide a means to capitalize on the advanced state of classical kinetic theory, for applications to quantum systems. In the following paper (hereafter referred to as II), the formalism described here is applied to the calculation of electrical conductivity for a hot, dense, two-component plasma under conditions such that both quantum mechanical and strong coupling effects are important (e.g., exploding wire plasmas).

The relationship of correlation functions to kinetic theory results from the fact that in many cases of experimental importance, the observables of interest, $A, B, \ldots$, are represented by sums of single-particle operators, i.e.,

$$A_N=\sum_{i=1}^{N} a(i)\,, \quad B_N=\sum_{i=1}^{N} b(i)\,. \tag{1.1}$$

The correlation function may then be expressed in terms of an average in the single-particle subspace. To show this consider a correlation function of the form,[19]

$$C_{BA}(t)=\langle (A-\langle A\rangle_0)B(t)\rangle_0\,, \tag{1.2}$$

$$B(t)=e^{iH_Nt}Be^{-iH_Nt}\,, \tag{1.3}$$

where H_N is the Hamiltonian for a system of N particles with pairwise additive forces. The average appearing in Eq. (1.2) is to be calculated by performing a trace over the appropriate subspace of the N-particle Hilbert space, that spanned by symmetrized states for bosons or that spanned by antisymmetrized states for fermions. However, it is convenient to suppress this restriction due to the statistics and enlarge the trace in Eq. (1.2) to the full Hilbert space, through the introduction of a projection operator onto the appropriate subspace,

$$C_{BA}(t)=\sum_{N\geq 0}\mathrm{Tr}_{1\ldots N}\rho_N(A-\langle A\rangle_0)B_N(t)\,, \tag{1.4}$$

$$\rho_N\equiv\frac{z^Ne^{-\beta H_N}}{\Xi}P_N\,. \tag{1.5}$$

Here P_N is the projection operator onto the subspace of symmetrized or antisymmetrized states,

z is the fugacity, and Ξ is the grand partition function. This choice to retain the statistics in the equilibrium density operator is convenient since it allows a discussion of the kinetic theory in operator form which is notationally compact and independent of representation. The trace in Eq. (1.4) is now over the full Hilbert space and a convenient representation is that with states formed from the direct product of complete sets of single-particle states,

$$|\alpha_1\cdots\alpha_N)\equiv|\alpha_1)x|\alpha_2)x\cdots x|\alpha_N)\,, \tag{1.6}$$

where α_i denotes the quantum numbers associated with a single-particle state. Therefore, using the time invariance of the average and carrying out a partial trace gives,

$$C_{BA}(t)=\mathrm{Tr}_1\, b(1)\psi^{(1)}(1;t)\,. \tag{1.7}$$

The single-particle operator, $\psi^{(1)}(1;t)$, is one of a class of operators defined by,

$$\begin{aligned}\psi^{(s)}(1\cdots s;t)&\\ &\equiv\sum_{N\geq s}\frac{N!}{(N-s)!}\,\mathrm{Tr}_{s+1\ldots N}\,\rho_N[A(-t)-\langle A\rangle_0]\\ &=\sum_{N\geq s}\frac{N!}{(N-s)!}\,\mathrm{Tr}_{s+1\ldots N}\,\rho_N e^{-itL(1\cdots N)}(A_N-\langle A\rangle_0)\,.\end{aligned} \tag{1.8}$$

The Liouville operator, $L(1\cdots N)$, is defined by its operation on an arbitrary N-particle operator O_N,

$$L(1\cdots N)O_N\equiv H_NO_N-O_NH_N\equiv[H_N,O_N]\,. \tag{1.9}$$

For pairwise additive forces, the Liouville operator may be expressed as

$$L(1\cdots N)=\sum_{i=1}^{N}L(i)+\sum\sum_{1\leq i<j\leq N}L_I(ij)\,, \tag{1.10}$$

where

$$\begin{aligned}L(i)O_N&\equiv\left(\frac{P_i^2}{2m},O_N\right),\\ L_I(ij)O_N&\equiv[\,V(ij),O_N]\,.\end{aligned} \tag{1.11}$$

The time dependence of the operator, $\psi^{(s)}$, is determined by the hierarchy of equations,[20]

$$\left(i\frac{\partial}{\partial t}-L(1\cdots s)\right)\psi^{(s)}(1\cdots s;t)=\mathrm{Tr}_{s+1}\sum_{i=1}^{s}L_I(i,s+1)\psi^{(s+1)}(1\cdots s+1;t)\,. \tag{1.12}$$

In particular, for $s=1$ and $s=2$, the equations are

$$\left(i\frac{\partial}{\partial t}-L(1)\right)\psi^{(1)}(1;t)=\mathrm{Tr}_2L_I(12)\psi^{(2)}(12;t)\,, \tag{1.13}$$

$$\left(i\frac{\partial}{\partial t}-L(12)\right)\psi^{(2)}(12;t)=\mathrm{Tr}_3[L_I(13)+L_I(23)]\psi^{(3)}(123;t)\,. \tag{1.14}$$

These equations must be supplemented by the initial values determined from Eq. (1.8) with $t=0$. In classical mechanics the equations for $\psi^{(s)}$ are the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy and provide the basis for most discussions of kinetic equations. The term kinetic equation will be used to refer to a closed equation for the single-particle operator $\psi^{(1)}$. In Sec. II the existence of a formally exact kinetic equation for time-correlation functions is established along lines familiar from the study of classical fluids. The basic starting point is a formal closure of the first equation of the quantum BBGKY hierarchy, Eq. (1.13). This is accomplished by recognizing that the $\psi^{(s)}(1\cdots s)$ are linear functionals of the single-particle operators $a(i)$, of Eq. (1.1). Inverting this functional relationship for $\psi^{(1)}$ allows $\psi^{(2)}$ to be written as a time-dependent functional of $\psi^{(1)}$. This functional relationship together with Eq. (1.13) provides the desired kinetic equation. The result of this analysis is that the correlation function may be determined in the single-particle subspace by Eq. (1.7) and a kinetic equation of the form

$$\left(i\frac{\partial}{\partial t}-L(1)-B(1)\right)\psi^{(1)}(1;t)=-i\int_0^t d\tau\, M(1;t-\tau)\psi^{(1)}(1;\tau)\,. \tag{1.15}$$

The focus in any particular application is therefore an appropriate analysis of the "collision operator", $M(1;t)$, and the "mean-field operator", $B(1)$. This equation represents the quantum-mechanical generalization of the classical equation derived by Lebowitz, Percus, and Sykes.[11] It is shown that much of the structure of the operators $B(1)$ and $M(1;t)$ may be obtained without approximation, just as in the classical case. The method of analysis is an algebraic approach generalizing that of Lebowitz *et al.* due to Mazenko[15] and Gross.[14]

The formal analysis is presented in Sec. II. In Sec. III the exact short-time kinetic equation is briefly discussed as a collisionless approximation that contains the random phase approximation and several generalized mean-field approximations. Also, the weakly coupled fluid is described along with two phenomenological "generalized Born approximations." The latter are a quantum-mechanical version of a classical kinetic equation recently used to describe a one-component plasma.[18] Finally, in Sec. IV the utility of the formalism for small parameter expansions is demonstrated for a low-density quantum gas. The resulting kinetic equation is of the Uehling-Uhlenbeck-type[21] with a degeneracy modified T matrix. The general discussions of Secs. III and IV are intended to illustrate the formalism, make contact with known limits, and provide the basis for future applications, such as that of II.

A study similar in method and objective to that presented here has been previously made by Boley and Smith.[22] However, these authors limit their investigation of the collision operator to an expansion to second order in the interaction (weak coupling), and as is demonstrated here, the degeneracy effects in a strongly coupled system lead to modifications of the collision cross section which do not appear in the second-order approximation. In another study by Valls, Mazenko, and Gould,[23] the collision operator for He^3 has been successfully modeled from known equilibrium properties and a single relaxation time determined from Fermi liquid theory. Application of quantum kinetic theory to spectral line broadening in plasmas has been described by the authors[24]; several other attempts at constructing quantum kinetic theories have also been given recently.[25]

II. FORMAL CLOSURE OF THE HIERARCHY

In this section the formal closure of Eq. (1.13), which was outlined briefly above, will be accomplished and an exact, closed kinetic equation for $\psi^{(1)}$ will be provided. As might be expected, the expression for the nonlocal collision operator appearing in the kinetic equation is quite complex. However, an analogous formal closure at the level of the second equation provides a more explicit representation of the collision operator in terms of the effective dynamics in the two-particle subspace. The results of formal closure at the second equation for classical time-correlation functions are due to Mazenko,[15] and this section provides their generalization to the quantum-mechanical case. The advantage of this reformulation of the BBGKY hierarchy is that much of the structure of the kinetic equation is exposed in a form such that simple approximations or expansions may be applied most efficiently. The formulation does not circumvent the many-body problem, but merely postpones it to a point more suitable for controlled approximation.

As noted above, the development of the classical kinetic theory is based on the recognition that for arbitrary times, the classical function, $\psi^{(s)}$, is a linear functional of the initial data. That this remains true in the quantum case is seen by using Eq. (1.1) to write Eq. (1.8) as,

$$\psi^{(s)}(1\cdots s;t)=\left(\sum_{N\geq s}\frac{N!}{(N-s)!}\mathrm{Tr}_{s+1\ldots N}\rho_N e^{-itL(1\cdots N)}\sum_{i=1}^{N}\mathcal{P}_{1,i}-f^{(s)}(1\cdots s)\mathrm{Tr}_{s+1}f(s+1)\mathcal{P}_{1,s+1}\right)a(1)$$
$$\equiv U(1\cdots s;t)a(1)\,, \tag{2.1}$$

where $U(1\cdots s;t)$, defined by the quantity in large parentheses, maps single-particle operators into s-particle operators. The operator $\mathcal{P}_{ij}$ appearing in Eq. (2.1) interchanges the labels i and j appearing in an operator, and the equilibrium reduced distribution operators, $f^{(s)}$, are defined by

$$f^{(s)}(1\cdots s)\equiv\sum_{N\geq s}\frac{N!}{(N-s)!}\mathrm{Tr}_{s+1\ldots N}\rho_N\,. \tag{2.2}$$

It is actually more convenient to construct the kinetic equation in terms of the Laplace transform of $\psi^{(s)}$. For $\mathrm{Im}\,z>0$ the Laplace transformed operators are

$$\tilde{\psi}^{(s)}(1\cdots s;z)\equiv -i\int_0^\infty dt\,e^{izt}\psi^{(s)}(1\cdots s;t)\,. \tag{2.3}$$

The equation for $\tilde{\psi}^{(s)}$ corresponding to Eq. (2.1) is therefore,

$$\tilde{\psi}^{(s)}(1\cdots s;z)=\tilde{U}(1\cdots s;z)a(1)\,, \tag{2.4}$$

where $\tilde{U}(1\cdots s;z)$ is the Laplace transform of $U(1\cdots s;t)$.

A. Closure of the first equation of the hierarchy

The equation determining $\tilde{\psi}^{(1)}(1;z)$ follows directly from the Laplace transform of the first equation of the hierarchy, Eq. (1.13),

$$[z-L(1)]\tilde{\psi}^{(1)}(1;z)-\mathrm{Tr}_2L_I(12)\tilde{\psi}^{(2)}(12;z)=\psi^{(1)}(1;t=0)\,. \tag{2.5}$$

To obtain a closed equation for $\tilde{\psi}^{(1)}$ it is necessary to express $\tilde{\psi}^{(2)}$ in terms of $\tilde{\psi}^{(1)}$. This may be accomplished formally by eliminating $a(1)$ between the Eq. (2.4) for $s=1$ and $s=2$ to get

$$\tilde{\psi}(12;z)=\tilde{U}(12;z)\tilde{U}^{-1}(1;z)\tilde{\psi}^{(1)}(1;z)\,. \tag{2.6}$$

Substitution of (2.6) into (2.5) provides the desired equation for $\tilde{\psi}^{(1)}(1;z)$,

$$[z - L(1) - \tilde{\Sigma}(1;z)]\,\tilde{\psi}^{(1)}(1;z) = \psi^{(1)}(1;0)\,, \tag{2.7}$$

where $\tilde{\Sigma}(1;z)$, defined by

$$\tilde{\Sigma}(1;z) \equiv \mathrm{Tr}_2 L_I(12)\,\tilde{U}(12;z)\,\tilde{U}^{-1}(1;z)\,, \tag{2.8}$$

contains all the many-body effects on the time evolution of a single-particle operator. The kinetic equation for $\psi^{(1)}(1;t)$ follows from inverting the transform of Eq. (2.7),

$$\left(i\frac{\partial}{\partial t} - L(1)\right)\psi^{(1)}(1;t) = -i\int_0^t d\tau\,\Sigma(1;t-\tau)\psi^{(1)}(1;\tau)\,, \tag{2.9}$$

where $\Sigma(1;t)$ is the inverse transform of $\tilde{\Sigma}(1;z)$.

The operator $\Sigma(1;t)$ actually has a singular part corresponding to the initial effect of correlations among the particles. To separate off this piece explicitly, define the operators $\tilde{K}(1\cdots s;z)$ by

$$\tilde{U}(1\cdots s;z)\,\tilde{U}^{-1}(1;z) \equiv U(1\cdots s;0)\,U^{-1}(1;0) + \tilde{K}(1\cdots s;z)\,. \tag{2.10}$$

Then, $\tilde{\Sigma}(1;z)$ may be written as the sum of a static (z-independent) part, $B(1)$, and a dynamic part, $\tilde{M}(1;z)$. Hence, with the substitution of Eq. (2.10) with $s=2$ into Eq. (2.8), $\tilde{\Sigma}(1;z)$ becomes

$$\tilde{\Sigma}(1;z) = B(1) + \tilde{M}(1;z)\,, \tag{2.11}$$

where

$$B(1) \equiv \mathrm{Tr}_2 L_I(12)\,U(12;0)\,U^{-1}(1;0) \tag{2.12}$$

and

$$\tilde{M}(1;z) \equiv \mathrm{Tr}_2 L_I(12)\tilde{K}(12;z)\,. \tag{2.13}$$

The static operator, $B(1)$, is an average of the interaction over the equilibrium correlations of the system. In the classical limit, it may be expressed exactly as a Vlasov-type operator with the potential replaced by the direct correlation function.[11] No such expression involving only two-body correlations is known to exist in the more general quantum case, but $B(1)$ may still be interpreted as a mean-field operator which contains the exact short-time behavior of $\psi^{(1)}$. The frequency-dependent operator, $\tilde{M}(1;z)$, vanishes as $|z| \to \infty$, and contains, therefore, the effects of collisions between a single particle and the particles of the surrounding medium requiring a finite time. This is somewhat more evident in the time representation where Eqs. (2.7) and (2.11) give

$$\left(i\frac{\partial}{\partial t} - L(1) - B(1)\right)\psi^{(1)}(1;t) = -i\int_0^t d\tau M(1;t-\tau)\psi^{(1)}(1;\tau)\,. \tag{2.14}$$

Here, the collision operator M is defined to be the inverse Laplace transform of $\tilde{M}(1;z)$.

B. Closure of the section equation of the hierarchy

The static operator $B(1)$ is expressed entirely in terms of the initial value operators $U(12;0)$ and $U(1;0)$. As discussed in Sec. III and in Appendix B these operators may be determined from the equilibrium reduced distribution operators. On the other hand, $M(1;t)$ is defined only formally in terms of the dynamical quantity, $\tilde{K}(12;z)$. To further expose the structure of $\tilde{K}(12;z)$ an equation of motion for it may be obtained by a formal closure of the second equation of the hierarchy. The operators $\tilde{U}(12;z)$ and $\tilde{U}(1;z)$ obey the transformed hierarchy equations [see Eqs. (1.13) and (1.14)]

$$[z - L(1)]\,\tilde{U}(1;z) - \mathrm{Tr}_2 L_I(12)\,\tilde{U}(12;z) = U(1;0)\,. \tag{2.15}$$

$$[z - L(12)]\,\tilde{U}(12;z) - \mathrm{Tr}_3[L_I(13) + L_I(23)]\,\tilde{U}(123;z) = U(12;0)\,. \tag{2.16}$$

Also, $\tilde{K}(1\cdots s;z)$ is related to the $\tilde{U}$ operators from Eq. (2.10) by

$$\tilde{U}(1\cdots s;z) = [U(12;0)\,U^{-1}(1;0) + \tilde{K}(12;z)]\,\tilde{U}(1;z)\,. \tag{2.17}$$

Therefore, an equation for $\tilde{K}(12;z)$ may be obtained from those for $\tilde{U}(12;z)$ and $\tilde{U}(1;z)$. Substitution of Eq. (2.17) into Eq. (2.16) and use of Eq. (2.15) yields, after a bit of algebra, the second hierarchy equation for $\tilde{K}(12;z)$,

$$\begin{aligned}&[z - L(12)]\tilde{K}(12;z)\\ &\quad + U(12;0)U^{-1}(1;0)\,\mathrm{Tr}_3 L_I(13)\tilde{K}(13;z)\\ &\qquad - \mathrm{Tr}_3[L_I(13) + L_I(23)]\tilde{K}(123;z) = K(12;0)\end{aligned} \tag{2.18}$$

The operators $\tilde{K}(1\cdots s;z)$ are analyzed in Appendix A and found to be of the form

$$\tilde{K}(1\cdots s;z) = \tilde{G}(1\cdots s;z)\,L_I(12)(1 + \mathcal{P}_{1,2})\,U^{-1}(1;0)\,. \tag{2.19}$$

Substitution of (2.19) into (2.18) gives the equation for $\tilde{G}(12;z)$,

$$\begin{aligned}&[z - L(12)]\,\tilde{G}(12;z)\\ &\quad + U(12;0)U^{-1}(1;0)\,\mathrm{Tr}_3 L_I(13)\,\tilde{G}(13;z)\\ &\qquad - \mathrm{Tr}_3[L_I(13) + L_I(23)]\tilde{G}(123;z) = G(12)\,,\end{aligned} \tag{2.20}$$

where $G(1\cdots s)$ is the initial value defined by $K(1\cdots s;t=0) = G(1\cdots s)\,L_I(12)(1 + \mathcal{P}_{1;2})U^{-1}(1,0)$.

As expected, this is not a closed equation for $\tilde{G}(12;z)$, since it is coupled to $\tilde{G}(123;z)$. However, a formal closure is possible at this two-particle level just as described above for the first BBGKY equation. Writing $\tilde{G}(123;z)$ as

$$\tilde{G}(123;z) = [\tilde{G}(123;z)\,\tilde{G}^{-1}(12;z)]\,\tilde{G}(12;z) \tag{2.21}$$

yields the equation for $\tilde{G}(12;z)$,

$$[z - L(12) - \tilde{\Sigma}^{(2)}(12;z)]\,\tilde{G}(12;z) = G(12)\,, \tag{2.22}$$

with

$$\tilde{\Sigma}^{(2)}(12;z) \equiv \mathrm{Tr}_3[L_I(13)+L_I(23)]\,\tilde{G}(123;z)\,\tilde{G}^{-1}(12;z) - U(12;0)U^{-1}(1;0)\,\mathrm{Tr}_3\,L_I(13)\mathcal{P}_{23}\,. \quad (2.23)$$

Also, in a manner very similar to the procedure leading to Eqs. (2.10) and (2.19), it is possible to show that,

$$\tilde{G}(123;z)\tilde{G}^{-1}(12;z) = G(123;0)G^{-1}(12;0) + \tilde{J}(123;z)(1+\mathcal{P}_{1,3}+\mathcal{P}_{2,3}) \times [L_I(13)+L_I(23)]\,G^{-1}(12;0)\,, \quad (2.24)$$

where the operator $\tilde{J}(123;z)$ vanishes in the limit $|z| \to \infty$. Just as in the single-particle case, therefore, $\tilde{\Sigma}^{(2)}$ may be written as the sum of a static, mean-field term and a dynamic collisional term,

$$\tilde{\Sigma}^{(2)}(12;z) = B^{(2)}(12) + \tilde{M}^{(2)}(12;z)\,, \quad (2.25)$$

where the operators $B^{(2)}$ and $\tilde{M}^{(2)}(12;z)$ are identified as

$$B^{(2)}(12) = \mathrm{Tr}_3[L_I(13)+L_I(23)]\,G(123;0)G^{-1}(12;0) - U(12;0)U^{-1}(1;0)\,\mathrm{Tr}_3\,L_I(13)\mathcal{P}_{13}\,, \quad (2.26)$$

$$\tilde{M}^{(2)}(12;z) = \mathrm{Tr}_3[L_I(13)+L_I(23)]\,\tilde{J}(123;z)(1+\mathcal{P}_{13}+\mathcal{P}_{23}) \times [L_I(13)+L_I(23)]G^{-1}(12;0)\,. \quad (2.27)$$

Therefore, by formally solving Eq. (2.22), the collision operator $\tilde{M}(1;z)$ is displayed in terms of the effective two-body dynamics contained in Eqs. (2.26) and (2.27),

$$\tilde{M}(1;z) = \mathrm{Tr}_2\,L_I(12)[z - L(12) - B^{(2)}(12) - \tilde{M}^{(2)}(12;z)]^{-1} \times G(12)\,L_I(12)(1+\mathcal{P}_{12})\,U^{-1}(1;0)\,. \quad (2.28)$$

With the exception of $\tilde{M}^{(2)}$, all of the operators in Eq. (2.28) may be analyzed in terms of equilibrium correlation functions or two-body dynamics. Although a further decomposition of the structure of $\tilde{M}^{(2)}$ along the lines of this section is possible,[26] it will not be pursued here. Instead, it is simply noted that weak-coupling, low-density, or small plasma-parameter kinetic equations follow directly from expansions of $B^{(2)}$ and $\tilde{M}^{(2)}$ to lowest order in the associated small parameter. Furthermore, the dynamical many-body difficulties have been relegated to the operator $\tilde{M}^{(2)}$, representing dynamical corrections to the two-body effective motion. The effect of static correlations are already contained in operators $G(12)$ and $B^{(2)}$ so that, for example, even neglecting $\tilde{M}^{(2)}$ altogether leads to a kinetic equation containing all the above mentioned small parameter results as limits while also giving correctly for strongly coupled systems the first four frequency moments of the time-correlation function and the exact conservation laws. This is not to suggest that $\tilde{M}^{(2)}$ is unimportant (for example, all of the interesting mode-coupling effects are contained in $\tilde{M}^{(2)}$), but rather to observe that the exact formulation of Eq. (2.28) provides a convenient representation from which to initiate a detailed analysis or phenomenological approximations.

III. SHORT-TIME AND WEAK-COLLISION EQUATIONS

A. Exact short-time equation (generalized mean-field approximation)

The removal of the singular part of $\Sigma(1;t)$ assures that $M(t)$ is well-behaved as a function of t near $t=0$. Consequently from Eq. (2.14) the approximate equation

$$\left(i\frac{\partial}{\partial t} - L(1) - B(1)\right)\psi^{(1)}(1;t) = 0\,, \quad (3.1)$$

is asymptotically exact at $t \to 0$. It follows also that the first two frequency moments are given exactly. The operator $B(1)$ represents the effects on the motion in the single-particle subspace due to initial correlations with other particles. This mean field is defined by the initial value operators $U(12;0)$ and $U(1;0)$ which are shown in Appendix B to be entirely determined from the one-, two-, and three-particle equilibrium reduced density matrices. In many cases of interest (e.g., Coulomb systems) the static mean-field effects dominate collisional effects and Eq. (3.1) may be taken as a first approximation for the calculation of correlation functions. The key advantage of this equation is its simplicity and the fact that it is applicable to strongly interacting fluids. To make contact with similar results of other many-body methods the relationship of the exact short-time kinetic equation to mean-field approximations, including the random phase approximation, is briefly described here.

To be more specific, the dielectric function for a dense degenerate gas is considered. The dielectric function may be defined in terms of the density-density response function,

$$\epsilon(k,\omega) \equiv [V(k)\chi(k,\omega)+1]^{-1}\,, \quad (3.2)$$

$$\chi(k,\omega) \equiv i\int_0^\infty dt\, e^{i(\omega+i\eta)t}\langle[n_{-k}, n_k(t)]\rangle_0\,, \quad (3.3)$$

where $V(k)$ is the Fourier-transformed potential. The brackets in Eq. (3.3) denote the commutator and n_k is the Fourier-transformed density,

$$n_k \equiv \Omega^{-1/2}\sum_{i=1}^{N} e^{-i\vec{k}\cdot\vec{r}_i}\,. \quad (3.4)$$

Also, Ω is the volume and η is an infinitesimal positive constant. The relationship to correlation functions and the kinetic theory follows from the cyclic invariance of the trace to write $\chi(k,\omega)$ as

$$\chi(k,\omega)=i\int_0^\infty dt\, e^{i(\omega+i\eta)t} \times[\langle n_{-k}n_k(t)\rangle_0-\langle n_{-k}n_k(t+i\beta)\rangle_0]\,. \tag{3.5}$$

The integrand is therefore expressed in terms of correlation functions of the type (1.2) with $a\equiv e^{i\vec{k}\cdot\vec{r}}$ and $b=e^{-i\vec{k}\cdot\vec{r}}$. The short-time kinetic equation may be solved in terms of $B(1)$ with the initial condition,

$$\psi^{(1)}(1;t=0)=U(1;0)\,e^{i\vec{k}\cdot\vec{r}_1} \tag{3.6}$$

to give

$$\begin{aligned}\chi(k,\omega)&=\frac{i}{\Omega}\int_0^\infty dt\, e^{i(\omega+i\eta)t}\mathrm{Tr}_1\, e^{-i\vec{k}\cdot\vec{r}_1}\left(e^{-it[L(1)+B(1)]}-e^{-i(t+i\beta)[L(1)+B(1)]}\right)U(1;0)\,e^{i\vec{k}\cdot\vec{r}_1}\\&=\frac{-1}{\Omega}\mathrm{Tr}_1\, e^{-i\vec{k}\cdot\vec{r}_1}[\omega+i\eta-L(1)-B(1)]^{-1}\left(1-e^{\beta[L(1)+B(1)]}\right)U(1;0)\,e^{i\vec{k}\cdot\vec{r}_1}\,.\end{aligned} \tag{3.7}$$

To further simplify, the operator $U(1;0)$ may be expressed in terms of $B(1)$ using the identity

$$\lim_{\omega\to\infty}\omega\tilde{\chi}(k,\omega)=\frac{1}{\Omega}\mathrm{Tr}_1\, e^{-i\vec{k}\cdot\vec{r}_1}[e^{i\vec{k}\cdot\vec{r}_1},f(1)]\,. \tag{3.8}$$

Comparison of (3.7) and (3.8) gives

$$U(1;0)e^{i\vec{k}\cdot\vec{r}_1}=\left(1-e^{\beta[L(1)+B(1)]}\right)^{-1}[f(1),e^{i\vec{k}\cdot\vec{r}_1}]\,. \tag{3.9}$$

Substitution of Eq. (3.9) into (3.7) then gives the dielectric function in the form,

$$\epsilon(k,\omega)=1-\frac{V(k)\chi(k,\omega)}{1+V(k)\chi(k,\omega)}\,, \tag{3.10}$$

$$\chi(k,\omega)=\frac{1}{\Omega}\mathrm{Tr}_1\, e^{-i\vec{k}\cdot\vec{r}_1}[\omega+i\eta-L(1)-B(1)]^{-1} \times[e^{i\vec{k}\cdot\vec{r}_1},f(1)]\,. \tag{3.11}$$

This is the result that will be referred to as the generalized mean-field approximation. To be more suggestive, define the "mean field," $\phi(k,\omega)$, by

$$\chi(k,\omega)=\frac{\chi_0(k,\omega)}{1-\phi(k,\omega)\chi_0(k,\omega)}\,, \tag{3.12}$$

where $\chi_0(k,\omega)$ is the ideal-gas value of $\chi(k,\omega)$ obtained from Eq. (3.11) by setting $B=0$. The dielectric function then takes the form

$$\epsilon(k,\omega)=1-\frac{V\chi_0}{1+(V-\phi)\chi_0}\,. \tag{3.13}$$

The mean field $\phi(k,\omega)$ is determined in terms of B by evaluation of Eq. (3.11). In the classical limit an exact evaluation is possible with the result

$$\phi(k,\omega)\xrightarrow[\text{classical}]{}-\beta C(k)\,, \tag{3.14}$$

where $C(k)$ is the direct correlation function. In the quantum case no such simple form is known, but there are kinetic modeling techniques that allow an approximate evaluation in terms of a finite set of matrix elements of B. In this way it is possible to make contact with the phenomenological parameters of Landau's Fermi liquid theory, for example, and use these to characterize B rather than make a direct calculation.[23] Here, however, we simply illustrate that the usual random phase approximation is recovered if B is evaluated to lowest order in the interaction or small plasma parameter. In this limit,

$$B(1)\to\mathrm{Tr}_2 L_I(12)f_i(12)(1+P_{12})f_i^{-1}(1)\,, \tag{3.15}$$

where $f_i(1)$ and $f_i(12)$ are the one- and two-particle equilibrium reduced distribution operators, respectively, for an ideal gas. This operator represents a quantum-mechanical generalization of the Vlasov operator of classical kinetic theory including first-order exchange contributions to the single-particle energies (Hartree-Fock). Neglecting the exchange terms, $\chi(k,\omega)$ is readily evaluated to give for ϕ and ϵ,

$$\begin{aligned}\phi(k,\omega)&=V(k)\,,\\ \epsilon(k,\omega)&=1-V\chi_0\,,\end{aligned} \tag{3.16}$$

which is the usual result. The exchange terms are somewhat more difficult to handle. It was suggested by Hubbard[27] that these terms may be partially incorporated with an approximate form for the exchange matrix elements. Use of Hubbard's approximation in the expression for $B(1)$, Eq. (3.15), reproduces his results for $\epsilon(k,\omega)$ as well,

$$\epsilon_H(k,\omega)=1-\frac{V(k)\chi_0(k,\omega)}{1+V_{\mathrm{eff}}(k)\chi_0(k,\omega)}\,, \tag{3.17}$$

where the effective field of Hubbard is given by

$$V_{\mathrm{eff}}(k)=\frac{2\pi e^2}{k^2+k_F^2}\,, \tag{3.18}$$

and k_F is the Fermi wave vector.

B. Weak-collision approximations

The preceding discussion was limited to the case for which the collision operator $M(1;t)$ could be neglected, but in most cases of interest collisions must be accounted for. Under circumstances such that momentum transfer is small, an expansion in the coupling constant may be possible. The form of M given by Eq. (2.28) is manifestly of second order in the interaction. Thus to lowest order, all other quantities in Eq. (2.28) may be replaced by their ideal-gas values,

$$\tilde{M}(1;z) \rightarrow \mathrm{Tr}_2 L_I[z - L(1) - L(2)]^{-1} G_i(12)\, L_I(12) \times (1+\mathcal{P}_{1,2}) U_i^{-1}(1;0)\,, \tag{3.19}$$

where the subscript i is used to indicate the ideal-gas value. This result together with the corresponding expansion of $B(1)$ provides a kinetic equation which is exact to second order in the coupling constant. The corresponding classical equation has been discussed in detail by Forster and Martin.[12] The quantum-mechanical result above has also been discussed recently by Boley and Smith.[22] It is straightforward to show that Eq. (3.19) may be expressed as a Boltzmann collision operator in the limit $z \rightarrow 0$ (long-time behavior) with the cross section given in the Born approximation. Since these results are discussed elsewhere no further comment will be given here. Instead, a generalization of this weak-coupling result will be discussed. To describe the generalization, note that the Born approximation describes a pair of two-body interactions coupled by a two-body propagator, $[z - L(1) - L(2)]^{-1}$. Each particle propagates independently of the other. A generalization is to preserve the independent propagation between the two interactions, but to replace the free-particle motion by the actual dynamics of the single-particle subspace. The result is still of the Born type since no direct interaction between the two particles occurs in the propagator. Doing this and retaining the initial correlations contained in $G(12)$ gives a quantum-mechanical generalization of the disconnected approximation (DA) employed by Gould and Mazenko and Wallenborn and Baus[18] to describe transport properties of a classical one-component plasma.

This generalization of the Born approximation is easy to describe but phenomenological in nature and therefore more difficult to justify. The disconnected approximation results from the replacement of the effective two-particle operator $\tilde{\Sigma}^{(2)}(12;z)$ by the sum of single-particle operators and neglect of the direct interaction, $L_I(12)$, in the propagator,

$$L(12) + \tilde{\Sigma}(12;z) \rightarrow L(1) + L(2) + \tilde{\Sigma}(1;z) + \tilde{\Sigma}(2;z)\,. \tag{3.20}$$

The resulting collision operator $\tilde{M}_{\mathrm{DA}}(1;z)$, is then given by,

$$\begin{aligned}\tilde{M}_{\mathrm{DA}}(1;z) &= \mathrm{Tr}_2 L_I(12)[z - L(1) - L(2) - \tilde{\Sigma}(1;z) - \tilde{\Sigma}(2;z)]^{-1} f^{(2)}(12)\mathcal{L}_I(12)(1+\mathcal{P}_{1,2})U^{-1}(1;0)\\ &= -i\int_0^\infty dt\, e^{izt}\, \mathrm{Tr}_2 L_I(12) U(1;t) U(2;t) U^{-1}(1;0) U^{-1}(2;0) f^{(2)}(12)\mathcal{L}_I(12)(1+\mathcal{P}_{1,2})U^{-1}(1;0)\,,\end{aligned} \tag{3.21}$$

where the effective interaction, $\mathcal{L}_I(12)$, has been introduced and is defined by

$$f^{(2)}(12)\mathcal{L}_I(12) \equiv G(12)\, L_I(12)\,. \tag{3.22}$$

Equation (3.21) is asymmetric in the sense that the left-hand interaction is different from the right-hand interaction, but in this form the disconnected approximation preserves the exact short-time behavior of the collision operator, which can be important even for long times.[18]

The dependence of $\tilde{M}_{\mathrm{DA}}(1;z)$ upon $\tilde{\Sigma}$ means, in practice, that determination of $\tilde{\Sigma}$ must be accomplished iteratively, beginning with an approximation for $\tilde{\Sigma}$ in Eq. (3.20). The simplest approximation would be $\tilde{\Sigma} \rightarrow 0$, but a better first choice is $\tilde{\Sigma} \rightarrow B$. The disconnected approximation with this substitution has been termed the effective interaction approximation (EIA),

$$\tilde{M}_{\mathrm{EIA}}(1;z) = \mathrm{Tr}_2 L_I(12)[z - L(1) - L(2) - B(1) - B(2)]^{-1} \times f^{(2)}(12)\mathcal{L}_I(12)(1+\mathcal{P}_{1,2})U^{-1}(1;0)\,, \tag{3.23}$$

which is equivalent to Eq. (3.21) with $U(1;t)$ determined from the short-time kinetic equation, Eq. (3.1). As mentioned above, Gould and Mazenko and Wallenborn and Baus[18] have applied the classical version of Eq. (3.23) to the one-component plasma. The two-component generalization of Eq. (3.23) is applied to the calculation of plasma electrical conductivity in II.

IV. LOW-DENSITY GAS

The inherent limitation to any Born approximation is that the strong collisions are not properly accounted for. Since most realistic interactions are strong at short range, this generally means that the rest of the Born series must be resummed to be adequate at small impact parameters. The cross section is then expressed in terms of an appropriate T matrix rather than the potential. The utility of the formalism of Sec. II is illustrated in this section by a density expansion to obtain the first-order kinetic equation for a quantum gas. The appropriate T matrix and exchange effects are identified and provide an important example of the effects to be expected when extending the generalized Born approximation to include strong collisions.

The formal kinetic equation is determined for a low-density gas with short ranged potential by retaining contributions to the operators B and M accurate to lowest order in na^3. Here n is the density and a is a length characterizing the force range, so that $(na^3)^{1/3}$ is a measure of the ratio of force range to average interparticle spacing, $l_0 \equiv n^{-1/3}$. The degree of degeneracy of the gas may be measured by $n\lambda^3$, where $\lambda \equiv (2\pi\hbar^2/mkT)^{1/2}$ is the thermal DeBroglie wavelength. Thus, in order to describe the gas at low temperatures it is necessary to allow $n\lambda^3$ to be large while maintaining na^3 small. The occurrence of these two distinct measures of the density makes the treatment of degenerate gases considerably more complex than in the classical case. The gas may be simultaneously dilute with respect to force range and very dense with respect to degeneracy, depending on whether the "size" of an atom is characterized by a^3 or λ^3. Expansions treating both na^3 and $n\lambda^3$ as small parameters lead to virial expansions (e.g., the Bethe-Uhlenbeck equation for the pressure)[28] that are valid only in the semiclassical region. Here only na^3 is taken to be small so that arbitrary degeneracy is permitted. Such an expansion is similar to the equilibrium binary collision expansions of Bloch and DeDominicis and Lee and Yang, and the ladder approximation to the finite temperature Bethe-Salpeter equation.[29] In the first part of this section the formal operators of Sec. II defining the kinetic equation are evaluated for small na^3 without any restrictions on the space and time scales (sometimes referred to classically as a generalized low-density Boltmann equation). In the second part, the Boltzmann limit is taken (space and time scales large compared to atomic dimensions) and a linearized Uehling-Uhlenbeck equation is obtained.[21]

A. Low-density kinetic equation

It can be demonstrated using scaling arguments that the operator $\tilde{\Sigma}(1;z)$ vanishes relative to the free-particle motion as $na^3 \to 0$. Hence, if only the leading order in na^3 is required, the contributions to $\tilde{\Sigma}(1;z)$ from $U^{-1}(1;0)$, $G(12)$, and $\tilde{\Sigma}^2$ need only be evaluated to zeroth order. Since the leading contribution to $\tilde{M}^2(12;z)$ is also at least of order na^3, the only possible low-density contribution to $\tilde{\Sigma}^{(2)}$ must come from $B^{(2)}(12)$.

The operators $U(1\cdots s;0)$ and $G(1\cdots s)$ each consist of two parts. In the first, the operand is multiplied by an s-particle distribution operator, while the second is a trace of the operand over the equilibrium correlations among $s+1$ and more particles. In the case of classical statistical mechanics, these correlations are due only to the potential, and in the low-density limit only the multiplicative parts of the operator need be retained. In quantum statistics, however, there are additional correlations due to exchange, and some of these correlations make contributions even in the limit $a/l_0 \to 0$. This is because these exchange terms have a leading density dependence involving the degeneracy parameter, a/λ, rather than a/l_0. In Appendix B, the contributions of such exchange terms to the initial-value operators with $s \leqslant 3$ are explicitly isolated and grouped with the multiplicative parts of the operators. The remainder of the "integral" operators, though still containing both potential and exchange correlations, are now at least of order a/l_0. Therefore, in the low-density limit only the exchange-modified multiplicative parts of the operators need be retained. Therefore, from the results of Appendix B and the above discussion, the following low-density approximations result

$$B(1)y(1) \to \mathrm{Tr}_2 L_I(12)[f_0^{(2)}(12)(1+\mathcal{P}_{12})f_0^{-1}(1)y(1) + \epsilon(1+\mathcal{P}_{12})y(1)\tilde{f}_0^{-1}(1)g_0^{(2)}(12)]\,, \tag{4.1}$$

$$B^{(2)}(12)y(12) \to \epsilon[f_0(1)+f_0(2)]V(12)y(12) - \epsilon y(12)V(12)[f_0(1)+f_0(2)]\,, \tag{4.2}$$

$$G(12)y(12) \to f_0(12)y(12)\{1+\epsilon[f_0(1)+f_0(2)] + \tfrac{1}{2}f_0(12)\}\,, \tag{4.3}$$

where $y(12)$ is an arbitrary two-particle operator. Substitution of these results into Eq. (2.28) with $M^{(2)}=0$ gives the desired result,

$$\tilde{\Sigma}(1;z)y(1) \to \mathrm{Tr}_2 L_I(12)[f_0^{(2)}(12)(1+\mathcal{P}_{12})f_0^{-1}(1)y(1) + \epsilon(1+\mathcal{P}_{12})y(1)\tilde{f}_0^{-1}(1)g_0^{(2)}(12)] + \mathrm{Tr}_2 L_I(12)[z-\hat{L}(12)]^{-1}f_0^{(2)}(12)[L_I(12)(1+\mathcal{P}_{12})f_0^{-1}(1)y(1)\tilde{f}_0^{-1}(1)][S_0(12)+\tfrac{1}{2}f_0^{(2)}(12)]\,, \tag{4.4}$$

where $\hat{L}(12)$ and $S_0(12)$ are defined by

$$\hat{L}(12) = L(1)+L(2)+\hat{L}_I(12)\,, \quad \hat{L}_I(12)0 \equiv \hat{V}(12)0 - 0\hat{V}^{\dagger}(12)\,, \quad \hat{V}(12) \equiv V(12)S_0(12)\,, \quad S_0(12) \equiv \tilde{f}_0(1)\tilde{f}_0(2) - f_0(1)f_0(2)\,, \quad \tilde{f}_0 = 1+\epsilon f_0 \tag{4.5}$$

and $\epsilon = (-)1$ for (fermions) bosons. The subscript "0" indicates that the operators are to be approximated by their low-density expressions.

Equation (4.4) is the principal result of this section. It yields a non-Markovian, nonlocal kinetic equation valid at low densities, but arbitrary degeneracy. It must be supplemented by a determination of the low-density equilibrium distribution operators $f_0(1)$ and $f_0^{(2)}(12)$. Making the same low-density assumptions in the first two equations of the equilibrium hierarchy yields

$$L(1)f_0(1) + \mathrm{Tr}_2 L_I(12) f_0^{(2)}(12) = 0 \tag{4.6}$$

and

$$\hat{L}(12) f_0^{(2)}(12) = 2[L(1) + L(2)] f_0(1) f_0(2) P_2 . \tag{4.7}$$

A consistent set of solutions to these equations is[30]

$$f_0(1) = (\mathfrak{z}^{-1} e^{\beta H(1)} - \epsilon)^{-1} \tag{4.8}$$

and

$$f_0^{(2)}(12) = 2F(\hat{H}) S_0(12) P_2 = 2S_0(12) F(\hat{H}^\dagger) P_2 , \tag{4.9}$$

with

$$F(\hat{H}) = \mathfrak{z}^2 e^{-\beta\hat{H}} (1 - \mathfrak{z}^2 e^{-\beta\hat{H}})^{-1} , \quad \hat{H} \equiv H_0 + \hat{V} \tag{4.10}$$

which are identical to results obtained through diagrammatic perturbation theory using a ladder resummation.

B. The Boltzmann limit

The result Eq. (4.4) simplifies considerably if attention is restricted to variations of the solution to the kinetic equation to times long compared to the collision time, t_0, and space variations large compared to the force range, a. Such a restriction will be referred to as the Boltzmann limit. Actually, to further simplify the description the case of a spatially uniform system will be considered. Using Eq. (4.4) in Eq. (2.7) and inverting the Laplace transform gives

$$\begin{aligned}\left(i\frac{\partial}{\partial t} - L(1)\right)\psi^{(1)}(1;t) = {} & \mathrm{Tr}_2 L_I(12)[f_0^{(2)}(12)(1+\mathcal{P}_{12}) f_0^{-1}(1)\psi^{(1)}(1;t) + \epsilon(1+\mathcal{P}_{12})\psi^{(1)}(1;t)\tilde{f}_0^{-1}(1) g_0^{(2)}(12)] \\ & + 2i\int_0^t d\tau\, \mathrm{Tr}_2 L_I(12) F(\hat{H}) \left(\frac{\partial\Omega(\tau)}{\partial\tau} S_0(12)(1+\mathcal{P}_{12})\chi(1;t-\tau)\Omega^\dagger(\tau)\right. \\ & \left. + \Omega(\tau) S_0(12)(1+\mathcal{P}_{12})\chi(1;t-\tau)\frac{\partial\Omega^\dagger(\tau)}{\partial\tau}\right)[1+F(\hat{H}^\dagger)]P_2 ,\end{aligned} \tag{4.11}$$

where the notation $\chi(1;t) = f_0^{-1}(1)\psi^{(1)}(1;t)\tilde{f}_0(1)^{-1}$ has been introduced, $g_0^{(2)}(12)$ is defined by Eq. (B1) and $\Omega(t)$ is defined by

$$\Omega(t) \equiv e^{-it\hat{H}} e^{itH_0} . \tag{4.12}$$

Since for short ranged potentials, $\Omega(t)$ approaches a limit for $t > t_0$, $\partial\Omega(t)/\partial t$ vanishes for times large compared to a collision time, and the domain of integration in the above is only over an interval on the order of a collision time. Therefore, in the Boltzmann limit, for $t \gg t_0$ the replacement

$$\chi(1;t-\tau) \to \chi(1;t)$$

is correct to order t_0/t, and the following equation results,

$$\begin{aligned}\left(i\frac{\partial}{\partial t} - L(1)\right)\psi^{(1)}(1;t) = {} & 2\epsilon\, \mathrm{Tr}_2 L_I(12)\{F(\hat{H})S_0(12), (1+\mathcal{P}_{12})\chi(1;t)f_0(1)\}P_2 - 2\,\mathrm{Tr}_2 L_I(12)F(\hat{H})S_0(12)(1+\mathcal{P}_{12})\chi(1;t)F(\hat{H}^\dagger)P_2 \\ & + 2\,\mathrm{Tr}_2 L_I(12)F(\hat{H})\hat{\Omega}(12)S_0(12)(1+\mathcal{P}_{12})\chi(1;t)\hat{\Omega}^\dagger(12)[1+F(\hat{H})]P_2 .\end{aligned} \tag{4.13}$$

Here the curly brackets denote the anticommutator, and $\hat{\Omega}(12)$ is the Möller operator associated with $\Omega(t)$,

$$\hat{\Omega}(12) \equiv \lim_{t\to\infty} e^{-it\hat{H}} e^{itH_0} . \tag{4.14}$$

The properties of $\hat{\Omega}$ and the related transition matrix, $\hat{T}$, are discussed in Appendix C. Since spatial homogeneity has been assumed, $\psi^{(1)}(1;t)$ is diagonal in the momentum representation

$$(p_1|\psi^{(1)}(1;t)|p_1') = \psi^{(1)}(p_1;t)\delta_{p_1,p_1'} , \tag{4.15}$$

and only the diagonal matrix elements of Eq. (4.13) need be considered. Therefore, the kinetic equation is

$$
\begin{aligned}
i\frac{\partial}{\partial t}\psi^{(1)}(p_1;t) = \epsilon \sum_{p_2} & (p_1p_2|[V,\{F(\hat{H})S_0(12),(1+\mathcal{P}_{12})\chi(1;t)f_0(1)\}]P_2|p_1p_2) \\
& - \sum_{p_2}(p_1p_2|[V,F(\hat{H})S_0(12)(1+\mathcal{P}_{12})\chi(1;t)F(\hat{H}^\dagger)]P_2|p_1p_2) \\
& + 2\sum_{p_2}(p_1p_2|\{V,\hat{\Omega}f_0(1)f_0(2)[1+F(H_0)](1+\mathcal{P}_{12})\chi(1;t)\hat{\Omega}^\dagger\}P_2|p_1p_2)\,, \qquad (4.16)
\end{aligned}
$$

where use has been made of the intertwining relation, Eq. (C9). With the aid of the "detailed balance" property

$$
\begin{aligned}
(p_1p_2|F(\hat{H})S_0(12)|p_1'p_2') &= (p_1'p_2'|F(\hat{H})S_0(12)|p_1p_2) \\
&= (p_1'p_2'|S_0(12)F(\hat{H}^\dagger)|p_1p_2)\,, \qquad (4.17)
\end{aligned}
$$

the first two terms in Eq. (4.16) may be shown to vanish. Hence, only the collisional term remains, and through the application of Eqs. (C5), (C6), and the optical theorem, Eq. (C8), it may be written in the form

$$
\begin{aligned}
\frac{\partial\psi(p_1;t)}{\partial t} = 4\pi \sum_{p_2,\bar{p}_1,\bar{p}_2}\sum\sum & |\langle p_1p_2|\hat{T}|\bar{p}_1\bar{p}_2\rangle|^2\delta(E-\bar{E})\,\tilde{f}_0(\bar{p}_1)\tilde{f}_0(\bar{p}_2)f_0(p_1)f_0(p_2) \\
& \times[\chi(\bar{p}_1;t)+\chi(\bar{p}_2;t)-\chi(p_1;t)-\chi(p_2;t)]\,, \qquad (4.18)
\end{aligned}
$$

where $|p_1p_2\rangle \equiv P_2|p_1p_2)$ and $E=(p_1^2+p_2^2)/2m$.

Aside from the degeneracy modified T matrix this linear kinetic equation is the same as that proposed by Uehling and Uhlenbeck.[21] The linearized form occurs here because the correlation functions are being described rather than the reduced distribution operators. The nondegenerate quantum Boltzmann equation follows with the replacements everywhere, $\tilde{f}_0 \to 1$ and $f_0(p) \to$ corresponding Boltzmann distribution. The occurrence of the two factors $\tilde{f}_0(\bar{p}_1)$ and $\tilde{f}_0(\bar{p}_2)$ may be understood by the arguments of Uehling and Uhlenbeck that the availability of states into which particles may be scattered will be modified by their relative occupation by other particles. The further modification of T to $\hat{T}$ follows from the same considerations if T is represented in terms of its Born series and each matrix element of the potential is corrected for the relative occupancy of the states. The correction factor is S_0 instead of $\tilde{f}_0\tilde{f}_0$ in this latter case because there can be both particle-particle and hole-hole matrix elements for the intermediate states whereas only particle states can occur as external states in the matrix element of T.

Returning to the dependence of the collision operator on na^3 and a/λ mentioned at the beginning of this section, it is instructive to consider the collision frequency associated with Eq. (4.18).

$$
\begin{aligned}
\nu(p_1) \equiv \frac{1}{f_0(p_1)\tilde{f}_0(p_1)} \sum_{p_2}\sum_{\bar{p}_1}\sum_{\bar{p}_2} & f_0(p_1)f_0(p_2)\tilde{f}_0(\bar{p}_1)\tilde{f}_0(\bar{p}_2) \\
& \times \delta(E-\bar{E})|\langle p_1p_2|\hat{T}|\bar{p}_1\bar{p}_2\rangle|^2\,. \qquad (4.19)
\end{aligned}
$$

This may be written in terms of a cross section using the identity

$$f_0(p) = \mathfrak{z}\,e^{-\beta p^2/2m}\tilde{f}_0(p)\,,$$

to give

$$
\begin{aligned}
\nu(p_1) &= \frac{\mathfrak{z}}{[\tilde{f}_0(p_1)]^2}\sum_{p_2} e^{-\beta p_2^2/2m}\sum_{\bar{p}_1}\sum_{\bar{p}_2}\delta(E-\bar{E})|\langle p_1p_2|[\tilde{f}_0(p_1)\tilde{f}_0(p_2)]^{1/2}\hat{T}[\tilde{f}_0(\bar{p}_1)\tilde{f}_0(\bar{p}_2)]^{1/2}|\bar{p}_1\bar{p}_2\rangle|^2 \\
&= \frac{\mathfrak{z}}{\tilde{f}_0(p_1)^2}\,\frac{1}{\Omega}\sum_{p_2} e^{-\beta p_2^2/2m}\frac{|p_1-p_2|}{m}\;\sigma_T(p_1,p_2)\,. \qquad (4.20)
\end{aligned}
$$

Here $\sigma_T(p_1p_2)$ is the two-particle total cross section modified by degeneracy effects such that the scattering occurs between initial and final particle states with both particle-particle and hole-hole intermediate states. To estimate ν consider the zero momentum value

$$\nu(0)=\frac{(\mathfrak{z}/\lambda^3)}{[\tilde{f}_0(0)]^2}\frac{\hbar}{m\lambda}\int\frac{d^3p}{(2\pi)^3}e^{-p^2}p\sigma_T(\hbar p/\lambda)\,. \tag{4.21}$$

The total cross section has units of area and is of the form,

$$\sigma_T=a^2F(\mathfrak{z},a/\lambda,p)\,, \tag{4.22}$$

where $F(z,a/\lambda)$ is dimensionless. In the nondegenerate limit, $z\rightarrow n\lambda^3$ so

$$\nu(0)\rightarrow na^3t_0^{-1}\int\frac{d^3p}{(2\pi)^3}e^{-p^2}pF(0,a/\lambda,p)\,. \tag{4.23}$$

Here $t_0=a/v_T$ is the time required to traverse the force range a with the thermal velocity $\hbar/m\lambda$. Thus, the collision frequency is proportional to na^3, as expected. The situation is more complicated in the degenerate case. Restricting attention to fermions, the range of the momentum integral at low temperatures is more appropriately scaled to the Fermi momentum, h/l_0,

$$\nu(0)\rightarrow na^3t_0^{-1}\frac{\mathfrak{z}}{[\tilde{f}_0(0)]^{-1}}\int\frac{d^3p}{(2\pi)^3}e^{-(\lambda p/l_0)^2}pF(z,a/l_0,p)\,. \tag{4.24}$$

Again, $t_0=a/v_T$, except that now the thermal velocity is $\hbar/ml_0$. The coefficient of na^3 is now no longer simply a function of a/λ; instead it is dependent on the density through $\mathfrak{z}=(n\lambda^3)$ as well as the dependence of the cross section of a/l_0. The $\mathfrak{z}$ dependence arises from the degeneracy modification of $\hat{T}$, but the scattering theory associated with this operator has not been studied in detail so that further comment on the form of F cannot be given here.

The main difficulty in obtaining the low-density result discussed here is the presence of a second dimensionless density, $n\lambda^3$, due to the degeneracy. Since the latter is not necessarily small, the density dependence even for $na^3\ll 1$ can be complicated. This problem is clearly illustrated in the ladder ressummations of diagrammatic perturbation theory. Each is a low-density expansion (in the sense $na^3\ll 1$), but lead to different results (for example, Bloch-deDominicis, Bethe-Salpeter, and Lee and Yang ladders give inequivalent pressure equations).[30] Here only na^3 has been taken small, with $n\lambda^3$ arbitrary, and the kinetic equation differs from the nondegenerate Boltzmann result by a modification of the two-particle interactions to account for exchange of the interacting pair with each other and the remaining particles of the system. For fermions these modifications account for the average occupation of two-particle states and includes both particles (excitation energies above the Fermi energy) and holes (excitation energies below the Fermi energy). A similar result occurs for bosons although the situation is complicated by the possibility of Bose-Einstein condensation. The kinetic equation is of the Uehling-Uhlenbeck form although the latter has the above modifications of the cross section only for the final states. The only other derivation of the Uehling-Uhlenbeck equation known to the authors,[31] that of Kadanoff and Baym[32] using Green's functions, also leads to a complete modification of all interactions in the cross section. The $\hat{T}$ matrix defining the degeneracy modified cross section is discussed in Appendix C, but its properties are largely unknown.[33] For example, there may be bound states for fermions due to the formation of Cooper pairs and the consequences of condensation for bosons have not been studied.

V. DISCUSSION

Generalizing classical kinetic theory for time-correlation functions to apply quantum mechanically in the nondegenerate case is largely a matter of extending notation, and causes no real difficulty. In the degenerate case, the situation is complicated further by the imposition of Fermi or Bose statistics which couples the many-body system in a way not present for classical or semiclassical calculations. There are additional fundamental length and time scales associated with the degeneracy that complicate order of magnitude estimates associated with small parameter expansions and other approximation methods. Much of the difficulty associated with symmetry and statistics has been suppressed here by retaining all such effects in the equilibrium density operator and the associated equilibrium reduced density operators. This could also have been accomplished by using a second quantization representation, but the use of reduced distribution operators in the form given here allows a closer parallel with the classical case, and in its operator form is notationally more compact than the corresponding second quantization representation. The identification of the operators involved in the formal closure of the hierarchy is then only slightly more complicated than for the classical case. Indeed, the major difficulty is recognizing certain exchange operators for degnerate systems that are not otherwise present, and the relevant results have been given in Appendix B. By construction, the formally exact kinetic equation is analogous to the corresponding classical result, with a mean-field operator determined by the initial equilibrium correlations, and a collision operator. The collision operator has a structure revealed by the second BBGKY hierarchy equation and is represented in terms of effective two-particle dynamics modified by the equilibrium correlations among the particles. Explicit kinetic equations may be obtained by appropriately approximating this effective two-body problem. The primary example provided here is that of the low-density degenerate gas in which the operators determining the two-particle dynamics are approximated to zeroth order in a/l_0. Paper II applies these results to the calculation of the electrical conductivity for a multicomponent plasma and illustrates its utility for systems in which long-range forces are present.

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APPENDIX A: ANALYSIS OF $\tilde{K}(1\cdots s;z)$

The purpose of this appendix is the justification of Eq. (2.19) for $\tilde{K}(1\cdots s;z)$. Use of the operator identity

$$[z-L(1\cdots N)]^{-1}=1/z\{1+[z-L(1\cdots N)]^{-1}L(1\cdots N)\}\,, \tag{A1}$$

allows $\tilde{U}(1\cdots s;z)$ to be written as

$$\begin{aligned} U(1\cdots s;z)a(1)&=\frac{1}{z}U(1\cdots s;0)a(1)+\frac{1}{z}\sum_{N\geqslant s}N!/(N-s)!\,\mathrm{Tr}_{s+1\cdots N}\,\rho_N(z-L)^{-1}\left(\sum_{i=1}^{N}L(i)a(i)+\sum_{i<j}L_I(ij)[a(i)+a(j)]\right)\\ &=\frac{1}{z}[U(1\cdots s;0)+\tilde{U}(1\cdots s;z)L(1)+\tilde{W}(1\cdots s;z)L_I(12)(1+\mathcal{P}_{1,2})]a(1)\,. \end{aligned} \tag{A2}$$

The new operator $\tilde{W}(1\cdots s;z)$ is similar to $\tilde{U}(1\cdots s;z)$ except that it maps two-particle operators into s-particle operators. It is defined by

$$\begin{aligned} \tilde{W}(1\cdots s;z)\equiv{}&\sum_{N\geqslant s}N!/(N-s)!\,\mathrm{Tr}_{s+1\cdots N}\,\rho_N[z-L(1\cdots N)]^{-1}\sum_{i<j}\mathcal{P}_{1,i}\mathcal{P}_{2,j}\\ &-\frac{1}{2z}f^{(s)}_{(1\cdots s)}\,\mathrm{Tr}_{s+1,\,s+2}\,f^{(2)}_{(s+1,\,s+2)}\mathcal{P}_{1,\,s+1}\mathcal{P}_{2,\,s+2}\,, \end{aligned} \tag{A3}$$

where the subtracted term arises through use of the equilibrium BBGKY hierarchy for $f^{(s)}$. Equation (A2) may be inverted for $s=1$ to give

$$\tilde{U}^{-1}(1;z)=\{z-\tilde{U}^{-1}(1;z)[\tilde{U}(1;z)L(1)+\tilde{W}(1;z)L_I(12)](1+\mathcal{P}_{1,2})\}U^{-1}(1;0)\,. \tag{A4}$$

Combining Eqs. (A2) and (A4) gives the desired result

$$\begin{aligned} \tilde{K}(1\cdots s;z)&\equiv\tilde{U}(1\cdots s;z)\tilde{U}^{-1}(1;z)-U(1\cdots s;0)U^{-1}(1;0)\\ &=\tilde{G}(1\cdots s;z)L_I(12)(1+\mathcal{P}_{1,2})U^{-1}(1;0)\,, \end{aligned} \tag{A5}$$

where

$$\tilde{G}(1\cdots s;z)\equiv\tilde{W}(1\cdots s;z)-\tilde{U}(1\cdots s;z)\tilde{U}^{-1}(1;z)\tilde{W}(1;z)\,. \tag{A6}$$

APPENDIX B: REDUCED DISTRIBUTION OPERATORS AND INITIAL-VALUE OPERATORS, $U(1\cdots s;0)$ AND $G(1\cdots s)$

The initial-value operators $U(1\cdots s;0)$ and $G(1\cdots s)$ are completely determined by the equilibrium reduced distribution operators defined in Eq. (2.2). Generally, the operators $U(1\cdots s;0)$ and $G(1\cdots s)$ are of the form of a multiplicative operator plus another operator involving a trace over certain reduced distribution operators times the operand. Actually, some of the terms in the trace are also simply multiplicative operators, due to the presence of exchange. It is convenient for estimating density dependence and other approximations to separate off these latter terms so that $U(1\cdots s;0)$ and $G(1\cdots s)$ may be written as the sum of a multiplicative and an integral operator. To do so, it is convenient to make an Ursell-type expansion of the distribution operators. The functions, $g^{(s)}(1\cdots s)$, are defined by the set of equations

$$\frac{1}{2!}f^{(2)}(12)\equiv[f(1)f(2)+g^{(2)}(12)]P_2\,, \tag{B1}$$

$$\frac{1}{3!}f^{(2)}(123)\equiv[f(1)f(2)f(3)+(1+\mathcal{P}_{12}+\mathcal{P}_{13})f(1)g^{(2)}(23)+g^{(3)}(123)]P_3\,, \tag{B2}$$

$$\begin{aligned}\frac{1}{4!}f^{(4)}(1234)\equiv{}&[f(1)f(2)f(3)f(4)+(1+\mathcal{P}_{23}+\mathcal{P}_{24}+\mathcal{P}_{12}\mathcal{P}_{23}+\mathcal{P}_{12}\mathcal{P}_{24}+\mathcal{P}_{13}\mathcal{P}_{24})f(1)f(2)g^{(2)}(34)\\&+(1+\mathcal{P}_{23}+\mathcal{P}_{24})g^{(2)}(12)g^{(2)}(34)+(1+\mathcal{P}_{12}+\mathcal{P}_{13}+\mathcal{P}_{14})f(1)g^{(3)}(234)+g^{(4)}(1234)]P_4\,.\end{aligned} \tag{B3}$$

Since for $s\leqslant s'$, $P_{s'}P_s=P_{s'}$, each $g^{(s)}$ defined above is assumed to contain an implicit factor of P_s.

Exact expressions for the initial-value operators $U(1\cdots s;0)$ and $G(1\cdots s)$ in terms of the equilibrium distribution operators may now be developed. In particular, the multiplicative part of the operators, which include exchange contributions, will be explicitly separated from their integral parts. To see how such a separation occurs, consider first the simplest operator $U(1;0)$. By definition

$$U(1;0)y(1)\equiv f(1)y(1)+\mathrm{Tr}_2[f^{(2)}(12)-f(1)f(2)]y(2)\,. \tag{B4}$$

Using Eq. (B1) for $f^{(2)}$, Eq. (B4) becomes

$$\begin{aligned}U(1;0)y(1)&=f(1)y(1)+\mathrm{Tr}_2[f(1)f(2)\epsilon\hat{P}_{12}+g_\epsilon^{(2)}(12)]y(2)\\&=\phi(1)y(1)+\mathrm{Tr}_2\,g_\epsilon^{(2)}(12)y(2)\,,\end{aligned} \tag{B5}$$

where $\hat{P}_{ij}$ permutes the labels i and j in the direct product state, Eq. (1.6). Also, the multiplicative operator $\phi(1)$ is

$$\phi(1)\equiv f(1)y(1)\tilde{f}(1)\,, \tag{B6}$$

and $g_\epsilon^{(s)}(1\cdots s)\equiv s!g^{(s)}(1\cdots s)$. The operator $U^{-1}(1;0)$ is also required. This inverse may be defined in terms of the "direct correlation operator," $C(12)$, as

$$U^{-1}(1;0)\equiv\phi^{-1}(1)[1-\mathrm{Tr}_2C(12)\phi^{-1}(2)\mathcal{P}_{1,2}]\,. \tag{B7}$$

Then by requiring $U(1;0)U^{-1}(1;0)=U^{-1}(1;0)U(1;0)=1$, one finds

$$\begin{aligned}C(12)&=g_\epsilon^{(2)}(12)-\mathrm{Tr}_3\,g_\epsilon^{(2)}(13)\phi^{-1}(3)C(32)\\&=g_\epsilon^{(2)}(12)-\mathrm{Tr}_3C(13)\phi^{-1}(3)g_\epsilon^{(2)}(32)\,.\end{aligned} \tag{B8}$$

This equation is a quantum generalization of the Ornstein-Zernike relation for the classical direct correlation function.

Similarly, one discovers that

$$U(12;0)y(1)=\phi^{(2)}(12)y(1)+\mathrm{Tr}_3[g_\epsilon^{(3)}(123)+(1+\mathcal{P}_{1,2})f(2)g_\epsilon^{(2)}(13)(1+\epsilon\hat{P}_{1,2})]y(3)\,, \tag{B9}$$

where

$$\phi^{(2)}(12)y(1)\equiv f^{(2)}(12)(1+\mathcal{P}_{1,2})y(1)\tilde{f}(1)+\epsilon(1+\mathcal{P}_{1,2})f(1)y(1)g_\epsilon^{(2)}(12)\,. \tag{B10}$$

A slightly more involved calculation yields

$$\begin{aligned}U(123;0)y(1)=\phi^{(3)}(123)y(1)+\mathrm{Tr}_4[&g^{(4)}(1234)+(1+\mathcal{P}_{1,2}+\mathcal{P}_{1,3})f(1)g_\epsilon^{(3)}(234)(1+\epsilon\hat{P}_{1,2}+\epsilon\hat{P}_{1,3})\\&+(1+\mathcal{P}_{2,3}+\mathcal{P}_{1,2}\mathcal{P}_{2,3})f^{(2)}(12)g_\epsilon^{(2)}(34)(1+\epsilon\hat{P}_{1,3}+\epsilon\hat{P}_{2,3})]y(4)\,,\end{aligned} \tag{B11}$$

with

$$\begin{aligned}\phi^{(3)}(123)y(1)\equiv{}&f^{(3)}(123)(1+\mathcal{P}_{1,2}+\mathcal{P}_{1,3})y(1)\tilde{f}(1)+\epsilon(1+\mathcal{P}_{1,2}+\mathcal{P}_{1,3})f(1)y(1)g_\epsilon^{(3)}(123)\\&+\epsilon(1+\mathcal{P}_{2,3}+\mathcal{P}_{1,2}\mathcal{P}_{2,3})f^{(2)}(12)[y(1)g_\epsilon^{(2)}(13)+y(2)g_\epsilon^{(2)}(23)(1+\epsilon\hat{P}_{1,3})]\,.\end{aligned} \tag{B12}$$

Clearly, the expressions are becoming involved. Therefore, since knowledge of $U(1\cdots s;0)$ for values of $s>3$ is not required they will not be considered here. It is important, however, to evaluate the combination $U(12;0)U^{-1}(1;0)$ since it appears in the mean-field operator $B(1)$. Combining (B9) and (B7) and using the explicit expressions for $\phi(1)$, $\phi(12)$, and $C(12)$, one obtains

$$\begin{aligned}U(12;0)U^{-1}(1;0)y(1)={}&f^{(2)}(12)(1+\mathcal{P}_{1,2})f^{-1}(1)y(1)+\epsilon(1+\mathcal{P}_{1,2})y(1)\tilde{f}^{-1}(1)g_\epsilon^{(2)}(12)\\&+\mathrm{Tr}_3\lambda^{(3)}(12;3)\phi^{-1}(3)y(3)\,,\end{aligned} \tag{B13}$$

with

$$\lambda^{(3)}(12;3) \equiv g_c^{(3)}(12;3) - \mathrm{Tr}_4 g_c^{(3)}(12;4)\phi^{-1}(4)C(43) - \epsilon(1+\mathcal{P}_{12})C(13)\tilde{f}^{-1}(1)g_\epsilon^{(2)}(12) \tag{B14}$$

and

$$g_c^{(3)}(12;3) \equiv g_\epsilon^{(3)}(123) - g_\epsilon^{(2)}(12)(1+\mathcal{P}_{1,2})f^{-1}(1)g_\epsilon^{(2)}(13)\,. \tag{B15}$$

In an exactly analogous manner one finds

$$U(123;0)U^{-1}(1;0)y(1) = \phi(123)\phi^{-1}(1)y(1) + \mathrm{Tr}_4\lambda^{(4)}(123;4)\phi^{-1}(4)y(4)\,, \tag{B16}$$

with

$$\begin{aligned}\lambda^{(4)}(123;4) \equiv{}& g_c^{(4)}(123;4) - \mathrm{Tr}_5 g_c^{(4)}(123;5)\phi^{-1}(5)C(54) - \epsilon(1+\mathcal{P}_{1,2}+\mathcal{P}_{1,3})C(14)\tilde{f}^{-1}(1)g_\epsilon^{(3)}(123)\\ &- \epsilon(1+\mathcal{P}_{2,3}+\mathcal{P}_{1,2}\mathcal{P}_{2,3})f^{(2)}(12)[\phi^{-1}(1)C(14)g_\epsilon^{(2)}(13) + \phi^{-1}(2)C(24)g_\epsilon^{(2)}(23)(1+\epsilon\hat{P}_{1,3})]\end{aligned} \tag{B17}$$

and

$$\begin{aligned}g_c^{(4)}(123;4) \equiv{}& g_\epsilon^{(4)}(1234) - (1+\mathcal{P}_{1,2}+\mathcal{P}_{1,3})\{g_\epsilon^{(3)}(123)f^{-1}(1)g_\epsilon^{(2)}(14)\\ &- f(1)[g^{(3)}(234) - (1+\mathcal{P}_{2,3})g^{(2)}(23)f^{-1}(2)g^{(2)}(24)]\,(1+\epsilon\hat{P}_{12}+\epsilon\hat{P}_{13})\}\,.\end{aligned} \tag{B18}$$

The calculation of $G(12)$ and $G(123)$ is similar to the above, but much more involved. Therefore, only the results will be presented here. The result for $G(12)$ operating upon an arbitrary two-particle operator $y(12)$, is

$$G(12) = \gamma(12)y(12) + \Lambda(12)y(12)\,, \tag{B19}$$

where the multiplicative operator, $\gamma(12)$, is

$$\gamma(12)y(12) = f^{(2)}(12)y(12)[1+\epsilon f(1)+\epsilon f(2)+\tfrac{1}{2}f^{(2)}(12)]\,, \tag{B20}$$

and the integral operator is

$$\begin{aligned}\Lambda(12)y(12) \equiv{}& (1+\mathcal{P}_{1,2})\,\mathrm{Tr}_3[g_\epsilon^{(3)}(123) - g_\epsilon^{(2)}(12)f^{-1}(1)g_\epsilon^{(2)}(13) + \epsilon f(2)g_\epsilon^{(2)}(13)(\hat{P}_{12}+\hat{P}_{23})\\ &\quad + f(1)g_\epsilon^{(2)}(23)(1+\epsilon\hat{P}_{13})]y(13)\\ &+\tfrac{1}{2}\,\mathrm{Tr}_{3,4}[g_\epsilon^{(4)}(1234) - (1+\mathcal{P}_{1,2})g_\epsilon^{(2)}(12)f^{-1}(1)g_\epsilon^{(3)}(134) + \epsilon(1+\mathcal{P}_{12})f(1)g_\epsilon^{(3)}(234)(\hat{P}_{1,3}+\hat{P}_{1,4})\\ &\quad + (1+\mathcal{P}_{3,4})f(3)g_\epsilon^{(3)}(124)(1+\epsilon\hat{P}_{23}+\epsilon\hat{P}_{13}+\epsilon\hat{P}_{34})\\ &\quad - (1+\mathcal{P}_{34})(1+\mathcal{P}_{12})f(3)g_\epsilon^{(2)}(12)f^{-1}(2)g_\epsilon^{(2)}(24)(1+\epsilon\hat{P}_{23}+\epsilon\hat{P}_{34}) + (1+\mathcal{P}_{34})(1+\mathcal{P}_{12})f(1)f(3)g_\epsilon^{(2)}(24)\\ &\quad \times(\epsilon\hat{P}_{14}+\epsilon\hat{P}_{13}+\epsilon\hat{P}_{13}\hat{P}_{12}+\hat{P}_{13}\hat{P}_{34}+\hat{P}_{13}\hat{P}_{14}+\epsilon\hat{P}_{13}\hat{P}_{12}\hat{P}_{34})]y(34)\\ &+\tfrac{1}{2}\,\mathrm{Tr}_{3,4}(1+\mathcal{P}_{1,2})g_\epsilon^{(2)}(13)g_\epsilon^{(2)}(24)(1+\hat{P}_{1,2}+\hat{P}_{1,4}+\hat{P}_{2,3}+\hat{P}_{3,4}+\hat{P}_{1,2}\hat{P}_{3,4})y(34)\\ &- \mathrm{Tr}_{3,4}\lambda^{(3)}(123)f^{(2)}(34)y(34) - \tfrac{1}{3}\,\mathrm{Tr}_{4,5}\lambda^{(3)}(123)[f^{(3)}(345) - f(3)f^{(2)}(45)]y(45)\\ &- (1+\mathcal{P}_{12})\tfrac{1}{2}\,\mathrm{Tr}_{3,4}[f(1)g_\epsilon^{(2)}(34)(\epsilon\hat{P}_{1,3}+\epsilon\hat{P}_{1,4}) + (1+\mathcal{P}_{3,4})f(3)g_\epsilon^{(2)}(14)(1+\epsilon\hat{P}_{1,3}+\epsilon\hat{P}_{3,4}) + g^{(3)}(134)]y(34)\,.\end{aligned} \tag{B21}$$

Similarly, one finds

$$G(123)y(12) = \gamma(123)y(12) + \Lambda(123)y(12)\,, \tag{B22}$$

$$\begin{aligned}\gamma(123)y(12) \equiv{}& f^{(3)}(123)(1+\mathcal{P}_{2,3}+\mathcal{P}_{1,2}\mathcal{P}_{2,3})y(12)\{1+\epsilon[f(1)+f(2)]+\tfrac{1}{2}f^{(2)}(12)\}\\ &+ (1+\mathcal{P}_{2,3}+\mathcal{P}_{1,2}\mathcal{P}_{2,3})f^{(2)}(12)y(12)g_\epsilon^{(3)}(123)\\ &+ \epsilon(1+\mathcal{P}_{12}+\mathcal{P}_{13})[(1+\mathcal{P}_{23})f^{(2)}(12)y(12)\tilde{f}(1)g^{(2)}(23) + \epsilon f^{(2)}(23)y(23)\tilde{f}(2)g^{(2)}(13)\hat{P}_{12}]\,.\end{aligned} \tag{B23}$$

The integral operator $\Lambda(123)$ is determined from an equation like Eq. (B21), but since knowledge of its exact form is not required, it will not be written down explicitly here. Its basic structure is the same as $\Lambda(12)$; i.e., it involves traces of y and $g^{(s)}$ operators.

By writing

$$G^{-1}(12)=\gamma^{-1}(12)-G(12)\Lambda(12)\gamma^{-1}(12)\,, \tag{B24}$$

one obtains an expression for the important combination $G(123)G^{-1}(12)$,

$$G(123)G^{-1}(12)=\gamma(123)\gamma^{-1}(12)+[\Lambda(123)-G(123)G^{-1}(12)\Lambda(12)]\gamma^{-1}(12)\,. \tag{B25}$$

The multiplicative part of (B25) is explicitly given by

$$\gamma(123)\gamma^{-1}(12)y(12)=f^{(3)}(123)(1+\mathcal{P}_{23}+\mathcal{P}_{12}\mathcal{P}_{23})[f^{(2)}(12)]^{-1}y(12)+(1+\mathcal{P}_{23}+\mathcal{P}_{12}\mathcal{P}_{23})f^{(2)}(12)[\gamma^{-1}(12)y(12)]g_\epsilon^{(3)}(123)$$

$$+(1+\mathcal{P}_{12}+\mathcal{P}_{13})\{(1+\mathcal{P}_{23})f^{(2)}(12)[\gamma^{-1}(12)y(12)]\tilde{f}(1)g_\epsilon^{(2)}(23)$$

$$+\epsilon f^{(2)}(23)[\gamma^{-1}(23)y(23)]\tilde{f}(2)g_\epsilon^{(2)}(13)\hat{P}_{12}\}\,. \tag{B26}$$

Since $f^{(2)}(12)$ contains a factor of P_2, its inverse is defined according to

$$[f^{(2)}(12)]^{-1}f^{(2)}(12)=f^{(2)}(12)[f^{(2)}(12)]^{-1}=P_2\,,$$

which is the identity on the two-particle (anti) symmetrized subspace. This explicit dependence upon P_2 may be ignored if every s-particle operator is assumed to contain a factor of P_s, implicitly.

Upon substituting Eq. (B26) into Eq. (2.25) for $B^{(2)}(12)$, one discovers that $B^{(2)}$ may be expressed in the form

$$B^{(2)}(12)y(12)=\epsilon\{[f(1)+f(2)]V(12)y(12)-y(12)V(12)[f(1)+f(2)]\}$$

$$+(1+\mathcal{P}_{1,2})\{\mathrm{Tr}_3[V(13)(1+\epsilon\hat{P}_{1,3}),f(3)y(12)]+\mathrm{Tr}_3V(13)(1+\mathcal{P}_{1,2})\hat{P}_{13}f(1)y(23)$$

$$-(1+\mathcal{P}_{1,2})f(1)y(23)\hat{P}_{13}V(13)+\mathrm{Tr}_3L_I(13)(1+\hat{P}_{12})[f(2)y(13)+f(1)y(23)]$$

$$+\mathrm{Tr}_3L_I(13)(3!P_3)[f(2)g^{(2)}(13)+f(1)g^{(2)}(23)+g^{(3)}(123)][f^{(2)}(12)]^{-1}y(12)$$

$$+\mathrm{Tr}_3L_I(13)[\Lambda(123)-G(123)G^{-1}(12)\Lambda(12)]\gamma^{-1}(12)y(12)\}$$

$$-U(12;0)U^{-1}(1;0)\,\mathrm{Tr}_3L_I(13)y(13)\,. \tag{B27}$$

The only term in Eq. (B27) that survives in the limit $a/l_0\to 0$, $n\lambda^3=\text{const}$ is the first one, so the low-density limit to $B^{(2)}$ is

$$B^{(2)}(12)y(12)\to\epsilon\{[f_0(1)+f_0(2)]V(12)y(12)-y(12)V(12)[f_0^{(1)}(1)+f_0^{(1)}(2)]\}\,. \tag{B28}$$

Similarly, one finds the following low-density limits:

$$U(12;0)U^{-1}(1;0)y(1)\to f_0^{(2)}(12)(1+\mathcal{P}_{1,2})f_0^{-1}(1)y(1)+2\epsilon y(1)\tilde{f}_0^{-1}(1)g_0^{(2)}(12)\,, \tag{B29}$$

$$G(12;0)y(12)\to f_0^{(2)}(12)y(12)\{1+\epsilon[f_0^{(1)}(1)+f_0^{(1)}(2)]+\tfrac{1}{2}f_0^{(2)}(12)\}\,. \tag{B30}$$

Equation (B29) can now be used in the definition, Eq. (2.12), to obtain the low-density estimate of the mean-field operator,

$$B(1)y(1)\to\mathrm{Tr}_2L_I(12)[f_0^{(2)}(12)(1+\mathcal{P}_{1,2})f_0^{-1}(1)y(1)+2\epsilon(1+\mathcal{P}_{1,2})y(1)f_0^{-1}(1)g_0^{(2)}(12)]\,. \tag{B31}$$

Combining $L_I(12)$ with $B^{(2)}(12)$ in the low-density limit yields,

$$[L_I(12)+B^{(2)}(12)]\to\hat{L}_I(12)\,, \tag{B32}$$

where $\hat{L}_I(12)$ is defined in Eq. (4.5). Using this and the low-density expression, Eq. (B30), in Eq. (2.13) gives the low-density approximation to the collision operator,

$$\tilde{M}(1;z)y(1)\to\mathrm{Tr}_2L_I(12)[z-\hat{L}(12)]^{-1}f_0^{(2)}(12)[L_I(12)(1+\mathcal{P}_{1,2})f_0^{-1}(1)y(1)\tilde{f}_0(1)][S_0(12)+\tfrac{1}{2}f_0^{(2)}(12)]\,. \tag{B33}$$

APPENDIX C: THE $\hat{\Omega}$ AND $\hat{T}$ OPERATORS

The effective Möller operator introduced in Sec. IV may be expressed as

$$\hat{\Omega}\equiv\lim_{t\to\infty}e^{-it\hat{H}}e^{itH_0}$$

$$=1-i\int_0^\infty dt\,e^{-it\hat{H}}\hat{V}e^{itH_0}e^{-\eta t}\,, \tag{C1}$$

where η is a positive infinitesimal introduced to ensure convergence. When operating upon the *ket* $|p_1p_2)$, Eq. (C1) becomes

$$\hat{\Omega}\,|p_1p_2)=[1+(E+i\eta-\hat{V})^{-1}\hat{V}]\,|p_1p_2)\,, \tag{C2}$$

with $E=(p_1^2+p_2^2)/2m$, the energy of the state $|p_1p_2)$. The $\hat{T}$ operator is now defined for arbitrary complex z by the relation

$$(z-\hat{H})^{-1}\hat{V} \equiv (z-H_0)^{-1}S_0(12)\hat{T}(z)\,, \tag{C3}$$

which is equivalent to the Lippmann-Schwinger equation

$$\begin{aligned}\hat{T}(z) &= V+\hat{T}(z)S_0(z-H_0)^{-1}V \\ &= V+V(z-H_0)^{-1}S_0\hat{T}(z)\,.\end{aligned} \tag{C4}$$

From Eqs. (C2)–(C4) one obtains the useful relations

$$\hat{\Omega}\,|\,p_1p_2) = [1+(E+i\eta-H_0)^{-1}S_0(12)\hat{T}(E+i\eta)]\,|\,p_1p_2) \tag{C5}$$

and

$$V\hat{\Omega}\,|\,p_1p_2) = \hat{T}(E+i\eta)\,|\,p_1p_2)\,. \tag{C6}$$

By eliminating the potential from Eq. (C4) and its adjoint equation, one obtains the optical theorem

$$\hat{T}^\dagger(z)-\hat{T}(z) = \hat{T}^\dagger(z)S_0(12)[(z^*-H_0)^{-1}-(z-H_0)^{-1}]\hat{T}(z)\,. \tag{C7}$$

On the energy shell, in the momentum representation, Eq. (C7) becomes

$$\begin{aligned}&(p_1p_2\,|[T^\dagger(E+i\eta)-T(E+i\eta)]|\,p_1p_2) \\ &\quad = 2\pi i\sum_{\bar{p}_1\bar{p}_2}\left|(p_1p_2\,|\,\hat{T}\,|\,\bar{p}_1\bar{p}_2)\right|^2 S_0(\bar{p}_1\bar{p}_2)\delta(E-\bar{E})\,.\end{aligned} \tag{C8}$$

Finally, it is useful to take note of the so-called intertwining relation between $\hat{\Omega}$ and the Hamiltonian

$$\hat{H}\hat{\Omega} = \hat{\Omega}H_0\,. \tag{C9}$$

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