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Homomorphism between SO(4,2) and SU(2,2)

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By stereographically projecting the four coordinates which transform the hydrogen atom into a four-dimensional harmonic oscillator into a six-dimensional space, the homomorphism between SO(4,2) and SU(2,2) is explicitly demonstrated.

Recently, it has been shown that the hydrogen atom can be transformed into a harmonic oscillator in four-dimensional space.[1-3] This was done in Ref. 3 by first multiplying the Schrödinger equation $[-(\hbar^2/2m)\nabla^2 - Ze^2/r - E]\Psi = 0$ from the left by $4r/a_0$,[4] where a_0 has the dimension of length, and then using the coordinates[5] $s_1 = s\cos\alpha\cos\beta$, $s_2 = s\cos\alpha\sin\beta$, $s_3 = s\sin\alpha\cos\gamma$, $s_4 = s\sin\alpha\sin\gamma$ in the resulting equation. In so doing, we obtain[6]

$$[-(\hbar^2/2ma_0)\nabla_4^2 + \tfrac{1}{2}m\omega^2 a_0 s^2 - 2n\hbar\omega]\Psi = 0\,, \qquad (1)$$

where we have set $-4E/a_0^2 = \frac{1}{2}m\omega^2$ and $4Ze^2/a_0 = 2n\hbar\omega$, with ∇_4^2 as the four-dimensional Laplacian and $s^2 = s_1^2 + s_2^2 + s_3^2 + s_4^2$. The solutions of Eq. (1) are the products of Hermite polynomials and are the basis functions in a four-dimensional Hilbert space for a realization of the ladder representation of the Lie algebra $u(2,2)$ of the group U(2,2).[7] This realization can be obtained by defining the boson annihilation and creation operators as follows:

$$a_i = \left(\frac{\partial}{\partial y_i} + y_i\right)\frac{1}{\sqrt{2}}\,,\quad a_i^+ = \left(-\frac{\partial}{\partial y_i} + y_i\right)\frac{1}{\sqrt{2}}\,,\quad i = 1,2$$
$$b_j = \left(\frac{\partial}{\partial y_j} + y_j\right)\frac{1}{\sqrt{2}}\,,\quad b_j^+ = \left(-\frac{\partial}{\partial y_j} + y_j\right)\frac{1}{\sqrt{2}}\,,\quad j = 3,4 \qquad (2)$$

where $y_i = (m\omega a_0/\hbar)^{1/2} S_i$, $i = 1,2,3,4$. The boson operators satisfy the commutation relations $[a_i^+, a_j^+] = [b_i, b_j^+] = \delta_{ij}$ with all other commutators vanishing. The generators of U(2,2) can be realized in terms of these boson operators accordingly. For the purpose of demonstrating the homomorphism, we set

$$z_1 = (y_1 + iy_2)\frac{1}{\sqrt{4n}}\,,\quad z_2 = (y_3 + iy_4)\frac{1}{\sqrt{4n}}\,,$$
$$z_3 = \left(\frac{\partial}{\partial y_1} + \frac{i\partial}{\partial y_2}\right)\frac{1}{\sqrt{4n}}\,,\quad z_4 = \left(\frac{\partial}{\partial y_3} + \frac{i\partial}{\partial y_4}\right)\frac{1}{\sqrt{4n}}\,. \qquad (3)$$

Equation (1) then takes the form $z_1z_1^* + z_2z_2^* - z_3z_3^* - z_4z_4^* = 1$, which is invariant under SU(2,2) transformations.[8]

We now make a stereographic projection such that the points y_1, y_2 and y_3, y_4 on the equatorial planes correspond, respectively, to the points x_1, x_3, x_4 and x_2, x_5, x_6 on two orthogonal unit hyperboloids in a six-dimensional space. The formulas for making the projection are

$$\begin{aligned}
x_1 &= (1 + y_1^2 + y_2^2)/[1 - (y_1^2 + y_2^2)]\,,\\
x_2 &= (1 + y_3^2 + y_4^2)/[1 - (y_3^2 + y_4^2)]\,,\\
x_3 &= 2y_1/[1 - (y_1^2 + y_2^2)]\,,\\
x_4 &= 2y_2/[1 - (y_1^2 + y_2^2)]\,,\\
x_5 &= 2y_3/[1 - (y_3^2 + y_4^2)]\,,\\
x_6 &= 2y_4/[1 - (y_3^2 + y_4^2)]\,.
\end{aligned} \qquad (4)$$

To prove the homomorphism between SO(4,2), which leaves invariant the quadratic form $x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2$ and SU(2,2), we first construct the antisymmetric matrices of the form

$$A = \begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & u_3^* & -u_2^* \\ -u_2 & -u_3^* & 0 & u_1^* \\ -u_3 & u_2^* & -u_1^* & 0 \end{pmatrix},$$

where $u_1 = x_1 + ix_2$, $u_2 = x_3 + ix_4$, and $u_3 = x_5 + ix_6$. We then form Kronecker products of the Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$ and the unit matrix $\sigma_4 = I_2$ and obtain sixteen basic matrices of the form $U_{ij} = \sigma_i \oplus \sigma_j$.[9] These 4×4 complex matrices are unitary and unimodular and they transform the set of complex matrices A above according to $A' = UA\tilde{U}$ such that A' remains antisymmetrical. It can be shown that $\mathrm{Tr}(gA'^+gA') = \mathrm{Tr}(gA^+gA) = 4(x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2)$, where

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The matrices U also leave invariant the quadratic form $z_1^*z_1+z_2^*z_2-z_3^*z_3-z_4^*z_4=z^+gz$ where z^+ and z are row and column matrices. Since $U^+gU=\pm g$, we have $z'^+gz'=z^+U^+gUz=\pm z^+gz$.[10] Thus the homomorphism between SO(4,2) and SU(2,2) is explicitly demonstrated.[11]

As has been pointed out, the coordinates in Eq. (4) stereographically project a four-dimensional Euclidean space onto unit hyperboloids in six dimensions. They are analogous to the Fock coordinates[12] which project stereographically the momentum space onto a unit sphere, unit paraboloid, or unit hyperboloid in four dimensions for the cases $E<0$, $E=0$, or $E>0$, respectively. If Fock's projection can be viewed[13] as an "exercise in geometrizing the Coulomb field," then our result gives a new way of geometrizing the Coulomb field.

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[1]J. Cizek and J. Paldus, Int. J. Quantum Chem. 12, 875 (1977), showed a transformation relating the radial equations of the hydrogen atom and the oscillator.

[2]A. O. Barut, C.K.E. Schneider, and R. Wilson, J. Math. Phys. 20, 2244 (1979), related the two quantum systems by the Kustaanheimo-Steifel transformation of classical mechanics.

[3]A. C. Chen, Phys. Rev. A 22, 333 (1980).

[4]The left multiplication by r makes the equation linear in the group generators of SO(2,1).

[5]The coordinates in Ref. 2 correspond to ours as follows: $u_1=s_3$, $u_2=-s_2$, $u_3=-s_1$, $u_4=s_4$. Their boson operators can be expressed as a linear combination of ours given in Eq. (2), e.g., $a_1\to(b_3-ib_4)/\sqrt{2}$, $b_1\to-(a_1+ia_2)/\sqrt{2}$.

[6]The equation is also valid for $E=0$, in which case it becomes a differential equation leading to Bessel functions and yields the zero-energy wave function.

[7]R. L. Anderson, J. Fisher, and R. Raczka, Proc. R. Soc. London Ser. A302, 491 (1968).

[8]The mathematical structures of SU(2,2) have been studied by many authors. See, for example, T. Yao, J. Math. Phys. 12, 315 (1971).

[9]B. Kurşunoğlu, *Modern Quantum Theory* (Freeman, San Francisco, 1962), p. 233. The generator $U_{44}=\sigma_4\oplus\sigma_4=I_4$ belongs to U(2,2). The remaining 15 generators satisfy the same commutation relations as L_{ab}, the canonical generators of SO(4,2). The U_{ij} and the L_{ab} correspond to each other as follows: $U_{1i}=2iL_{5i}$, $U_{14}=-2iL_{46}$, $U_{2i}=2iL_{i6}$, $U_{24}=-2iL_{45}$, $U_{3i}=2L_{i4}$, $U_{34}=2L_{56}$, $U_{4i}=2\epsilon_{ijk}L_{jk}$.

[10]The top sign is generated by U_{3j} and U_{4j} and the lower sign by U_{1j} and U_{2j}.

[11]See A. O. Barut, in *De Sitter and Conformal Group and their Applications*, edited by A. O. Barut and W. E. Brittin (Colorado Assoc. University Press, Boulder, Colorado, 1971), p. 12.

[12]V. Fock, Z. Phys. 98, 145 (1935); M. Bander and C. Itzykson, Rev. Mod. Phys. 38, 330 (1966); 38, 346 (1966); A. C. Chen, J. Math. Phys. 19, 1037 (1978).

[13]L. C. Biedenharn, J. Math. Phys. 2, 433 (1961).