Quantization of motion in a velocity-dependent field: The v^2 case

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The case of the quantum unidimensional motion of a point particle subject to a viscous force proportional to the square of the speed of the particle is treated. Once a suitable Lagrangian has been chosen, the canonical quantization procedure is applied leading to a Schrödinger equation for quadratic friction and an arbitrary potential. The cases of an exponentially decreasing potential plus a linearly increasing term is explicitly solved.

I. INTRODUCTION

Friction is the form in which the interaction of an object with a medium, whose basic components are far less than it in size and energy, is macroscopically manifested. We also know that this interaction is vehiculated by a braking force whose intensity, for sufficiently low velocities (Reynolds numbers), is first proportional to the speed of the object, then to its square.

Although the idea of friction is essentially classical and macroscopic, there are some cases in which it can be called into play also for the microscopic world: an example is the radiation damping; other examples can be found in the interaction of a single quantum object with a many-body system whose excitation energies are much lower than that of the incoming particle. The latter case needs to be treated semiclassically anyway, in that the bullet must be considered as distinct from the target during the course of the interaction, or, at least, one must be able to identify something that is being "braked".

These practical reasons and also a general theoretical interest for possible microscopic continuous interactions gives sense to the problem of quantizing the motion in a viscous field. When considering a point massive particle the classical equations of motion in a linear or quadratic friction field are, in order,

$$m\frac{d^2\vec{r}}{dt^2}+m\gamma\frac{d\vec{r}}{dt}=\vec{F},$$
$$m\frac{d^2\vec{r}}{dt^2}+m\gamma\left(\frac{dr}{dt}\right)^2\frac{d\vec{r}}{dt}\Big/\frac{dr}{dt}=\vec{F}, \tag{1}$$

where m and $\vec{r}$ are the particle mass and position, γ is the friction coefficient, and $\vec{F}$ is any external force. Both Eqs. (1) cannot be derived from a Lagrangian[1] in their present form, but it is possible, under suitable conditions,[1] to find a Lagrangian leading to a form equivalent to (1). This has already been done for the linear case[1-4] and will be shown also for the quadratic case in the present paper.

Once a Lagrangian has been found, the way is open for canonical quantization. This procedure has been applied to the linear case,[2,3] leading to some problems with the interpretation of the uncertainty principle. Alternative methods for quantization have been proposed, introducing nonlinearities into the Schrödinger equation.[5-7] An extensive list of references on the subject of quantizing linear friction may be found in Ref. 8.

Anyway, once we agree to consider the quantization of friction as a meaningful problem, there is no reason that we can't further pursue the analogy with the classical picture studying the quadratic regime too. This is precisely the content of the present work. The special case of the motion in the absence of any external force has already been solved[9]; what is treated now is the general problem of the unidimensional motion in any external potential $V(r)$ in the presence of a viscous force proportional to the squared speed of the particle. The classical equation of motion reads

$$m\frac{d^2r}{dt^2}+m\gamma\left(\frac{dr}{dt}\right)^2+\nabla V=0. \tag{2}$$

The quantization procedure we follow is the canonical one.

II. CLASSICAL CANONICAL TREATMENT

Following Ref. 1 we look for a Lagrangian for the equation of motion:

$$f(r,dr/dt,t)\left[m\frac{d^2\vec{r}}{dt^2}+m\gamma\left(\frac{d\vec{r}}{dt}\right)^2+\frac{dV}{dr}\right]=0 \tag{3}$$

in which f is any solution of

$$\left[\gamma\left(\frac{dr}{dt}\right)^2+\frac{V'}{m}\right]\frac{\partial\ln f}{\partial\, dr/dt}+2\gamma\frac{dr}{dt}=\frac{\partial\ln f}{\partial r}\frac{dr}{dt}+\frac{\partial\ln f}{\partial t}, \tag{4}$$

where primes denote differentiation with respect to r.

It is readily verified that

$$f=\exp(2\gamma r) \tag{5}$$

satisfies Eq. (4) and consequently a Lagrangian for (3) is

$$L=\frac{m}{2}\left(\frac{dr}{dt}\right)^2\exp(2\gamma r)-\int dr\, V'\exp(2\gamma r)\,. \qquad (6)$$

With the position (5), Eq. (3) is indeed equivalent to (2), as we see that f has neither poles nor zeros on the real axis.

From (6) we obtain the canonical momentum p and Hamiltonian H of the system:

$$p=\frac{\partial L}{\partial\, dr/dt}=m\frac{dr}{dt}\exp(2\gamma r), \qquad (7)$$

$$H=\frac{p^2}{2m}\exp(-2\gamma r)+\int dr\, V'\exp(2\gamma r)\,. \qquad (8)$$

p does not manifestly coincide with the kinetic momentum $(m\cdot dr/dt)$ of the particle and H, which is conserved, when V is independent of time, does not correspond to the total energy of the particle.

III. CANONICAL QUANTIZATION

The quantization of the system we are studying can be performed via the usual correspondence rule

$$\hat{r}=r,$$
$$\hat{p}=-i\hbar\frac{\partial}{\partial r},$$

that substitutes operators for canonical variables.

In the case of the Hamiltonian the problem arises of the ordering of the operators in the first term of (8), so we are naturally led to introduce a symmetrized Hamiltonian operator. One must, however, be careful in the symmetrization procedure if the desired Hamiltonian is to formally reproduce Hamilton's equations of motion. The correct result is obtained when substituting for the classical expression $p^2\exp(-2\gamma r)$ an operational expression like, for instance $[\hat{p}\exp(-\gamma r)][\hat{p}\exp(-\gamma r)]$. The symmetrized Hamiltonian is then

$$\hat{H}_s=\frac{1}{4m}[\hat{p}\exp(-\gamma r)\hat{p}\exp(-\gamma r)+\exp(-\gamma r)\hat{p}\exp(-\gamma r)\hat{p}]+\int dr\, V'\exp(2\gamma r)$$
$$=-\frac{\hbar^2}{2m}\exp(-2\gamma r)\left(\frac{\partial^2}{\partial r^2}-2\gamma\frac{\partial}{\partial r}+\gamma^2\right)+\int dr\, V'\exp(2\gamma r) \qquad (9)$$

and, using Heisenberg's equation of motion for operators, it correctly reproduces an equation like (2) and satisfies Ehrenfest's theorem.

Now we can write down the Schrödinger equation for the wave function $\psi(r,t)$ of the particle:

$$\hat{H}_s\psi=i\hbar\frac{\partial\psi}{\partial t}\,.$$

Whenever V is independent of time, $\hat{H}_s$ admits eigenvalues which we shall call A. The time dependence of the wave function can then be factorized into a function $\Theta(t)$, whose expression is obviously

$$\Theta(t)=\exp\left(-i\frac{A}{\hbar}t\right)\,.$$

The spatial part $\chi(r)$ of the wave function must satisfy the equation

$$\frac{\partial^2\chi}{\partial r^2}-2\gamma\frac{\partial\chi}{\partial r}+\left(\gamma^2-\frac{2m}{\hbar^2}\exp(2\gamma r)\int dr\, V'\exp(2\gamma r)+\frac{2m}{\hbar^2}A\exp(2\gamma r)\right)\chi=0\,. \qquad (10)$$

Equation (10) can be transformed by the substitution $\chi=\exp(\int u\, dr)$, obtaining

$$u'+u^2-2\gamma u+\gamma^2-\frac{2m}{\hbar^2}\exp(2\gamma r)\int dr\, V'\exp(2\gamma r)+\frac{2m}{\hbar^2}A\exp(2\gamma r)=0\,. \qquad (11)$$

Let us look for solutions of (11) of the type

$$u=g(r)\exp(2\gamma r)+w(r)\,.$$

Substituting into (11) and rearranging we get

$$w'\exp(-2\gamma r)+g'+w^2\exp(-2\gamma r)+g^2\exp(2\gamma r)+2gw-2\gamma w\exp(-2\gamma r)+\gamma^2\exp(-2\gamma r)$$
$$-\frac{2m}{\hbar^2}\int dr\, V'\exp(2\gamma r)+\frac{2m}{\hbar^2}A=0\,, \qquad (12)$$

with the obvious remark that both A and V' are real.

IV. SOME EXAMPLES

The solution of (10) [or (11) or (12)] obviously depends on the explicit form of the potential. Analytical solutions can be found in some cases; we just quote a couple of examples. The constant potential case has been solved in Ref. 9. An exactly soluble class of potentials is

$$V = V_0 + a\exp(-4\gamma r) + br\,, \tag{13}$$

where a and b are nonnegative constants.

The wave function of the system is, in this case, of the type

$$\psi \sim \exp\left(\alpha r + \beta\exp(2\gamma r) - i\frac{A}{\hbar}t\right) \tag{14}$$

with

$$\alpha = \gamma \pm \sqrt{4ma/\hbar^2}\,,$$

$$\beta = \pm\sqrt{mb/4\hbar^2\gamma^3}\,,$$

$$A = \pm\hbar\sqrt{b\gamma/m} \pm 2\sqrt{ab/\gamma}\,. \tag{15}$$

A simple special case of (13) is that for $a = 0$, i.e., the linear potential. "Pseudopolynomial" potentials of the same kind as (13) with odd powers of r can also be exactly solved.

V. CONCLUSIONS

The present treatment of the quadratic friction does not avoid the troubles already encountered when canonically quantizing the linear case. Heisenberg uncertainty is indeed satisfied, as $[\hat{r}, \hat{p}] = i\hbar$, but the commutator between $\hat{r}$ and the particle kinetic momentum is spatially vanishing. In fact, if we choose to represent the kinetic momentum by the symmetrized operator

$$\hat{p}_k = \tfrac{1}{2}[\hat{p}\exp(-2\gamma r) + \exp(-2\gamma r)\hat{p}]$$
$$= -i\hbar\exp(-2\gamma r)\frac{\partial}{\partial r} + i\hbar\gamma\exp(-2\gamma r)\,, \tag{16}$$

then we obtain

$$[\hat{r}, \hat{p}_k] = i\hbar\exp(-2\gamma r)\,. \tag{17}$$

One way to get around this disturbing feature is to assume that the treated particle has a varying mass,[10] namely,

$$m = m_0\exp(2\gamma r)\,.$$

We must however recognize that the general validity of the interaction of a particle with a friction field is questionable: As the particle moves through the field and loses energy, the picture itself fades away.

Another important remark is that in any case the friction field must be considered limited in space in order to avoid divergencies of the wave functions; the typical problem to be dealt with will be that of the piercing of a viscous potential barrier.

The problem to be explored now is that of the limits at the validity of the linear or quadratic quantum friction picture. In this respect, particularly illuminating is the comment[11] by Stevens that treats quantum friction as an apparent effect in a finite nondissipative system studied for a sufficiently short time; this is perfectly consistent with the procedure by which Ford, Kac, and Mazur[12] derived their equation for an infinite harmonic oscillators medium. Provided all these remarks and limits are taken into account, the canonical quantization can be assumed to be valid and constitutes a logically closed and reasonably simple tool to investigate the properties of a dissipative system.

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