Ultraviolet dynamic renormalization group: Small-scale properties of a randomly stirred fluid

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(Received 19 June 1980)

A dynamic renormalization-group method is developed to study the ultraviolet properties of velocity correlations generated by the Navier-Stokes equation with a random stirring force with the correlator decreasing as $k^{-y}(y \geq d)$ at $k \to \infty$. It is shown that the elimination of modes from a shell near the infrared cut-off $k_0 \approx 1/L \to 0$ results in two major effects: transition to a frame of reference moving with random velocity $\vec{v}_L$ ($v_L^2 a L^{2/3}$ if $y = d$) and the appearance of a *negative dissipation in the Navier-Stokes equation proportional to* $k^{2/3}$. All the divergent terms are summed up into a kinematic effect of a transfer of small eddies by large ones and the dynamics is determined by convergent series. Long-time, large-scale behavior of a fluid is identical with the one obtained by Martin and de Dominicis, Lucke, and Fournier and Frisch. It is shown that the theory is asymptotically free in the ultraviolet region for any $\epsilon = d - y \leq 0$. The energy spectrum of a stirred fluid is $E(k)\; a[k^{-5/3}/\ln(kL)]$ for $(d = y)$ and the Kolmogorov spectrum without corrections never exists in the ultraviolet limit.

I. INTRODUCTION

The role of renormalization-group methods in hydrodynamics is becoming more and more important for it has been shown during the last years that the Navier-Stokes equation for an incompressible fluid under a Gaussian random forcing $\vec{f}$, which is defined by its correlator as

$$\langle \vec{f}(\vec{k},\omega)\vec{f}(\vec{k}',\omega')\rangle \alpha \frac{1}{k^y}\,\delta(\vec{k}+\vec{k}')\delta(\omega+\omega')\ , \quad (y=d) \tag{1.1}$$

is capable of modeling long-time large-scale properties of a turbulent fluid.[1-4] Dynamic renormalization-group methods, originally developed by Ma and Mazenko[5] for the theory of critical phenomena, have been successfully applied by Forster, Nelson, and Stephen for investigation of Navier-Stokes equation for a randomly stirred fluid yielding many new and interesting results.[6] Their ideas were used by authors[1-4] who wanted to study different problems of hydrodynamic turbulence.

It is natural that many attempts have been made to investigate ultraviolet properties of a randomly stirred fluid, using the renormalization-group (RG) methods, since one can expect scale-invariant solutions in the limit $k\to\infty$. These attempts failed because the standard renormalization-group procedure led to the growing of infinity dimensionless coupling parameters, which made it impossible to use schemes based on perturbation expansion. This situation is disturbing, for it has been shown that perturbation expansion in powers of y [see (1.1)] does not have an infinite radius of convergence; it is divergent for any $y \geq d$.[1,2] Thus $y=d$ corresponds to the boundary of convergence. As a result of these problems in the ir region, one would expect stable fixed point or asymptotic freedom in the uv limit $k\to\infty$.

This paper is devoted to the investigation of the ultraviolet behavior of a randomly stirred fluid. It is organized in the following manner. In Sec. II, we discuss the Navier-Stokes equation with the random forcing on the right-hand side of it. We added to the left-hand side of this equation, the enrgy source

$$-\Gamma_0 k^{2/3}\vec{v}(\vec{k},\omega)\ , \tag{1.2}$$

which appears (see Sec. III) after the first iteration. It is our opinion that this term *must* be written in the Navier-Stokes equation when a random force, having nonzero component in the limit $k\to 0$, is added into the rhs of it.

In Sec. III, we apply the RG method, developed by Forster, Nelson, and Stephen,[6] to investigate long-time large-scale properties of a fluid. It is shown that the results of Refs. 1–4 hold, and the negative dissipation (1.2) does not affect them whatsoever.

In Sec. IV, we develop ultraviolet renormalization-group procedure. It is shown that the perturbation expansion contains infinite number of divergent terms which, however, can be summed up as resulting in a pure kinematic effect of transfer of small eddies by large ones. This result was anticipated by Kadomtzev[7] and V. S. L'vov,[8] who was even able to indicate the type of divergent terms which should yield this transfer. It is also shown that elimination of modes from a shell near the infrared cutoff $k_0\to 0$ results, in addition to this kinematic effect, in the energy source (1.2). The $\frac{2}{3}$ power is universal external-forcing independent, while the information about the force $\vec{f}(\vec{k},\omega)$ in the limit $k\to 0$ is hidden in the proportionality coefficient Γ_0.

After the diverging terms are removed by a corresponding coordinate transformation the RG method leads to a dimensionless coupling constant decreasing to zero with each step of renormalization or, in other words, we show that the theory is asymptotically free. *The $\frac{5}{3}$ Kolmogorov spectrum does not exist in the limit $k\to\infty$.* It is interesting that the force (1.1), which yields the $\frac{5}{3}$ result in the long-time large-scale regime, leads to the energy spectrum

$$E(k)\propto k^{-5/3}[\ln(kL)]^{-1}$$

in the limit $k\to\infty$. In Sec. VI, we discuss some problems concerning the nature of intermittencies in the inertial range.

II. EQUATIONS OF MOTION

We are interested in an incompressible fluid under action of a Gaussian random force $\vec{f}(\vec{r},t)$. It is tempting to describe this system by the Navier-Stokes equation with the force $\vec{f}(\vec{r},t)$ on the right-hand side of it, although validity of a naive addition of a force into the dynamic equation of motion is not clear. We know, for example, from a Zwanzig-Mori theory,[9] that a random-force appearance in an equation of motion usually results from a loss of information originally contained in a detailed microscopic description. The second important outcome of a general theory[9] is that random force is always connected to corresponding dissipative terms in the equations of motion. This is necessary for the basic conservation laws to survive elimination of part of the degrees of freedom which is the essential approximation of the theory.[9] The simplest illustration of this principle is the appearance of a viscous dissipation simultaneously with the corresponding random noise in a system, originally described by deterministic equations, after all the degrees of freedom related to the microscopic motions are eliminated. It is important to stress that the proportionality of the viscous term to k^2 does not depend on the details of a small-scale property of a fluid which are hidden in the viscosity coefficient ν_0. We shall show in this work that the long-scale $(k\to 0)$ components of an arbitrary random force are responsible for the appearance of *a universal negative dissipation* term in the Fourier-transformed Navier-Stokes equation $\sim\Gamma_0 k^{2/3}\vec{V}(k,\omega)$ and all the details of a random stirring force are contained in the constant factor Γ_0.

Let us consider the Fourier-transformed Navier-Stokes equation for the lth component of velocity field $v_l(\vec{k},\omega)$:

$$v_l(\vec{k},\omega)=G_0(k,\omega)f_l(\vec{k},\omega)-\tfrac{1}{2}i\lambda_0 G_0(k,\omega)P_{lmn}(\vec{k})\times\int_{\vec{q}\Omega} v_m(\vec{q},\Omega)v_n(\vec{k}-\vec{q},\omega-\Omega)\ , \qquad (2.1)$$

where we define the unrenormalized propagator

$$G_0(k,\omega)=\frac{1}{-i\omega-\Gamma_0 k^{2/3}+\nu_0 k^2}\ . \qquad (2.2)$$

The function $P_{lmn}(\vec{k})$ is

$$P_{lmn}=P_{lm}(\vec{k})k_n+P_{ln}(\vec{k})k_m\ , \qquad (2.3)$$

where $P_{ij}(\vec{k})$ is the transverse projection operator

$$P_{ij}(\vec{k})\equiv\delta_{ij}-\frac{k_i k_j}{k^2}\ .$$

In Eq. (2.1) we have adopted a standard convention by defining

$$\int_{\vec{q}\Omega}\equiv\int\frac{d^dq}{(2\pi)^d}\int_{-\infty}^{+\infty}\frac{d\Omega}{(2\pi)}\ . \qquad (2.4)$$

In this work we shall develop a renormalization-group procedure suitable for investigation of both infrared and ultraviolet properties of a system described by Eqs. (2.1) and (2.2). To perform our program, we introduce an ultraviolet cut off Λ and infrared cut off $k_0\approx 1/L\to 0$ when the size of the system L tends to infinity. Thus depending on our aims the wave-vector integration in (2.4) is to be carried out in either of the intervals $q<\Lambda$ or $q>k_0\to 0$. Proportionality constant λ_0 will be set equal to unity after all the calculations are completed. The main difference between equations of motion (2.1) and (2.2) and the Navier-Stokes equation intensively used in literature, is the negative dissipation term $\propto\Gamma_0 k^{2/3}$, which we added on the basis of our results. This term will be justified in what follows.

One can, in principle, using the zeroth-order solution of (2.1)

$$v_l(\vec{k},\omega)=G_0(k,\omega)f_l(\vec{k},\omega)\ , \qquad (2.5)$$

construct perturbation expansions in powers of the nonlinear mode-mode coupling term proportional to λ_0. This problem was formally solved in a classic paper by Wyld,[10] although it is quite hard to sum up the diagrammatic series derived in Ref. 10 to obtain reliable conclusions from (2.1). It is easier, however, to elucidate the effect of the modes in a shell $\Lambda e^{-l}<q<\Lambda$ on the dynamics of the remaining ones. This is what the renormalization group does. One can ask a different question: How do the modes in a shell near the infrared cut off $k_0<q<k_0e^l$ $(k_0\to 0)$ effect the remaining modes for $k\to\infty$. To answer this question we shall develop an approach which

is similar to the usual infrared renormalization-group treatment[6] although it is not identical to it. It was shown by Ma and Mazenko[5] in the theory of critical phenomena that the dynamic renormalization group consists of two main steps. First, we eliminate from (2.1) either the modes from a shell $\Lambda e^{-l} < q < \Lambda$ if we are interested in the infrared properties, or modes from a region $k_0 < q < k_0 e^{+l}$ if we want to study the ultraviolet behavior of the system. This can be done by substituting the zeroth-order solution (2.5) into (2.1) with the subsequent averaging over the part of the random force which acts in the shell $\Lambda e^{-l} < q < \Lambda$ (ir case) or $k_0 < q < k_0 e^{l}$ (uv case). This redefines all the coefficients of the remaining modes which enter the reduced equation of motion. The fluctuating terms which cannot be associated with the coefficients of the Navier-Stokes equation are added to a random noise $f_i(\vec{k},\omega)$. This renormalizes the spectrum of the original random force in Eq. (2.1). The last step of RG procedure consists of rescaling space, time, and the remaining velocities and forces in order to make the new set of equations look as much as possible like the original Navier-Stokes equation which is defined on a somewhat larger space of variables $v_l(k,\omega)$.

III. LONG-TIME LONG-DISTANCE BEHAVIOR OF A RANDOMLY STIRRED FLUID

This problem was formulated and solved in a paper by Forster, Nelson, and Stephen.[6] They have shown that in order to carry out the RG program outlined above, one can use a diagrammatic expansion developed in Ref. 10 with the only difference that the integration over the internal momenta be carried out in a shell $\Lambda e^{-l} < q < \Lambda$. In this manner the intermediate values (before rescaling) of the parameter can be derived in the limit when external momentum k and frequency ω tend to zero. After that, rescaling is performed.

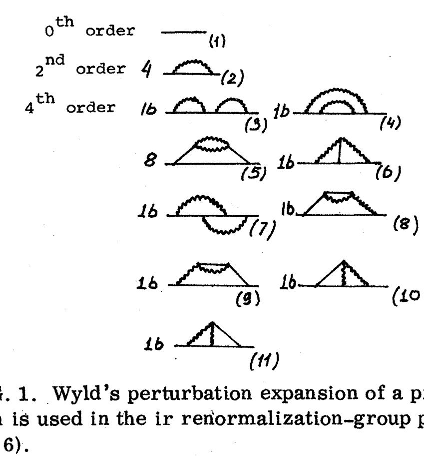

FIG. 1. Wyld's perturbation expansion of a propagator which is used in the ir renormalization-group procedure (Ref. 6).

Wyld's diagrams up to the fourth order for the proporgator and the velocity correlation function $U(\vec{k},\omega)[\delta(\omega+\omega')\delta(\vec{k}+\vec{k}'')U(\vec{k},\vec{\omega})=\langle v(\vec{k}\omega)v(k',\omega')\rangle]$ are shown in Figs. 1 and 2. The diagrams are constructed of three elements: (a) a thin straight line (propagator) $- G_0(k_\perp\omega)$, (b) a point (vertex)$-$ proportional to $i\lambda_0 P_{lmn}(\vec{k})$, and (c) a wavy line $-$, the unrenormalized random-force correlator $\langle f^{\alpha}_{\vec{k}\omega} f^{\beta}_{\vec{k}'\omega'}\rangle = D(k)\delta(\vec{k}+\vec{k}')\delta(\omega+\omega')(\delta_{\alpha\beta} - k_\alpha k_\beta/k^2)$.

Proceeding exactly as was prescribed by Forster, Nelson, and Stephen, we calculate in the second order of perturbation expansion ($k\to 0$ and $\omega \to 0$).

$$D_l = D_0 ,$$
$$\Gamma_l = \Gamma_0 , \tag{3.1}$$
$$\lambda_l = \lambda_0 ,$$
$$\nu_l = \nu_0\left(1 + N_d \frac{\lambda_0^2 D_0}{\nu_0^3} l\right) \quad (l \to 0)$$

where N_d is the constant depending on the spacial dimensionality. The fact that D_0 does not vary upon the small-scale elimination is clear, for the second-order correction to the correlator, proportional to k^2, is irrelevant in the limit $k \to 0$ for the model $D(k) \propto k^{-y}$. The second step of the ir RG treatment consists of time, space, and velocity rescaling:

$$k' = ke^{l}, \quad \omega' = e^{\alpha(l)}\omega, \quad \vec{v}^{<}(\vec{k},\omega) = \vec{v}'(k',\omega')\xi , \tag{3.2}$$

where $\vec{v}^{<}$ is the velocity defined on the smaller k space $0 < k < \Lambda e^{-l}$. The recursion relations cor-

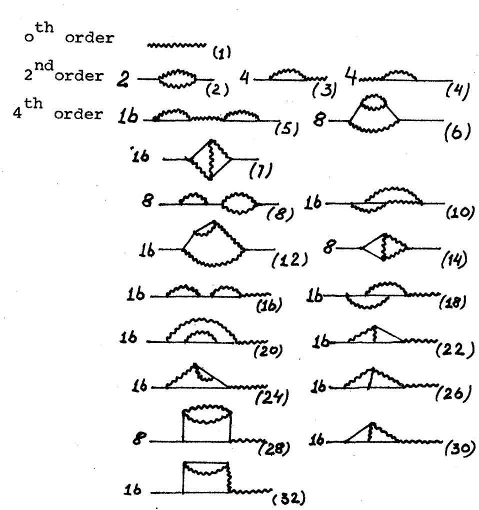

FIG. 2. Diagrams for the correlation function $U(k)$ to fourth order (Ref. 10).

responding to (3.1) and (3.2) are

$$\frac{d\nu(l)}{dl}=\nu(l)(z-2+N_d\bar{\lambda}^2)\ ,$$
$$\frac{d\Gamma(l)}{dl}=\Gamma(l)(z-\tfrac{2}{3})\ ,$$
$$\frac{dD}{dl}=0\ , \qquad (3.3)$$
$$\frac{d\lambda}{dl}=\lambda\left(\tfrac{3}{2}z-1-\frac{d-y}{2}\right)\ .$$

Deriving relations (3.3) we set

$$\xi=\exp\left(\tfrac{3}{2}\alpha(l)+\frac{d+y}{2}l\right) \qquad (3.4)$$

and

$$\frac{d\alpha(l)}{dl}\equiv z\ .$$

Equations (3.3) suggest that the actual expansion parameter is not λ_0 we began with, but the dimensionless parameter $\bar{\lambda}^2=\lambda^2 D/\nu^3$ and

$$\frac{\partial\bar{\lambda}}{\partial l}=\frac{\bar{\lambda}}{2}[4-(d-y)-3N_d\bar{\lambda}^2]\ . \qquad (3.5)$$

Solving (3.5) we find that at the fixed point

$$\bar{\lambda}^*=\left(\frac{4-(d-y)}{3N_d}\right)^{1/2}\ \text{or}\ \left(\frac{4}{3N_d}\right)^{1/2}\quad(d=y) \qquad (3.6)$$

and

$$z=\tfrac{2}{3}\ \text{when}\ d=y\ . \qquad (3.7)$$

It is interesting to notice that both ν and Γ are finite in the limit $l\to\infty$. This is a somewhat unexpected result since we could assume on intuitive grounds that the viscous term would be irrelevant in the limit $k\to 0$ in comparison with the negative dissipation $\propto k^{2/3}$. The energy spectrum corresponding to this fixed point can be calculated readily[6]

$$E(k)=k^{d-1}\int_{-\infty}^{\infty}U(\vec{k},\omega)d\omega\propto k^{-5/3}\quad(y=d). \qquad (3.8)$$

It is clear from our development that this result was obtained for a region where

$$\nu_0 k^2\gg\Gamma_0 k^{2/3} \qquad (3.9)$$

or in the limit $\Gamma_0\to 0$.

Fourier, Nelson, and Stephen performed their calculations for the model in which $\Gamma_0=0$.[6] It can be shown that in this case the results (3.3)–(3.5) stay intact and that the effective viscosity becomes k dependent: $\nu(k)\propto k^{-4/3}$ so that $\nu(k)k^2\propto k^{2/3}$. This in turn justifies the dissipation term proportional to $k^{2/3}$ which we introduced from the begining.

In the opposite case, $\nu_0\to 0$ or $\Gamma_0\to\infty$, which corresponds to infinite Reynolds numbers R we derive instead of (3.1) $(y=d)$:

$$\Gamma_l=\Gamma_0\left(1+M_d\frac{\lambda_0^2 D_0 k^{4/3}}{\Gamma_0^3}l\right)\ ,$$
$$D_l=D_0\ , \qquad (3.10)$$
$$\lambda_l=\lambda_0\ .$$

We notice that in this case there is no k-independent dimensionless coupling parameter. The effective coupling is now k dependent and can be made as small as desired in the limit $k\to 0$. Recursion relations are

$$\frac{d\Gamma}{dl}=\Gamma(l)(z-\tfrac{2}{3})\ ,$$
$$\frac{dD}{dl}=0\ ,$$
$$\frac{d\lambda}{dl}=\lambda(\tfrac{3}{2}z-1)\ , \qquad (3.11)$$
$$\xi=\exp[\tfrac{3}{2}\alpha(l)+dl]\ .$$

The only fixed point corresponding to $z=\frac{2}{3}$ is marginal and the energy spectrum (3.8) holds although the marginality of the fixed point raises doubts about its stability.

IV. ULTRAVIOLET RENORMALIZATION-GROUP PROCEDURE

In this section, we will be interested in the ultraviolet properties of a system described by Eqs. (2.1) and (2.2). Let us assess the influence which modes from a shell near the infrared cut off have on the dynamics of the remaining modes with $k\to\infty$. First of all, we shall show that one cannot adopt Wyld's perturbation expansion without alternations which are already evident in the second order.

The graphic representation of the equation of motion (2.1) is given in Fig. 3, where a heavy

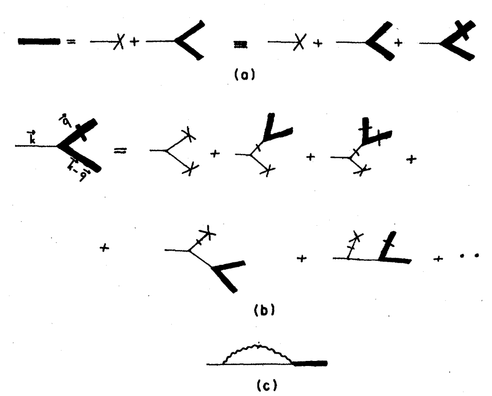

FIG. 3. (a) Equation of motion and (b) and (c) derivation of the correction to propagator.

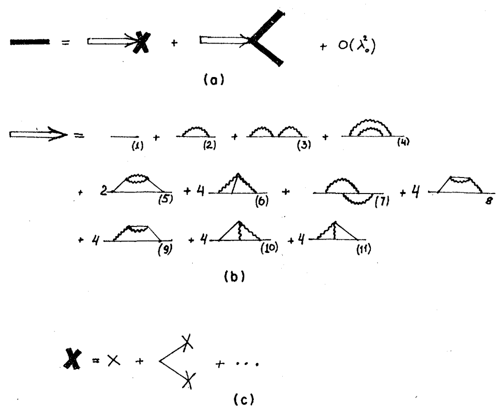

FIG. 4. (a) Modified equation of motion (b), expansion of the propagator in the uv renormalization-group procedure, and (c) series for the renormalized force.

line denotes $\vec{v}_l(\vec{k},\omega)$ and the slashed heavy line corresponds to modes $k_0<k<k_0e^l$, which must be eliminated. The symbol X stands for a bare random force f. We are interested in a case $\nu_0\to 0$, or in other words, $\Gamma_0k^{2/3}\gg\nu_0k^2$ everywhere in k space. Expanding the third term on the rhs of the graphic equation in Fig. 3(a), we obtain the only diagram providing correction to propagator. The procedure is shown on Figs. 3(b) and 3(c). It is worth mentioning that there were four corresponding graphs in the second order of perturbation expansion in the ir renormalization-group treatment outlined above (see also Ref. 10). This holds for higher orders of the series: That is, all geometric elements of Wyld's diagrammatic expansion present in our ultraviolet renormalization-group procedure, but with different proportionality coefficients.

Repeating the arguments described in Wyld's paper,[10] we can express an equation of motion in terms of the renormalized propagator and random force. This equation is represented in Fig. 4 where the open arrow stands for the renormalized Green's function and the thick cross denotes a renormalized random force. Diagrammatic expansion of the propagator and of a random force are given in Figs. 4(b) and 4(c), respectively. To sum up the series for G [Fig. 4(b)], let us notice that each branching of a lower leg of the graphic term of Fig. 3(b) leads to a factor $k\to\infty$ while any branching of *a shashed leg* is proportional to $k_0\to 0$. We shall see that the main contribution to a diagrammatic expansion stems from the expansion of a lower leg ($k\to\infty$) of the nonlinear term represented by Fig. 3(b), although we shall take into account the *other diagrams in an explicit way.*

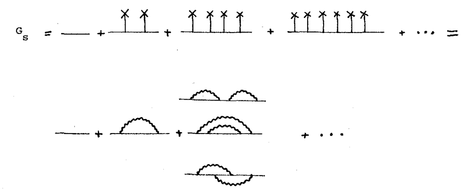

FIG. 5. Principle subset G_s of diagrams for the renormalized propagator G.

Let us introduce function G_s, which is the sum of the subset of the diagrams for the propagator, represented in Fig. 5. In what follows, we shall analyze the function G_s in great detail. We begin investigation of the series for G_s by evaluating the second-order correction to the propagator, keeping in mind that all the internal momenta integrations are to be carried out over a shell $k_0<q<k_0e^l$, while external $k\to\infty$. We write analytic expression I_1 corresponding to diagram

$$I_1\propto k^2\lambda_0^2\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{k_0}^{k_0e^l}\frac{D(q)}{[\Omega^2+(\Gamma_0q^{2/3}-\nu_0q^2)^2][-i(\omega-\Omega)+\nu_0|\vec{k}-\vec{q}|^2-\Gamma_0|\vec{k}-\vec{q}|^{2/3}]}\;. \tag{4.1}$$

Simple integration over Ω yields ($\nu_0\to 0$):

$$I_1\propto\frac{k^2U^2(k_0l)}{-i\omega-\Gamma_0k^{2/3}}+\frac{\lambda_0^2k^2D_0\Gamma^3(k_0,l)}{(-i\omega-\Gamma_0k^{2/3})^2}+O\left(\frac{1}{L^{2/3}}\right)\,, \tag{4.2}$$

where

$$U^2(k_0l)\propto\frac{\lambda_0^2}{\Gamma_0}\int_{k_0}^{k_0e^l}\frac{D(q)}{q^{2/3}}\,\alpha\lambda_0^2L^{2/3} \tag{4.3}$$

and

$$M_2l\equiv\Gamma^3(k_0l)\propto\int_{k_0}^{k_0e^l}D(q)\quad(l\to 0)\;. \tag{4.4}$$

Investigating the ultraviolet properties of a turbulent fluid we need information about the asymptotic behavior of the random force in the limits $k\to\infty$. We postulate that $D(q)$ decreases when $k\to\infty$ as

$$D(q)\propto q^{-y},\quad y\geq d\quad(k\geq k_0;\ k_0\to 0\propto 1/L)\;. \tag{4.5}$$

It is our belief that the asymptotics (4.5) represents the situation somewhat close to reality though we leave discussion of this point to the following sections. Substituting (4.5) into (4.3) and (4.4) we see that $U^2(k_0l)\,\alpha\,L^{2/3}$ is divergent in the limit $L\to\infty$. At the same time result (4.4) for $\Gamma^3(k_0 l)$ does not cause any trouble since, if $d=y$, it is cut-off independent. Another observation following from formula (4.2) is that in the limit $k\to\infty$,

$$I_1 \approx -\frac{U^2(k_0 l)}{\Gamma_0}k^{4/3} + \frac{\lambda_0^2 D_0 \Gamma^3(k_0 l)}{\Gamma_0^2}k^{2/3} \qquad (4.6)$$

and we can anticipate on the basis of general considerations that the actual expansion parameter is k dependent, similar to one derived in the previous section, which grows to infinity when $k\to\infty$. For example, evaluating the contribution of the first from the top diagram of the fourth order represented in Fig. 5, we obtain for $[U(k_0 l)\equiv U]$:

$$I_2 = G_0(k,\omega)I_1^2 \simeq \frac{-U^4k^2}{\Gamma_0^3} + \frac{2\lambda_0^2 D_0\Gamma^3 U^2}{\Gamma_0^4}k^{4/3} - \frac{\lambda_0^4 D_0^2\Gamma^6}{\Gamma_0^5}k^{2/3} \qquad (4.7)$$

in the limit $k\to\infty$.

Combining (4.6) and (4.7), we learn that the expansion of the propagator consists of all powers of both large parameter $U^2 \propto L^{2/3}\to\infty$ and small parameter λ_0 and, in addition, all powers of $k^{2/3}\to\infty$. It is important, however, to mention that as we see from (4.6) and (4.7), contributions not containing powers of U^2 are proportional to $k^{2/3}$ in all orders of the perturbation series, for they can be written in an explicit form:

$$I_n^0 \propto G_0^{n-1}(k,\omega)\frac{\lambda_0^{2n}D_0^n\Gamma^{3n}k^{2n}}{(-i\omega-\Gamma_0 k^{2/3})^{2n}} \propto k^{2/3} \quad (n\geq 1)\,. \qquad (4.8)$$

As we shall see, those are the terms which determine ultraviolet behavior of a turbulent fluid while all the divergent contributions proportional to the powers of $L^{1/3}$ are irrelevant and can be summed up to a convergent expression.

To illustrate the main ideas of our method, let us single out the most divergent terms in each order n of the diagrammatic expansion of Fig. 5 which are proportional to $k^{2n}U^{2n}$. We denote the sum of these terms as $G_{\rm s.d.}$. Through construction, which is easily understood from Fig. 5, we have

$$G_{\rm s.d.} \propto G_0(k,\omega) + G_0^2(k,\omega)\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n}\frac{k^{2n}U^{2n}(k_0,l)(-1)^{n-1}}{(-i\omega-\Gamma_0 k^{2/3})^{2n-1}}\,. \qquad (4.9)$$

Introducing a Gaussian random velocity field $\vec{v}_L$ of the large eddies we have so far eliminated

$$U^2 = \langle \vec{v}_L^{\,2}\rangle \propto \int v_L^2 \exp\left(-\frac{v_L^2}{2U^2}\right)d^d v_L\,. \qquad (4.10)$$

$G_{\rm s.d.}$ can be written as

$$G_{\rm s.d.} = \left\langle \frac{1}{-i(\omega+\vec{k}\cdot\vec{v}_L)-\Gamma_0 k^{2/3}}\right\rangle\,, \qquad (4.11)$$

where the average is taken over all $\vec{v}_L$ according to perscription (4.10). It is clear that (4.9) and (4.11) are identical.

The next step is to try to generalize (4.11) to include all the divergent terms of the perturbation series into a mere frequency shift as we managed to do deriving (4.11). We made a guess that

$$G_s = \left\langle \frac{1}{-i(\omega+\vec{k}\cdot\vec{v}_L)-\Gamma_0 k^{2/3} + \frac{\bar{\lambda}_0^2 k^2\Gamma_0^3 M_2 l}{[-i(\omega+\vec{k}\cdot\vec{v}_L)-\Gamma_0 k^{2/3}]^2} - \frac{9l^2M_2^2\Gamma_0^6\bar{\lambda}_0^4k^4}{[-i(\omega+\vec{k}\cdot\vec{v}_L)-\Gamma_0k^{2/3}]^5}+\cdots}\right\rangle. \qquad (4.12)$$

Equivalence between expression (4.12) and diagrammatic expansion Fig. 5 was checked by an empirical study of the diagrams, which we have carried out up to and including some of the *eighth order diagrams*. Namely, one can check after a simple but bulky calculation that both (4.12) and expansion of Fig. 5 are identical and equal to:

$$G_s = \sum_{n=0}^{\infty} G_0^{2n+1}\frac{(2n-1)!!}{2^n}k^{2n}U^{2n} - \bar{\lambda}_0^2k^2G_0^3(k,\omega)[1+10k^2U^2G_0^2(k,\omega)+105k^4U^4G_0^4(k,\omega)+1260k^6U^6G_0^6(k,\omega)+\cdots] + \bar{\lambda}_0^4k^4G_0^6(k,\omega)[10+280k^2U^2G_0^2(k,\omega)]+\cdots\,. \qquad (4.13)$$

Writing expression (4.13) we set constant $M_2 l=1$ for the sake of simplicity and introduced dimensionless coupling parameter

$$\bar{\lambda}_0 = \frac{\lambda_0 D_0^{1/2}}{\Gamma_0^{3/2}}\,.$$

We are working now on the proof of the expression (4.12) to all orders of perturbation expansion but, at the present stage, the *exact* coincidence of (4.12) and of the diagrammatic series of Fig. 5 with (4.13) achieved in few first orders makes us to accept that

$$G_s=\left\langle\frac{1}{-i(\omega+\vec{k}\cdot\vec{v}_L)-\Gamma_0 k^{2/3}-\sum_n d_n\bar{\lambda}_0^{2n}k^{2n}l^{2n}G_0^{3n-1}(k,\omega+\vec{k}\cdot\vec{v}_L)}\right\rangle\ , \tag{4.14}$$

where coefficients $d_1=-1$, $d_2=9$ are calculated, while all the others have yet to be found. We would like to reiterate once more that expression (4.14) is an exact sum of the diagrams of Fig. 5 to all orders of $k^{2n}U^{2n}$ and to zeroth order of small parameter $\bar{\lambda}_0$ or, in other words, in the limit $\bar{\lambda}_0\to 0$.

Formula (4.14) teaches us that the main effect of the large-scale eliminations we performed is the transition to a frame of reference moving with the random velocity of large eddies $\vec{v}_L$. This result can be visualized easily if it is realized that the small eddies we are interested in are *confined* within the large ones in a manner in which the parts of a rigid body are. Thus, elimination of the modes describing the largest-scale motions in a system should necessarily amount to transition into a moving frame of reference associated with the large eddies we have gotten rid of. We conclude that the strong interaction corresponding to a coupling constant proportional to $L^{1/3}$ in the perturbation theory results exclusively in *the kinematic effect* described by formulas (4.12) and (4.14), provided the diagrams, not included in the subset G_s (Fig. 5), do not change this result. We shall take them into account in an explicit way in what follows.

The first diagrams we did not take into account in G_s appear in fourth order of perturbation expansion. They can be classified in two groups. Group 1 included diagrams 5, 8, and 9 of Fig. 4, while graphs 6, 10, and 11 are combined in the second subdivision. It is easy to check that none of the diagrams 6, 10 or 11 of Fig. 4 contain divergent terms $O(L^{1/3})$ since all are proportional to $k^{2/3}$. We are interested here in the role of the divergent terms and for the time being, let us disregard contributions from these graphs. Consider now, for example, diagram 5 of Fig. 4. It is estimated that (see the Appendix) its contribution I_4 is

$$I_4\approx-\tfrac{1}{8}\bar{\lambda}_0^2M_2lI_1\ , \tag{4.15}$$

where I_1 stems from the only graph of second order evaluated above [see Eqs. (4.2) and (4.6)]. Using (4.15) we can sum up another subset of graphs which includes *all* fourth-order contributions not taken into account so far, and some from the higher orders.

A procedure is presented in Fig. 6 with the obvious results:

$$G\approx G_s-\tfrac{5}{4}\bar{\lambda}_0^2M_2l(G_s-G_0)+O(\bar{\lambda}_0^4)\ . \tag{4.16}$$

Evaluating expression (4.16) we took into account all ten diagrams of fourth order (diagrams 5, 8, and 9), each of which is $\approx\frac{1}{8}\bar{\lambda}_0^2M_2lI_1$. Factor $\frac{5}{4}$ in (4.16) is the result of an approximate calculation, but it is clear that its exact value is >1 (see the Appendix).

The next step of our ultraviolet renormalization-group procedure consists of such coordinate transformation which brings us back to the laboratory nonmoving frame of reference. We use the Galilean transformation[11]

$$\omega'=\omega+\vec{k}\cdot\vec{v}_L \tag{4.17}$$

and write the expression for the Green's function which follows from (4.12, 4.14, and 4.16):

$$G\approx\frac{1-\frac{5}{4}\bar{\lambda}_0^2M_2l}{-i\omega-\Gamma_0k^{2/3}+\dfrac{\bar{\lambda}_0^2M_2l\Gamma_0^3k^2}{(-i\omega-\Gamma_0k^{2/3})^2}-\dfrac{9M_2^2l^2\Gamma_0^6\bar{\lambda}_0^4k^4}{(-i\omega-\Gamma_0k^{2/3})^5}+O(\bar{\lambda}_0^6)}\ . \tag{4.18}$$

Assuming from the begining that λ_0 is small, and neglecting terms proportional to $\bar{\lambda}_0^4$ which are justified in the limit $k\to\infty$, we derive readily that

$$G\approx\frac{1}{-i\omega(1+\frac{13}{4}\bar{\lambda}_0^2M_2l)-k^{2/3}\Gamma_0(1+\frac{1}{4}\bar{\lambda}_0^2M_2l)}\ . \tag{4.19}$$

Analyzing (4.19) we write the intermediate fre-

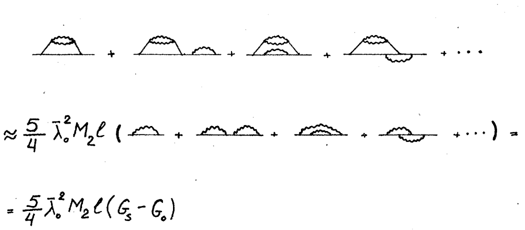

FIG. 6. Resummation based on fourth-order graphs not included into the principle subset G_s.

quency Ω_1 and Γ_1 as

$$\Omega_l = \omega(1 + \tfrac{13}{4}\bar{\lambda}_0^2 M_2 l) \ ,$$

$$\Gamma_l = \Gamma_0(1 + \tfrac{1}{4}\bar{\lambda}_0^2 M_2 l) \ , \qquad (4.20)$$

and

$$\lambda_l = \lambda_0 \ .$$

The second of the formulas in (4.20) justifies the negative dissipation term $\alpha\Gamma_0 k^{2/3}$ we have added into the equation of motion (2.1) from the beginning. Physical meaning of this term is clear: The energy is pumped into a system in the region $k_0 \approx 1/L \to 0$. The stable steady state can exist only if this energy is dissipated in the viscous range where the term $\nu_0 k^2$ is dominant. Eliminating the modes from the shell near k_0 we introduce the universal energy source $\Gamma_0 k^{2/3}\vec{v}(\vec{k},\omega)$ which compenstates the influence of the part of the energy-supplying modes we have gotten rid of. Without this negative dissipation, the energy balance would be destroyed which, of course, is forbidden. It is remarkable, however, that the large-scale elimination results in appearance of a universal term proportional to $k^{2/3}$ while the information about the structure of the stirring force in the region $k \to 0$ is hidden in the proportionality coefficient Γ_0. It is tempting to state that $\Gamma_l^3 \approx \bar{\epsilon}$.

We shall evaluate now the intermediate force correlation function. The only relevant diagrams are given in Fig. 7. Simple integration yields after the coordinate transformation (4.17):

$$D_l = D_0\left[1 - \bar{\lambda}_0^2 M_2 l \left(\frac{2\Gamma_0^4 k^{8/3}}{(\omega^2 + \Gamma_0^2 k^{4/3})^2} - \frac{\Gamma_0^2 k^{4/3}}{\omega^2 + \Gamma_0^2 k^{4/3}}\right)\right] \qquad (4.21)$$

or

$$D_l \approx D_0(1 - \bar{\lambda}_0^2 M_2 l) \qquad (4.20')$$

in the limit $k \to \infty$. The last step of the procedure consists, as usually, of rescaling:

$$k' = k e^{-l}, \quad \omega' = \omega e^{-\alpha(l)}, \quad \vec{v}^{>} = \vec{v}\,'\xi \ . \qquad (4.22)$$

Assuming that $\bar{\lambda}_0$ is small, we introduce the function

$$\alpha'(l) = \alpha(l) + \tfrac{13}{4}\bar{\lambda}_0^2 M_2 l$$

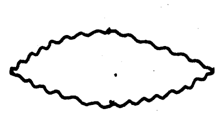

FIG. 7. Second-order correction to the force correlator.

and derive, keeping the coefficient in front of $-i\omega$ in (4.19) equal to unity:

$$\Gamma(l) = \Gamma_l e^{-\alpha(l) + 2/3l} \ ,$$

$$\lambda(l) \approx \lambda_0 \xi(1 - \tfrac{13}{4}\bar{\lambda}_0^2 M_2 l) e^{(d+1)l} \ , \qquad (4.23)$$

$$f' = \frac{f^{>}}{\xi} e^{-\alpha'(l)} \ ,$$

and consequently

$$D(l) = D_l(1 + \tfrac{13}{4}\bar{\lambda}_0^2 M_2 l)\frac{e^{-3\alpha'(l) - (d+y)l}}{\xi^2} \ . \qquad (4.24)$$

Choosing

$$\xi = \exp\left(-\frac{3\alpha'(l)}{2} - \frac{d+y}{2} l + \tfrac{9}{8}\bar{\lambda}_0^2 M_2 l\right) \ , \qquad (4.25)$$

we derive the differential recursion relations

$$\frac{dD}{dl} = 0 ,$$

$$\frac{d\lambda}{dl} = \lambda\left(-\tfrac{3}{2}z + 1 + \frac{d-y}{2} - 7\bar{\lambda}^2\right) \ , \qquad (4.26)$$

$$\frac{d\Gamma}{dl} = \Gamma(-z + \tfrac{2}{3} - \bar{\lambda}^2) \ ,$$

where we have set the dimensionality dependent factor in front of the nonlinear terms $\bar{\lambda}^2$ in (4.24) equal to unity. This factor does not enter further consideration. An equation for the dimensionless coupling parameter $\bar{\lambda}_1$,

$$\bar{\lambda}_1 = \frac{\lambda D^{1/2}}{\Gamma^{3/2}}$$

is obtained readily

$$\frac{d\bar{\lambda}_1}{dl} = \tfrac{1}{2}\bar{\lambda}_1(d - y - 11\bar{\lambda}_1^2) \ . \qquad (4.27)$$

Since we are interested exclusively in the cases $y \geq d$, we conclude that the theory described by Eq. (2.1) is asymptotically free in the ultraviolet region. Let us first take $d = y$. In this case, $\bar{\lambda}^2$ slowly tends to zero when $l \to \infty$ as $1/l$. We find from Eq. (4.25) that

$$z \approx \tfrac{2}{3} + O\left(\frac{1}{l}\right) \to \tfrac{2}{3} \ . \qquad (4.28)$$

Using (4.25 and 4.28), we calculate the energy spectrum:

$$E(k) \propto \frac{k^{-5/3}}{\ln(kL)} \ . \qquad (4.29)$$

If $\epsilon = d - y < 0$ the effective coupling parameter $\bar{\lambda}_1 \to 0$ exponentially with l. The value for z can be extracted from (4.26) if we impose the natural, in our opinion, condition

$$\lambda \to \text{const} \quad \text{at} \quad l \to \infty \ .$$

This yields

$$z = \frac{2}{3} + \frac{\epsilon}{3} \ .$$

The energy spectrum in this case is

$$E_1(k) \propto k^{-5/3 + 2\epsilon/3} \qquad (4.30)$$

and the renormalized constant Γ becomes k dependent:

$$\Gamma(k) \propto k^{|\epsilon|/3} . \qquad (4.31)$$

It can be checked readily that this fact does not influence the main steps of our ultraviolet renormalization-group procedure. We can, in principle, begin with an equation of motion similar to (2.1) but with the k dependent negative dissipation which, in this case, is proportional to $k^{2/3 + |\epsilon|/3}$. As long as we keep λ constant, all the results derived above hold, and the divergent terms can be removed by a proper coordinate transformation exactly as we did above, since the only necessary property of the system we need for it is

$$P_{lmn}(\vec{k}) = (1/a) P_{lmn}(a\vec{k}) .$$

Relations (4.30) and (4.31) should be taken with great caution. It can be seen from (4.4) that M_2 is finite only when $y \leq d$. This is most essential for our theory, for it is based on the fact that divergent terms result in a mere kinematic effect while the energy cascade is determined by *convergent* expressions. This means that $D(k)$ cannot be described by power $y > d$. One can imagine the somewhat more realistic situation when fluid is stirred by a force bounded in k space [$D(k) = \delta(k - k_0)$, for example]. If this trial force generates the force having nonzero components in the entire spectral range it must be proportional to k^{-d}.

V. ASYMPTOTIC FREEDOM IN A VISCOUS RANGE ($\Gamma_0 k^{2/3} \ll \nu_0 k^2$ AT $k \to \infty$)

In this case, formulas similar to (4.12), (4.18) (4.21) and (4.20′), but with $\nu_0 k^2$ instead of $\Gamma_0 k^{2/3}$, can be derived readily with use of the procedure introduced above. This alternation of the dissipative term in the formulas produces very large changes in our conclusions in the limit $k \to \infty$. We can see that

$$D_I = D_0 , \quad \Gamma_I = \Gamma_0 , \quad \nu_I = \nu_0 , \quad \lambda_I = \lambda_0 . \qquad (5.1)$$

After rescaling, we obtain the linear recursion relations

$$\frac{dD}{dl} = 0 , \quad \frac{d\nu}{dl} = \nu(-z + 2) , \quad \frac{d\Gamma}{dl} = \Gamma(-z + \tfrac{2}{3}) , \quad \frac{d\lambda}{dl} = \lambda\left(-\frac{3}{2} z + 1 + \frac{d - y}{2}\right) . \qquad (5.2)$$

The differential equation for the effective dimensionless coupling parameter $\bar{\lambda}$ (3.5):

$$\frac{d\bar{\lambda}}{dl} = \frac{1}{2} \bar{\lambda} \left(-2 + \frac{(d - y)}{2}\right) \qquad (5.3)$$

confirms asymptotic freedom in the viscous range.

VI. SUMMARY AND CONCLUSIONS

This work is based on the statement that formula (4.12) represents the correct dependence of the propagator upon large parameter $U^2 = \langle v_L^2 \rangle$. Expression (4.12) was checked by the direct calculations which involved evaluation of the diagrams, including 15 graphs of the sixth order and *105 graphs* of the eighth order. In principle, one cannot rule out the possibility that Eq. (4.12) fails in some very high orders of the perturbation series, but it seems quite unlikely after we have checked the *exactness* of (4.12) up to and including the eighth order of perturbation expansion.

As was mentioned above, the main result of the large-scale elimination is the transition to frame of reference moving in the physical space with the random velocity $\vec{v}_L$. Transition back to a nonmoving frame of reference which involved random Galilean transformation removed all the divergent terms. It was shown that dynamics is determined by finite contributions. These results were anticipated a long time ago by Kraichnan,[12,13] who wrote that the large-scale effect on the small ones consists of the formation of a random "field which convects all the structures without distortion in every realization and therefore has no effect at all on energy transfer. In other words, energy transfer is invariant to random Galilean transformation of a statistically homogenious turbulence." In his later work Kratchnan developed a formal perturbation expansion which did not contain contain convective terms.[14] This expansion, based on random Galilean transformation (RGT) must be related to the one introduced in the present work.

One of the results obtained in this work was the appearance of negative dissipation in the Navier-Stokes equation under random stirring. In a way, this broadens the range of the scales which can be described within a framework of macroscopic hydrodynamics. We know that the existence of the microscale is manifested by the viscous term in the Navier-Stokes equation and the microscopic motion of fluid molecules can be treated in terms of the small-scale, small-time random force. It is clear from our development that the large-scale random motion can be taken care of by introducing negative dissipation $\alpha k^{2/3}$. Moreover, it is our conviction that the large-scale random motion of atmosphere and ocean be treated in a sufficient way only if this negative dissipation term, which serves as a measure of our ignorance of what is going on the largest scale, is taken into account. It is quite probable that the negative-viscosity phenomena experimentally observed in fluids under certain conditions can be explained in terms of this contribution to the equation of motion.

The role of this negative dissipation is important for it introduces a new dimensionless parameter into the theory which is not equivalent to the Reynolds number we are all used to. From definition (4.4) and dimensionality considerations, we conclude that

$$\Gamma_0 \alpha \epsilon^{-1/3} \quad (d=y) \tag{6.1}$$

and we see that our ultraviolet renormalization-group procedure brings parameter $\bar{\epsilon}$ into the dynamic equation of motion. It is easy to form a dimensionless parameter, not including $\nu_0 \to 0$:

$$Y = \frac{U^3}{\bar{\epsilon} L} \ , \tag{6.2}$$

which can serve as a basis for constriction of perturbation expansion when the Reynolds number

$$R = \frac{UL}{\nu_0} \to \infty \tag{6.3}$$

is inconvenient. If $\bar{\epsilon}$ is finite, which represents the real experimental situation, we derive readily from (4.4) and (4.5) that $y \geq d$.

The most important outcome of our theory is the logarithmic correction (4.29) to the Kolmogorov spectrum. It follows from (4.27) that the $K-41$ expression $E(k) \propto k^{-5/3}$ does not exist in the limit $k \to \infty$. The very fact that K–41 corresponds to the boundary of convergence in the $k \to 0$ limit serves as indication that the logarithmic correction (4.29) persists in the entire k space. As we see from (4.27) this addition to the K–41 is responsible for the *non-Gaussian* statistic of the velocity field $\vec{v}(k,\omega)$ which is excited by a *Gaussian* random force $\vec{f}(k,\omega)$. It is clear that this in turn manifests the intermittent nature of fully developed turbulence initiated by the stirring force with the correlator (1.1).

However, we must ask the following question[15]: Does the expression (4.29) describe intermittencies generated by the models (2.1) and (1.1) in an accurate way? Our procedure is approximate and we should review it to discuss this important problem. Elimination of modes from a shell $k_0 < k < k_0 e^l$ yielded corrections to propagator $G_0(k,\omega)$ and force $\vec{f}(k,\omega)$ which have been calculated in the limit $k \gg k_0 e^l$. The renormalized functions G_l and $\vec{f}_l$ have then been substituted into the equation of motion and the next step of the procedure consisted of elimination of the modes from an adjacent shell $k_0 e^l \leq q \leq k_0 e^{2l}$. It is clear that among those modes were few with momenta $k \approx q$, with k belonging to a shell eliminated at a previous step. In other words, we have been treating interactions between scales of similar size in an approximate manner, while taking into account interactions between modes of sharply different wave vectors in an accurate way. It is clear that this approximation is uncontrollable within the framework of dynamic renormalization-group methods, and it is hard to judge whether those interactions contribute to intermittencies leading to the finite-power corrections to the K–41 result. It is to be emphasized that exactly the same problem arises in the regular infrared RG technique which is based on asymptotic ($k \to 0$, $\omega \to 0$) expressions for propagator and force.[6] It is to be stressed, however, that if there exists a finite correction to $\frac{5}{3}$ power it must be derived in both $k \to 0$ and $k \to \infty$ limits. The problem (2.1) and (1.1) does not have a length scale and thus the energy spectrum of randomly stirred fluid must have $k \to 0$ and $k \to \infty$ asymptotics different by not more than logarithmic factor. Keeping this in mind the finite correction to $\frac{5}{3}$ seems to be unlikely because even interactions between different scales, as we learned from Refs. 1 and 2, lead to expression K–41 corresponding to a boundary of convergence.

We would like to conclude this paper by discussion of the assumption about statistical independence of the velocity fields $\vec{v}(k,\omega)$ and $\vec{v}_L$. As we stated above, field $\vec{v}_L$ describes the motion of the largest scale of the problem and thus can be treated as decoupled from $\vec{v}(\vec{k},\omega)$ when $k\to\infty$. However, calculating the energy spectrum, we dealt with the integral

$$E(k)\,\alpha\, k^{d-1}\int U\left(k,\frac{\omega}{k^z}\right)d\omega\,, \tag{6.4}$$

where $-\infty<\omega<\infty$. It is clear that in the region $\omega\to 0$ one should be careful treating

$$\langle\vec{v}(\vec{k},\omega)\rangle_{\vec{v}_L}=\vec{v}(\vec{k},\omega)\,, \tag{6.5}$$

which was used in performing coordinate transformation (4.17) in the renormalized propagator [(4.14) and (4.16)], because the time scales of $\vec{v}_L$ and $\vec{v}(\vec{k},\omega)$ become of the same order. In this region of frequencies Eq. (6.5) does not hold and this can lead to additional corrections to the energy spectrum provided this frequency interval is wide enough to contribute to the integral (6.4). These corrections can, in principle, lead to intermittent behavior in the inertial range spectra although at the present stage, we have no grounds to accept or to reject this statement.

ACKNOWLEDGMENTS

This work was initiated in close cooperation with E. Levich and many of the ideas which emerged in our lengthy and most stimulating discussions evolved into this paper. I would like to express my profound gratitude to R. H. Kraichnan who influenced my understanding of the problems of turbulence. Helpful discussions with P. C. Martin, B. Levich, J. Sak, M. Stephen and L. Pismen are gratefully acknowledged. This work was supported in part by the U. S. Department of Energy under Grant No. DE-AC02-80ER10559.

APPENDIX

In this appendix we are mainly interested in the evaluation of some integrals involved in derivation of relation (4.15). We begin with the calculation of the integral (4.1) in the limit $\nu_0\to 0$. The Ω integration is readily done with the result

$$I_1\propto\lambda_0^2\int\frac{q^{-d}d^dq}{2q^{2/3}(-i\omega-q^{2/3}-|\vec{k}-\vec{q}|^{2/3})}\quad(\Gamma_0=1,D_0=1) \tag{A1}$$

which yields (4.2).

An analytic expression corresponding to the fourth-order diagram 5 of Fig. 4 is

$$I_4\propto\int\frac{q_1^2\,\dfrac{d\Omega}{2\pi}\,\dfrac{d\Omega_2}{2\pi}\,d^dq_1d^dq_2q_1^{-d}q_2^{-d}}{[-i(\omega-\Omega_1)-|\vec{k}-\vec{q}_1|^{2/3}](-i\Omega_1-q_1^{2/3})^2[(\Omega_1-\Omega_2)^2+|\vec{q}_1-\vec{q}_2|^{4/3}](\Omega_2^2+q_2^{4/3})}\,. \tag{A2}$$

In writing (A1) and (A2), we set $D(q)=q^{-d}$ in accordance with the asymptotic behavior of the random-force correlator (4.5) in the limit $q\to 0$. Constant $\Gamma_0=1$ is taken for the sake of beauty. The frequency integration in (A2) yields

$$I_4\propto\lambda_0^4\int\frac{q_1^2d^dq_1d^dq_2q_1^{-d}q_2^{-d}}{4|\vec{q}_1-\vec{q}_2|^{2/3}q_2^{2/3}(-i\omega-q_1^{2/3}-|\vec{k}-\vec{q}_1|^{2/3})(q_1^{2/3}+q_2^{2/3}+|\vec{q}_1-\vec{q}_2|^{2/3})^2}\,. \tag{A3}$$

It can be readily checked that region $q_1\ll q_2\to 0$ does not contribute to the integral (A3). It is a good approximation to evaluate (A3) in the limit $q_2\ll q_1\to 0$. The result is

$$I_4\propto\frac{\lambda_0^4}{16}\int\frac{d^dq_1d^dq_2q_1^{-d}q_2^{-d}}{q_2^{2/3}(-i\omega-q_1^{2/3}-|\vec{k}-\vec{q}_1|^{2/3})}\,. \tag{A4}$$

A comparison of (A4) with (A2) proves formula (4.15) if one takes into account the definition of M_2 which follows from formula (4.4).

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