Incoherent effects in the optical up-conversion process

W. J. Mielniczuk
Institute of Physics, Polish Academy of Sciences, Al. Lotników 32/46, 02-668 Warsaw, Poland

J. Chrostowski
Institute of Experimental Physics, University of Warsaw, Hoża 69, 00-681 Warsaw, Poland and LROL, Department of Physics, Laval University, Quebec G1K 7P4 Canada
(Received 11 June 1980)

The theoretical treatment of the optical frequency up-conversion process with stochastic pumping is presented. This process consists of exchange of photons between two modes of different frequencies. It is assumed that the two modes are coupled by a short memory stochastic function. Statistically averaged occupation numbers of both modes were calculated exactly by means of the theory of multiplicative stochastic processes. The obtained nonperturbative solution sums secular terms and is valid for arbitrary times. The oscillatory-damped nature of the exchange of energy between two modes and the role of partial incoherence of the pump is discussed. It is shown that stochastic coupling causes the equipartition of energy between two modes for sufficiently long times. Analogies between an up-converter and a two-level atom driven by a single mode electromagnetic field are indicated. Optical Bloch equations for an up-conversion process are derived.

I. INTRODUCTION

Optical frequency up-conversion process belongs to a group of nonlinear phenomena of great practical importance[1] and has also very interesting theoretical aspects. The process consists in the generation of photons of the higher frequency ω_a from photons of the lower frequency ω_b by means of a parametric interaction of a pump with a nonlinear medium.

A quantum description of the conversion process is given by Louisell.[2] Signal and idler modes interact with each other with assistance of the pump mode. An interaction Hamiltonian is proportional to $g\exp[-i(\omega_a-\omega_b)t]$. Tucker and Walls[3] assumed g to be a constant. Lu[4] solved the problem for g as an ordinary function of time. Crosignani *et al.*[5] assumed to the coupling function to be a stochastic Gaussian process with a Lorentzian power spectrum, corresponding to a phase-diffusion model of an amplitude-stabilized laser pump. Kryszewski and Chrostowski[6] analyzed statistical properties of converted light with the pump amplitude as a random function of time. They predicted an antibunching effect—the negative Hanbury-Brown and Twiss effect for the signal mode. The theory of multiplicative stochastic processes was applied to the frequency up-conversion process by Mielniczuk.[7]

In frequency-conversion experiments one deals usually with strong, multimode, pulsed lasers whose amplitude undergoes substantial fluctuations often comparable to, if not stronger than, those of a chaotic field. It is the purpose of this paper to present analytic solutions for statistically averaged occupation numbers of interacting modes in the case when the partially incoherent pump is randomly amplitude modulated.

The role of partial incoherence of a laser pump in atomic transitions has been studied recently by many authors.[8] Influenced by them we were able to find a formal similarity between a two-level atom interacting with the electromagnetic (EM) field and the frequency up-conversion system. Such a correspondence enables us to introduce a Bloch equation description to the parametric frequency conversion phenomena. The paper is organized as follows. In Sec. II we explain all physical assumptions of the up-converter with a stochastic coupling. In Sec. III for completness of our discussion we present a case of a perfectly coherent pump. In Sec. IV we discuss a case when a coupling function is a sum of a constant and a white-noise process. In Sec. V we indicate similarities between the up-converter and a two-level atom driven by a single mode EM field. In Sec. VI we generalize results of Sec. IV for a case when the up-converter relaxes with the longitudinal time T_1 and the transversal time T_2. In Sec. VII we compare results of Sec. IV with results of a case when the pump is a random stochastic process described by the phase-diffusion model. In Sec. VIII we present some concluding remarks.

II. THE STOCHASTIC MODEL OF THE UP-CONVERSION PROCESS

The frequency up-conversion process is modeled by an exchange of photons between two optical modes of different frequencies ω_a and ω_b ($\omega_a > \omega_b$) (Fig. 1). These two modes are called the signal mode (mode A) and the idler mode (mode B). The effective Hamiltonian describing that process has the form

$$H=\hbar\omega_a a^\dagger a+\hbar\omega_b b^\dagger b +\hbar\{g(t)\exp[-i(\omega_a-\omega_b)t]a^\dagger b+\text{H.c.}\}, \quad (2.1)$$

where $a^\dagger$ and a and $b^\dagger$ and b denote the creation and anihilation operators for modes A and B.

The first two terms of the Hamiltonian represent free energies of mode A and B. The interaction terms of the Hamiltonian (2.1) are schematically depicted in Fig. 1. It is worth noticing that in the frequency up-conversion process three electromagnetic modes are coupled. However, in a derivation of the Hamiltonian (2.1) one uses the well-known parametric approximation[3] that reduces the problem to the two-mode interaction. The parametric approximation is well justified for a case when the three following assumptions are fulfilled:

(i) An amplitude of a pump mode corresponding to the frequency $\omega_p=\omega_a-\omega_b$ is much stronger than amplitudes of other modes.

(ii) Radiative transitions depicted in Fig. 1 take place from a real molecular or atomic level to a virtual one.

(iii) The phase matching condition

$$\vec{k}_b+\vec{k}_p=\vec{k}_a$$

is fulfilled where $\vec{k}_a$, $\vec{k}_b$, and $\vec{k}_p$ denote the wave vectors of a photon of the frequency ω_a, ω_b, and ω_p, respectively.

The assumption (i) allows us to treat the pumping mode classically and to neglect the reaction of the two other modes on it.

The creation and annihilation operators satisfy the boson commutation rules

$$[a,a^\dagger]=[b,b^\dagger]=1, \quad [a,b]=0. \quad (2.2)$$

The stochastic character of the function $g(t)$ may have two sources:

(i) fluctuations of classically treated amplitude and phase of the pumping mode;

(ii) fluctuations of a nonlinear polarizability of a medium in which the frequency up-conversion occurs.

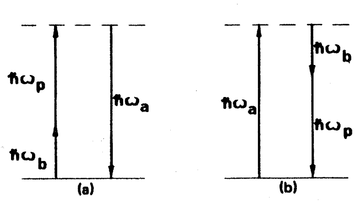

FIG. 1. Frequency conversion of light. ω_a, ω_b, and ω_p are frequencies of the signal, idler, and pump modes, respectively; (a) illustrates the first part of the interaction Hamiltonian (2.1), and (b) the Hermitian conjugate term.

If stochasticity originates in the fluctuations of the pump amplitude we may take it into account in an additive way, writing

$$E_{\text{pump}}(t)=E_0+E_1(t),$$

where $E_1(t)$ is a stochastic function of time. Such an assumption is well motivated by the signal-plus-noise model of a laser operating above threshold.[9] This model leads to the intensity fluctuations of laser light contrary to the phase-diffusion model applied in Sec. VII of our paper.

We assume that the coupling function $g(t)$ fluctuates around its time-independent mean value g_0

$$g(t)=g_0+g_1(t), \quad (2.3)$$

and $g_1(t)$ is a stochastic stationary Gaussian process for which

$$\begin{aligned}&\langle g_1(t)\rangle=\langle g_1^*(t)\rangle=0\,,\\&\langle g_1(t)g_1^*(t+\tau)\rangle=2D\delta(\tau)\,,\\&\langle g_1(t)g_1(t+\tau)\rangle=\langle g_1^*(t)g_1^*(t+\tau)\rangle=0,\end{aligned} \quad (2.4)$$

where $D\geq 0$ is the spectral density of the stochastic process $g(t)$. The symbol $\langle\cdots\rangle$ denotes the statistical average over the random variables of the stochastic process $g(t)$. The process $g(t)$ is a limiting case of a band-limited white-noise process[10] with the correlation time much smaller than other characteristic times involved. D scales incoherence of the pump. In the limit $D\to 0$ we get the perfectly coherent pump. It is easy to show that statistically averaged populations of modes depend on g_0 only via $|g_0|$. Since we are interested in populations only, we assume that $|g_0|=g_0\geq 0$, i.e., g_0 is a real non-negative number.

We assume that modes A and B are initially independent and described by the density operator

$$\rho(0)=\rho_B(0)|0\rangle_{AA}\langle 0|\,,$$

where $|0\rangle_A$ is the vacuum state of mode A and $\rho_B(0)$ is an arbitrary density operator for mode B. Mode A is generated during the frequency-conversion process.

III. PERFECTLY COHERENT PUMP

Equations of motion for annihilation operators a and b are[2]

$$\frac{d}{dt}a(t)=-i\omega_a a(t)-ig_0\exp[-i(\omega_a-\omega_b)t]b(t) \quad (3.1)$$

$$\frac{d}{dt}b(t)=-i\omega_b b(t)-ig_0\exp[i(\omega_a-\omega_b)t]a(t)\,. \quad (3.2)$$

This set of equations can be solved exactly

$$a(t)=\exp(-i\omega_a t)[a_0\cos(g_0 t)-ib_0\sin(g_0 t)], \quad (3.3)$$

$$b(t)=\exp(-i\omega_b t)[-ia_0\sin(g_0 t)+b_0\cos(g_0 t), \quad (3.4)$$

where a_0 and b_0 are annihilation operators of modes A and B at the initial moment $t=0$.

Taking into account initial conditions, we arrive at the mean photon numbers of modes A and B in the form

$$\langle a^\dagger(t)a(t)\rangle=N_0\sin^2(g_0 t), \quad (3.5)$$

$$\langle b^\dagger(t)b(t)\rangle=N_0\cos^2(g_0 t), \quad (3.6)$$

where

$$N_0=\mathrm{Tr}[\rho_B(0)b^\dagger(0)b(0)] \quad (3.7)$$

denotes the number of photons of mode B at the initial moment $t=0$. Populations of modes are depicted in Fig. 2.

It is easy to see that the total photon number $\langle a^\dagger(t)a(t)+b^\dagger(t)b(t)\rangle$ is preserved during the frequency-conversion process. The inversion W, i.e., the difference of populations of modes, is

$$W=-N_0\cos(2g_0 t). \quad (3.8)$$

The period of oscillations of the inversion W between modes A and B is

$$T=\pi/g_0. \quad (3.9)$$

while the period of oscillation of each mode is twice as large.

IV. PARTIALLY COHERENT PUMP

In the case when g_1 is a stochastic function of time, Heisenberg equations of motion for the annihilation operators are

$$\frac{d}{dt}a(t)=-i\omega_a a(t)-ig^*(t)\exp[-i(\omega_a-\omega_b)t]b(t), \quad (4.1)$$

$$\frac{d}{dt}b(t)=-i\omega_b b(t)-ig(t)\exp[i(\omega_a-\omega_b)t]a(t). \quad (4.2)$$

Equations for creation operators can be obtained by the Hermitian conjugate of Eqs. (4.1) and (4.2).

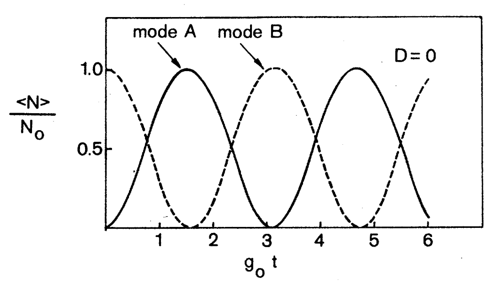

FIG. 2. Populations of signal mode A and idler mode B for the case of the perfectly coherent pump. N_0 is the total photon number in both modes; g_0 is the frequency of population oscillations.

The system of these four equations cannot be solved explicitly because of the stochastic nature of the coupling function $g(t)$. Fortunately we are interested in ensemble average quantities bilinear in creation and annihilation operators. However, we still have to solve a closed system of four differential equations for following quantities: $\langle a^\dagger(t)a(t)\rangle$, $\langle b^\dagger(t)b(t)\rangle$, $\langle a^\dagger(t)b(t)\rangle$, and $\langle a(t)b^\dagger(t)\rangle$, where $\langle\cdots\rangle$ denotes from this moment the average over the stochastic variables and subsequent quantum mechanical one. The system of these four equations can be reduced to the closed system of three equations by noting that the total photon number is a constant of motion even in the case of the partially coherent pump.

Equations of motion for operators $W=a^\dagger a-b^\dagger b$, $a^\dagger b$ and $ab^\dagger$ are of the form

$$\frac{d}{dt}W=-2ig^*\exp[i(\omega_b-\omega_a)t]a^\dagger b - 2ig\exp[-i(\omega_b-\omega_a)t]ab^\dagger, \quad (4.3)$$

$$\frac{d}{dt}(a^\dagger b)=i(\omega_a-\omega_b)a^\dagger b-ig\exp[i(\omega_a-\omega_b)t]W, \quad (4.4)$$

$$\frac{d}{dt}(ab^\dagger)=-i(\omega_a-\omega_b)ab^\dagger+ig^*\exp[-i(\omega_a-\omega_b)t]W. \quad (4.5)$$

Introducing a transformation to the rotating frame

$$\tilde{a}=a\exp(i\omega_a t), \quad (4.6a)$$

$$\tilde{b}=b\exp(i\omega_b t), \quad (4.6b)$$

we obtain

$$\frac{d}{dt}W=2ig\tilde{b}^\dagger\tilde{a}-2ig^*\tilde{a}^\dagger\tilde{b}, \quad (4.7)$$

$$\frac{d}{dt}(\tilde{a}^\dagger\tilde{b})=-igW, \quad (4.8)$$

$$\frac{d}{dt}(\tilde{a}\tilde{b}^\dagger)=ig^*W. \quad (4.9)$$

These equations can be rewritten in a vector notation

$$\frac{d}{dt}\Psi=M_0\Psi+ig_1M_1\Psi+ig_1^*M_2\Psi, \quad (4.10)$$

where

$$\Psi = \begin{pmatrix} W \\ \tilde{a}^\dagger \tilde{b} \\ \tilde{b}^\dagger \tilde{a} \end{pmatrix}, \quad M_0 = \begin{pmatrix} 0 & -2ig_0 & 2ig_0 \\ -ig_0 & 0 & 0 \\ -ig_0 & 0 & 0 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 0 & 0 & 2 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Applying theory of multiplicative stochastic processes[11-13] we obtain the system of the first-order differential equations for ensemble averaged vector $\langle\Psi\rangle$ (see the Appendix)

$$\frac{d}{dt}\langle\Psi(t)\rangle = M_0\langle\Psi\rangle - DM_1M_2\langle\Psi\rangle - DM_2M_1\langle\Psi\rangle. \quad (4.11)$$

The initial condition for the column vector $\langle\Psi(t)\rangle$ is

$$\langle\Psi(0)\rangle = \begin{pmatrix} -N_0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.12)$$

Consequently we get

$$\frac{d}{dt}\langle\tilde{a}^\dagger\tilde{b} + \tilde{b}^\dagger\tilde{a}\rangle = -2D\langle\tilde{a}^\dagger\tilde{b} + \tilde{b}^\dagger\tilde{a}\rangle, \quad (4.13)$$

$$\frac{d}{dt}\langle\tilde{a}^\dagger\tilde{b} - \tilde{b}^\dagger\tilde{a}\rangle = -2ig_0\langle W\rangle - 2D\langle\tilde{a}^\dagger\tilde{b} - \tilde{b}^\dagger\tilde{a}\rangle, \quad (4.14)$$

$$\frac{d}{dt}\langle W\rangle = -4D\langle W\rangle - 2ig_0\langle\tilde{a}^\dagger\tilde{b} - \tilde{b}^\dagger\tilde{a}\rangle. \quad (4.15)$$

Writing

$$\langle\tilde{a}\tilde{b}^\dagger\rangle = \tfrac{1}{2}N_0(u + iv), \quad (4.16)$$

$$\langle W\rangle = N_0 w, \quad (4.17)$$

we get

$$\dot{u} = -2Du, \quad (4.18)$$

$$\dot{v} = -2Dv + 2g_0 w, \quad (4.19)$$

$$\dot{w} = -4Dw - 2g_0 v, \quad (4.20)$$

with the initial condition

$$u^2(0) + v^2(0) + w^2(0) = 1. \quad (4.21)$$

From Eqs. (4.18)–(4.20) we find that

$$u^2(t) + v^2(t) + w^2(t) \leq \exp(-2Dt) \quad (4.22)$$

for arbitrary times $t \geq 0$.

Defining $\vec{s} = [u, v, w]$ we have a formal analog of a Bloch vector[14] lying on the Bloch sphere (Fig. 3). We see that the Bloch sphere shrinks in an exponential way as a function of time. During the evolution, the Bloch vector $\vec{s}$ moves upwards and downwards. The north pole orientation corresponds to the maximal population of mode A.

The quantities u and v are in-phase and out-of-phase components of the transition moment of the up-converting system during its interaction with the pump of the frequency $\omega_a - \omega_b$. Let us concentrate on the time evolution of the inversion $w(t)$. We notice that Eq. (4.18) decouples from Eqs. (4.19) and (4.20). Then the general form of the solution for the inversion $w(t)$ is

$$w(t) = \exp(-3Dt)\{A\cos[(4g_0^2 - D^2)^{1/2}t] + B\sin[(4g_0^2 - D^2)^{1/2}t]\}. \quad (4.23)$$

From the initial condition (4.12)

$$w(0) = -1. \quad (4.24)$$

From Eqs. (4.20) and (4.24) we have

$$\frac{d}{dt}W(t)\bigg|_{t=0} = 4D. \quad (4.25)$$

Then from Eqs. (4.18)–(4.20) and (4.23)–(4.25) we get

$$A = -1, \quad B = D/(4g_0^2 - D^2)^{1/2}. \quad (4.26)$$

Using the fact that the total photon number is preserved we finally obtain expressions for populations

$$\langle a^\dagger(t)a(t)\rangle = \tfrac{1}{2}N_0[1 + w(t)] \quad (4.27)$$

and

$$\langle b^\dagger(t)b(t)\rangle = N_0 - \langle a^\dagger(t)a(t)\rangle. \quad (4.28)$$

The solution given by Eqs. (4.27) and (4.28) is exact and valid for arbitrary times.

The population of mode A is depicted in Fig. 4 for values $D = 0$, $D = 0.1\,g_0$, and $D = g_0$, respectively. For the case $4g_0^2 - D^2 > 0$ we observe damping of the difference of populations with the damping rate equal to $3D$. This implies that only one

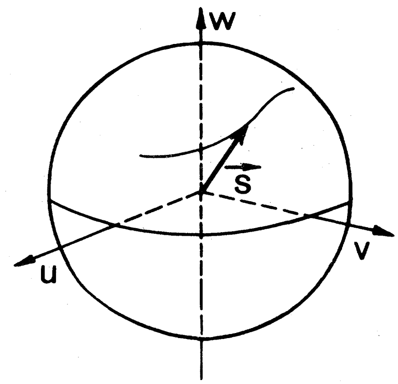

FIG. 3. Bloch vector description of the up-conversion process. The components of the vector $\vec{s} = [u, v, w]$ are the in-phase and out-of-phase component of the transition moment of the up-converter and the difference of populations of the modes, respectively.

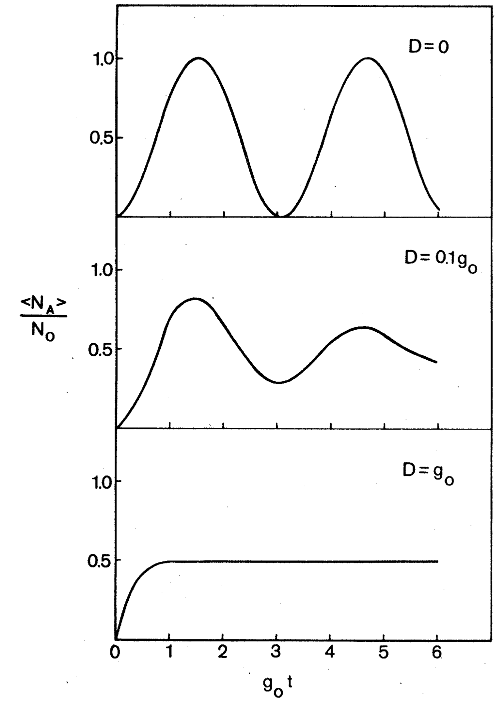

FIG. 4. Mean photon number of signal mode A, generated during up-conversion process as a function of time. Three cases correspond to the different values of the spectral density D of the incoherent part of the pump mode. N_0 denotes the initial photon number of idler mode B; g_0 is the expectation value of the stochastic coupling $g(t)$.

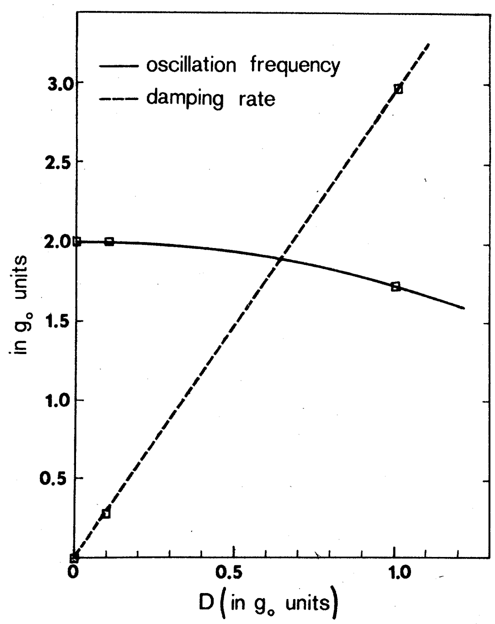

FIG. 5. Dependence of the frequency of energy oscillations and the damping rate on the spectral density D which scales incoherence of the pump. Squares on the curves mark points $D=0$, $D=0.1\ g_0$, and $D=g_0$.

half of the power of mode A can be converted asymptotically into mode B. Then in the steady state, equipartition of energy between two modes occurs, while in the coherent pump case a complete power exchange between two modes occurs periodically.

We observe that for the case $4g_0^2-D^2>0$ the frequency of inversion oscillations $\Omega=(4g_0^2-D^2)^{1/2}$ is smaller than the corresponding flopping frequency $\Omega_0=2g_0$ in the case of the perfectly coherent pump. The frequency of energy oscillations weakly depends on the spectral density D for small values of D and becomes significantly modified for D of the order of g_0 (Fig. 5). This fact is the origin of the significant differences between Figs. 4(b) and 4(c). For $D=g_0$ there is no oscillation of the inversion and the saturation limit is obtained for times t of the order of $1/g_0$. For the case $4g_0^2-D^2<0$ there is no energy oscillation.

V. TWO-LEVEL ATOM AND UP-CONVERTER ANALOGY

It is worth noting some analogies between a two-level atom driven by a single-mode resonant electromagnetic field and an up-converter. Let us denote by $|1\rangle$ and $|2\rangle$ a lower and a higher level of the atom, respectively; ω is the frequency of the incident field. The Hamiltonian of such a two-level atom is[2,14]

$$H=\tfrac{1}{2}\hbar\omega\sigma_3+\hbar\lambda\sigma\exp(i\omega t)+\hbar\lambda\sigma^\dagger\exp(-i\omega t), \qquad (5.1)$$

where $\sigma_3=|2\rangle\langle 2|-|1\rangle\langle 1|$ denotes the difference of populations of levels $|2\rangle$ and $|1\rangle$. $\sigma^\dagger=|2\rangle\langle 1|$ is the raising operator changing the state of an electron from the lower state $|1\rangle$ to the higher one $|2\rangle$ and causing simultaneous emission of a photon of the frequency $\omega=(E_2-E_1)/\hbar$. $\sigma=|1\rangle\langle 2|$ is the lowering operator, λ is a coupling constant proportional to the transition moment between levels $|1\rangle$ and $|2\rangle$. Correspondence between the operators governing the evolution of both considered systems is given in Fig. 6.

The difference of population operator $W=a^\dagger a-b^\dagger b$ plays the same role as the atomic operator σ_3. The operator $a^\dagger b$, corresponding to the oper ator $\sigma^\dagger$, describes annihilation of the photon of lower frequency mode B with the simultaneous creation of the photon of higher frequency mode A.

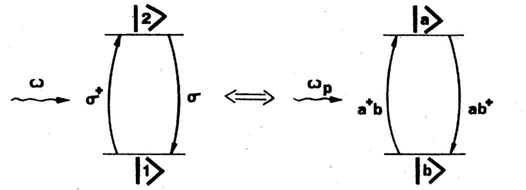

FIG. 6. Two-level atom and up-converter analogy.

Beside it the pump photon is annihilated. The operator $ab^\dagger$, corresponding to the atomic operator σ, is responsible for creation of the photon of lower frequency mode B with the simultaneous annihilation of the photon of higher frequency mode A. The commutation relation

$$[a^\dagger b, b^\dagger a] = a^\dagger a - b^\dagger b$$

resembles the commutation relation

$$[\sigma^\dagger, \sigma] = \sigma_3.$$

The flopping frequency $\Omega_0 = 2g_0$ corresponds to the Rabi frequency[14] of the two-level atom. The frequency $(\omega_a - \omega_b)$ is related to the characteristic frequency ω of the atom. The transformation

$$a^\dagger a \to \tilde{a}^\dagger \tilde{a} = a^\dagger a,$$
$$b^\dagger b \to \tilde{b}^\dagger \tilde{b} = b^\dagger b,$$
$$a^\dagger b \to \tilde{a}^\dagger \tilde{b} = \exp[i(\omega_a - \omega_b)t] a^\dagger b,$$
$$b^\dagger a \to \tilde{b}^\dagger \tilde{a} = \exp[i(\omega_a - \omega_b)t] b^\dagger a,$$

is similar to a transformation of atomic variables to the frame rotating with the frequency ω. Such an analogy between the frequency up-converter and the two-level atom has some limitations.

It has been shown by Walls and Barakat[15] that the trilinear interaction Hamiltonian of the form $H = g(ab^\dagger c^\dagger + a^\dagger bc)$, which describes parametric amplification or frequency up-conversion processes, can also be used to study the problem of coherent emission from a system of N two-level atoms interacting with a single mode of radiation field. The atomic angular momentum operators $J_z = \Sigma\sigma_3$, $J^+ = \Sigma\sigma^+$, and $J^- = \Sigma\sigma^-$ can themselves be represented in terms of two operators a and b obeying the boson commutation rules; $J^+ = ab^\dagger$, $J^- = b^\dagger a$, thus making both cases formally identical. For large $N \to \infty$ eigenfunctions and eigenvalues for both cases are equal.

Making use of the parametric approximation we reduced the problem to the two-mode interaction which is tractable analytically contrary to the trilinear case. However clearly now that the N atoms-converter analogy does not hold since the classically treated pump mode is not depleted and plays a quite different role in the interaction. Introducing a semiclassical description of the interaction between the atoms and the field we were able to recover a formal similarity between the up-converter and two-level atom instead, which amounts on the mathematical ground to a simple transformation of the c-number variables.

VI. DAMPED CONVERSION WITH PARTIALLY INCOHERENT PUMP

Now by analogy with the two-level atom we introduce two phenomenological relaxation times T_1 and T_2 to Eqs. (4.7)–(4.9). We obtain the system of Bloch equations

$$\dot{u} = (-2D - 1/T_2)u, \tag{6.1}$$
$$\dot{v} = (-2D - 1/T_2)v + 2g_0 w, \tag{6.2}$$
$$\dot{w} = (-4D - 1/T_1)w - 2g_0 v. \tag{6.3}$$

The decay times introduced in this manner can be related to the quantities of physical interest—the decay constants γ_a and γ_b of the amplitudes of the modes. The transversal relaxation time $T_2 = (\gamma_a + \gamma_b)^{-1}$ describes the dephasing between the modes due to noncoherent damping. The interpretation of T_1 is more complicated since in the damped case the total number of photons is not preserved. The evolution of the system is governed by four coupled equations instead of three and the decay of energy is in general not exponential. Only when $\gamma_a = \gamma_b = \gamma$ does the decay have a simple exponential form with $1/T_1 = 2\gamma$.

The equation responsible for the evolution of $u(t)$ decouples from the equations for $v(t)$ and $w(t)$ even in the case when the relaxation comes into play. The generalized Torrey equation is of the form

$$(-2D - 1/T_2 - \lambda)[(-2D - 1/T_2 - \lambda) \times (-4D - 1/T_1 - \lambda) + 4g_0^2] = 0. \tag{6.4}$$

The time evolution of the inversion w is given by

$$w(t) = \exp(-\Gamma t)[-\cos(\Omega' t) + C\sin(\Omega' t)], \tag{6.5}$$

where the damping constant Γ is defined by

$$\Gamma = 3D + \frac{1}{2}\left(\frac{1}{T_1} + \frac{1}{T_2}\right),$$

the frequency of oscillation

$$\Omega' = \left[4g_0^2 - \frac{1}{4}\left(\frac{1}{T_1} - \frac{1}{T_2} + 2D\right)^2\right]^{1/2}, \tag{6.6}$$

and

$$C = (4D - \Gamma + 1/T_1)/\Omega'. \tag{6.7}$$

Relaxation times for the incoherent pump are different from those in the coherent case. The longitudinal relaxation time T_1' for the case $D \neq 0$ is given by

$$\frac{1}{T_1'} = \frac{1}{T_1} + 4D, \tag{6.8}$$

while the transversal relaxation time T_2'

$$\frac{1}{T_2'} = \frac{1}{T_2} + 2D. \tag{6.9}$$

The spectral density D affects both the longitudinal and the transversal lifetimes but in a different way.

It is interesting to point out that the renormalized damping constant Γ depends on both damping rate $1/T_1$ and $1/T_2$ via their sum $(1/T_1 + 1/T_2)$, while the shifted frequency of oscillations Ω' depends on damping rates via their difference $(1/T_1 - 1/T_2)$.

For the case when both modes have the same noncoherent decay constants $\gamma_a = \gamma_b = \gamma$, i.e., $T_1 = T_2$, no shift in frequency of oscillation between the modes due to damping is present. The only contribution results from the noncoherence of the pump. The damping constant Γ of inversion has now a terms 2γ related to the noncoherent flow of energy from the system. This damping enhances the decay of inversion created by the noncoherent pump.

VII. UP-CONVERSION WITH A PHASE-DIFFUSED PUMP

Let us analyze some frequency up-conversion characteristics in the case of the laser pump with intensity stabilization. Such a light can be well described by a phase-diffusion model[9] leading to a Lorentzian line profile with a finite bandwidth. The coupling function $g(t)$ for the frequency conversion process is then given in the form

$$g(t) = g_0 \exp[i\Phi(t)], \tag{7.1}$$

where the phase $\Phi(t)$ is a stochastic process. We assume the following properties of the stochastic process $\Phi(t)$ and its time derivative $\dot{\Phi}(t)$

$$\langle \Phi(t) \rangle = 0, \tag{7.2}$$

$$\langle \dot{\Phi}(t)\dot{\Phi}(s) \rangle = 2\Gamma_L \delta(t-s), \tag{7.3}$$

where Γ_L is a laser bandwidth.

Applying the theory of the multiplicative stochastic processes[11-13] we obtain Bloch equations for the up-conversion process of the form

$$\dot{u} = -\Gamma_L u, \tag{7.4}$$

$$\dot{v} = -\Gamma_L v + 2g_0 w, \tag{7.5}$$

$$\dot{w} = -2g_0 v. \tag{7.6}$$

Similarly as in the previous case of Eqs. (4.18)–(4.20) the equation describing the evolution of the function u decouples from the system of two equations for v and w. The populations of modes are

$$\langle a^\dagger(t)a(t) \rangle = \tfrac{1}{2}N_0\left(1 - \exp(-\tfrac{1}{2}\Gamma_L t)\{\cos[4g_0^2 - (\tfrac{1}{2}\Gamma_L)^2]^{1/2} t\} + \tfrac{1}{2}\frac{\Gamma_L}{[4g_0^2 - (\tfrac{1}{2}\Gamma_L)^2]^{1/2}} \sin\{[4g_0^2 - (\tfrac{1}{2}\Gamma_L)^2]^{1/2} t\}\right), \tag{7.7}$$

$$\langle b^\dagger(t)b(t) \rangle = N_0 - \langle a^\dagger(t)a(t) \rangle. \tag{7.8}$$

The solution of the system of Eqs. (7.4)–(7.6) resembles very much the solution for the case of stochastic modulation of the pump amplitude. Instead of the previously obtained damping rate $3D$ we get for the case $[4g_0^2 - (\frac{1}{2}\Gamma_L)^2] > 0$ the damping rate equal to $\Gamma_L/2$. The oscillation frequency is shifted in a similar way to $[4g_0^2 - (\frac{1}{2}\Gamma_L)^2]^{1/2}$.

The reason for this similarity lies in the role played by incoherence parameters which appear only as diagonal elements of appropriate matrices [see, for example, Eqs. (4.18)–(4.20) and (7.4)–(7.6)]. The solution (7.7) shows also a qualitative agreement with approximate solution of Crosignani *et al.*[5] for the case of a broad-bandwidth laser, i.e., $\Gamma_L \gg g_0$.

VIII. CONCLUSION

We have analyzed the influence of incoherence of the pump mode on mean photon numbers of the signal and the idler mode in the up-conversion process. By means of the theory of multiplicative stochastic processes we obtained the nonperturbative solution for populations. This solution is valid for arbitrary times.

Two stochastic models of the coupling corresponding to two different physical situations were compared. We observed that the partially incoherent coupling causes damping of populations of the interacting modes. It is interesting to note that asymptotically the exchange of photons between the modes becomes insignificant and photon numbers in both modes become practically equal. This equipartition effect observed for both stochastic amplitude and phase modulations takes place regardless of how small the pump incoherence is. However the time needed for this to happen may be longer than the actual interaction time in which case this effect will not be observed. Nevertheless, such a result has more than a mere theoretical significance since it points towards clear trends of the physical processes involved.

ACKNOWLEDGMENTS

This work was partly supported by Polish Research Project MRI/5. The support of the Natural Sciences and Engineering Research Council, Canada and the Quebec Government is also aknowledged. We are greatly indebted to Professor Z. Birula-Bialynicka for a careful reading of the manuscript. One of us (W.J.M.) thanks Dr. A. Bandilla for his kind remarks on Ref. 7 and Professor S. K. Srinivasan for sending a copy of his paper prior to publication.

APPENDIX

To get Eq. (4.11) we have used the following indentity[16]:

$$\langle T \exp\{i \int_0^t dx[g(x)M_1(x) + g^*(x)M_2(x)]\} = \exp[-D \cdot M_1(t)M_2(t) - D \cdot M_2(t)M_1(t)] ,$$

where $M_1(t)$, $M_2(t)$ are arbitrary time-dependent matrices and $g(t)$ denotes the white noise process defined by Eq. (2.4). The symbol $\langle \cdots \rangle$ in accordance with a notation used previously denotes the statistical average over the random variables of the stochastic process $g(t)$, T denotes the time ordering.

[1]For a review see F. Zernike and I. E. Midwinter, *Applied Nonlinear Optics* (Wiley, New York, 1973); R. G. Byer, in *Nonlinear Optics*, edited by P. G. Harper and B. S. Wheerett (Academic, New York, 1977); I. Warner, in *Quantum Electronics* edited by H. Rabin and C. C. Tang (Academic, London, 1975), Vol. I, Part B.

[2]W. H. Louisell, *Radiation and Noise in Quantum Electronics*, (McGraw-Hill, New York, 1964).

[3]J. Tucker and D. F. Walls, Ann. Phys. (N.Y.) 52, 1 (1969); J. Tucker and D. F. Walls, Phys. Rev. 178, 2036 (1969).

[4]E. Y. Lu, Phys. Rev. A 8, 1053 (1973).

[5]B. Crosignani, U. Ganiel, S. Solimeno, and A. Yariv, Phys. Rev. Lett. 27, 237 (1971); B. Crosignani, P. Di Porto, U. Ganiel, S. Solimeno, and A. Yariv, IEEE J. Quant. Electron. 8, 731 (1972).

[6]S. Kryszewski and J. Chrostowski, J. Phys. A 10, L261 (1977).

[7]W. J. Mielniczuk, Opt. Acta 26, 1115 (1979).

[8]A. T. Georges, P. Lambropoulos, and P. Zoller, Phys. Rev. Lett. 42, 1609 (1979); P. Zoller, Phys. Rev. A 19, 1151 (1979); M. Raymer, J. Mostowski, and J. Carlsten, *ibid.* 19, 2304 (1979); J. Mostowski, and K. Rzazewski, Z. Phys. B 39, 183 (1980); J. L. F. de Meijere and J. H. Eberly, Phys. Rev. A 17, 1416 (1978); K. Wódliewicz, *ibid.* 19, 1686 (1979); J. Mostowski and M. Raymer (unpublished); for review see, *Multiphoton Processes*, edited by J. H. Eberly and P. Lambropoulos (Wiley, New York, 1978).

[9]J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968).

[10]W. E. Boyce, in *Probabilistic Methods in Applied Mathematics*, edited by A. T. Bharucha-Reid (Academic, New York, 1968).

[11]R. F. Fox, J. Math. Phys. 8, 1196 (1972).

[12]R. F. Fox, Phys. Rep. 48, 179 (1978).

[13]G. Agarwal, Phys. Rev. A 18, 1490 (1979).

[14]L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975).

[15]D. F. Walls and R. Barakat, Phys. Rev. A 1, 440 (1970); D. F. Walls, in *Quantum Optics*, edited by S. M. Kay and A. Maitland (Academic, London and New York, 1970).

[16]Ref. 12, pp. 243 and 244.