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Limits of superradiance as a process for achieving short pulses of high energy

J. C. MacGillivray* and M. S. Feld

Department of Physics and Spectroscopy Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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This paper augments previous theoretical descriptions of superradiance in an extended optically thick medium by giving expressions which limit the length and density of a high-gain system which can superradiate as a whole, or equivalently, the shortness of its output radiation pulses. Limits arising from experimental conditions such as finite inversion time, finite decay and dephasing times, finite transit time (cooperation length), feedback, diffraction, and Fresnel number not approximately one, are discussed. Modifications to the simple analytical expressions for the output radiation of the superradiant system due to each of these effects are described in detail. Detailed computer results are also given.

I. INTRODUCTION

The spontaneous emission rate of an assembly of N atoms or molecules can be much greater than that of a set of isolated atoms, due to mutual cooperation via their common radiation field.[1] In this effect, called superradiance, much of the stored energy is rapidly released by collective radiative damping. Superradiance is the optimal process for extracting coherent energy from an inverted system.

Superradiance has been observed in the far infrared (in HF^{2-4} and CH_3F^{5-7}), in the near infrared (in Na,[8,9] Tl,[9,10] Cs,[9,11-16] Li,[17] and Rb[9,18,19]), and recently in the visible (in Sr and Eu[19]) and submillimeter (in Cs and Na[20]). In all of the experiments a long (compared to λ), optically thick sample of N two-level atoms or molecules is prepared in the excited state, inverted indirectly so as to create a complete inversion between the two levels of interest. Feedback is absent, and there are no mirrors. After a sizable delay during which the system evolves into a superradiant state, it emits coherent radiation, a sequence of events sometimes referred to as superfluorescence.[21] In this process incoherent emission induces a small macroscopic polarization in the inverted medium which gives rise to a growing electric field and consequently an increasing polarization in space and time. After a long delay, a highly directional pulse of peak output power $\propto N^2$ is emitted, often accompanied by ringing.

In a previous paper[22] we presented a semiclassical theory of superradiance in an extended, optically thick medium formulated in terms of coupled Maxwell-Schrödinger equations which could be integrated numerically, and gave simple analytical expressions for observable output parameters. These expressions show that in a high-gain system where all decay and dephasing times are much longer than the collective radiation damping time, the peak intensity $\propto N^2$ and the pulse width $\propto 1/N$, opening the possibility of producing very short, high-intensity pulses. This paper considers features present in any realistic system, such as finite inversion and transit times, and diffraction and other geometrical effects, which limit the peak intensity and pulse shortness of a superradiant system. Other limiting features such as finite decay time and feedback are also discussed, as well as the effect of the interaction of forward and backward traveling waves. Both swept and uniform inversion configurations are considered. In addition to the computer results, analytical expressions are given for maximum length, maximum output power, and maximum energy in a superradiant pulse for several cases of interest. These limits should be of particular interest to those attempting to observe superradiance in other systems, and are relevant to the problem of x-ray laser system design[23,24] and ultrashort pulse generation.[25] Conditions which distinguish the regimes of superradiance and self-induced transparency in directly inverted two-level systems are also discussed.

The remainder of this paper is organized as follows. Section II: Simple expressions in the ideal limit. Section III: Experimental conditions which limit the length of a superradiant system. A. Inversion time. B. Loss. C. Uniform inversion: Cooperation length. Section IV: Other conditions needed to obtain superradiance. A. Decay and dephasing times. B. Feedback. C. Forward-backward wave interaction. D. Initial polarization at the superradiant transition. Section V: Deviations from plane-wave behavior. A. The plane-wave approximation. B. Small Fresnel number. C. Large Fresnel number. D. Initial nonuniform cross section. Section VI. Summary.

II. SIMPLE EXPRESSIONS IN THE IDEAL LIMIT

In the previous theoretical analysis of superradiance[22] the semiclassical approach (classical fields, quantized molecules)[1-3,26-32] was chosen, since it implicitly includes propagation effects. The coupled Maxwell-Schrödinger equations in the slowly varying envelope approximation [Eq. (11), Ref. 22] can be solved numerically to determine the behavior of any superradiant system which is consistent with the assumptions used to derive these equations: (1) the semiclassical model with a polarization source term to simulate spontaneous emission can be used instead of a quantized field model; (2) the plane-wave approximation is valid, i.e., effects associated with finite beam diameter can be neglected; and (3) the interaction of forward and backward traveling waves is negligible. These assumptions and their implications are discussed in detail in Ref. 25.

Although computer solutions of these Maxwell-Schrödinger equations should be used for precise comparisons with experimental data, approximate analytical solutions which are in close agreement with the computer results can be obtained in certain limiting cases. These results are useful in estimating relevant experimental parameters and as an aid to understanding the underlying physical processes.

In the "ideal superradiance" limit the following additional assumptions are made: (1) the system is inverted by a pulse traveling longitudinally through the medium at the speed of light (swept inversion), (2) it is inverted instantaneously, (3) it is prepared with no initial polarization at the superradiant transition, (4) all decay and dephasing rates are negligible, (5) there is no loss, (6) there is no feedback, (7) the system is nondegenerate, and (8) the polarization source term used to simulate spontaneous emission can be replaced by an equivalent step function input electric field.[33] The justification of assumptions (7) and (8) is discussed in Refs. 22 and 25.

Assumption (4) is equivalent to assuming that the small signal field gain coefficient at the superradiant transition,

$$\alpha = \frac{4\pi^2}{\lambda}\ \frac{\mu_z^2}{\hbar}\, n_0 T_2' = \left(\frac{T_2'}{T_{sp}}\right)\left(\frac{n_0\lambda^2}{8\pi}\right), \tag{1}$$

is infinite, since T_2', the inversion linewidth of the transition, is infinite. In Eq. (1), n_0 is the initial inversion density, λ is the wavelength of the transition, μ_z is the dipole moment component parallel to the direction of polarization, L is the length of the system, and T_{sp} is the spontaneous lifetime of the upper level of the transition.

In this ideal case the output intensity depends on two parameters: T_R, the characteristic radiative damping time of the collective system, and ϕ, a logarithmic function of θ_0, the initial tipping angle of the Bloch vector[34] corresponding to the state of the system[35] and also the "area" of the step-function input electric field[33]:

$$T_R = T_{sp}(8\pi/n_0\lambda^2 L) = T_2'/\alpha L, \tag{2}$$

$$\phi = \ln(2\pi/\theta_0), \tag{3}$$

$$\theta_0 = \mu_z T_R\, \mathcal{E}(x=0, T)/\hbar, \tag{4}$$

where $\mathcal{E}(x,t)$ is the complex envelope of the electric field propagating in the x direction, and $T = t - x/c$ is the retarded time. All of the results of Secs. III and IV, except those of Sec. III C, assume swept excitation and are written in terms of retarded time.

For any given value of θ_0 a single curve relates $T_R^2 I_p$ to T/T_R (see Fig. 4 of Ref. 22), where I_p is the peak output power. As shown in Ref. 22 approximate expressions can then be derived for the peak output power

$$I_p \approx \frac{4N\hbar\omega}{T_R\phi^2} \propto N^2, \tag{5a}$$

the width of the output pulse

$$T_W \sim T_R\phi \propto N^{-1}, \tag{5b}$$

and the energy contained in the first lobe of emitted radiation

$$E_p \approx 4N\hbar\omega/\phi \propto N, \tag{5c}$$

where $N = n_0 A L$ is the total number of initially inverted atoms in a system of cross-sectional area A, and ω is the frequency of the superradiant transition. The delay time from the inversion to I_p is

$$T_D \approx T_R\phi^2/4 \propto N^{-1}, \tag{5d}$$

so that $T_D \sim T_W\phi/4$. Equation (5d) was originally derived[2,3] in the semiclassical framework; the accuracy of this approximation has recently been verified by an analysis based on a series-expansion solution to the sine-Gordon equation.[36] This expression has also been derived from the quantized field treatment of Ref. 37, with $\phi = \ln(2\pi N)^{1/2}$.

Expressions for θ_0 and ϕ have been derived from both quantum and semiclassical considerations. Recently, Polder, Schuurmans, and Vrehen[37,38] and Glauber and Haake *et al.*[39-41] have developed quantized field treatments which analyze the initial stage of the superradiant emission proccess in an extended medium. These theories allow for spatial variations of the field amplitudes, and thus fully include propagation effects, unlike some earlier theories[42,43] which failed to include the spatial field amplitude variations which occur in any superradiant system.[22] They show that the initial fluctuations which trigger the superradiant pulse can be treated stochastically, using a fluctuating field source, a polarization source, or both, and confirm the validity of the semiclassical treatment of the subsequent behavior of the system.

The average initial tipping angle θ_0, calculated from either the Glauber-Haake polarization-fluctuation model or the Polder-Schuurmans-Vrehen field-fluctuation model, is

$$\theta_0 = 2/\sqrt{N}, \tag{5e}$$

$$\phi = \ln(\pi\sqrt{N}), \tag{5f}$$

so that typically $10 < \phi < 20$. Bonifacio and Lugiato[21] also obtained this result. (The more detailed calculation of Ref. 37 gives

$$\theta_0 = 2[\ln(2\pi N)^{1/8}/N]^{1/2}, \tag{5g}$$

which leads to an only slightly different value of ϕ.)

A somewhat different expression for ϕ has been given by MacGillivray and Feld,[22] using the semiclassical description and evaluating the amplitude of a polarization source term from detailed balance considerations similar to those used to derive the ratio of the Einstein A and B coefficients. For an initially totally inverted system, as in the experiments, this leads to an initial tipping angle

$$\theta_0 \approx N^{-1/2}(2\pi)^{-1/4}(\alpha L)^{-3/4}, \tag{5h}$$

$$\phi \approx \ln[N^{1/2}(2\pi)^{5/4}(\alpha L)^{3/4}], \tag{5i}$$

so that typically $10 < \phi < 25$. For $\lambda \gtrsim 50$ μm, N must be replaced by $N[1 + (e^{\hbar\omega/kT} - 1)^{-1}]$.[22]

For typical values of N, Eqs. (5f) and (5i) only differ by ~20%. Hence, it is difficult to decide experimentally between the two. A recent experiment by Vrehen and Schuurmans[14] has measured the effective tipping angle θ_0 using two cesium cells inverted in tandem by a pulse from a pump laser. An infrared attenuator placed between the cells permits a coherent pulse of small variable area θ, resonant with the superradiating transition, to be injected into a second, lower pressure cell. Studies of the delay time of the superradiant output from the second cell support the θ_0 expressions [Eqs. (5e) and (5g)] of Glauber and Haake[39-41] and Polder, Schuurmans, and Vrehen,[37,38] and give a value of θ_0 which differs by at least one order of magnitude from that of Eq. (5h). While this experiment does measure θ_0 directly, thus avoiding the logarithmic dependence entailed in measuring pulse delays, an accurate measurement of N is still required for comparison with theory.[44]

Other expressions for θ_0 and T_D are reviewed in Ref. 37. Analyses of pulse-to-pulse variations in the time delays of the output radiation caused by quantum fluctuations are given in Refs. 37 and 41. An expression similar to that of Ref. 41 was obtained earlier by Degiorgio.[45]

From Eqs. (5a)–(5d) it would appear that increasing the length or inversion density indefinitely would result in arbitrarily short, high-intensity pulses. The remainder of this paper examines effects which cause deviations from this ideal behavior, first those of particular importance in long systems and then those not related to length. Analytical expressions are derived and compared to computer results obtained by numerically integrating Eqs. (11) of Ref. 22, which include the randomly fluctuating polarization source.

III. EXPERIMENTAL CONDITIONS WHICH LIMIT THE LENGTH OF A SUPERRADIANT SYSTEM

As mentioned previously, the simple expressions derived in the ideal limit are useful in estimating experimental parameters and in understanding the underlying physical processes. In this section these formulas are examined for several cases of experimental interest in which the assumptions of the ideal limit are no longer satisfied. Wherever possible, as an aid to understanding, the results of computer solutions of the Maxwell-Schrödinger equations have been expressed in terms of simple formulas similar to those derived in the ideal limit.

A. Inversion time

Consider the case in which the process which populates the upper level of the superradiant transition occurs over a finite time τ, rather than instantaneously (dashed lines, Fig. 1). Computer results using the exact Maxwell-Schrödinger equations show that as long as this process is completed before the first superradiant pulse is emitted, the effect on the output is small [Fig. 1(b)], other than to increase the delay time [Eq. (6), below].

In the simple case where the inversion process creates population at a constant rate over a time $\tau < T_D$, Eq. (5d) still holds if T_D is measured from the midpoint in time of the excitation process [Fig. 1(b)]. The actual delay time T_D', measured in retarded time from the *start* of the inversion process, is greater than T_D [Eq. (5d)]: $T_D' \approx T_R\phi^2/4 + \tau/2$. In this case the inversion density which would exist in the absence of stimulated emission, $n^*(t)$, reaches a constant value which can be used in place of n_0 in Eqs. (2) and (5).

In the more general case where $n^*(t)$ does not grow at a constant rate, but still reaches a constant value before the output pulse is emitted (i.e., $\tau \lesssim T_D'$), a similar expression for the actual delay time T_D' can be written

$$T_D' \approx T_R\phi^2/4 + \tau f(\tau) \quad (\tau < T_D'), \tag{6}$$

where $\tau f(\tau)$ is the averaged additional delay,

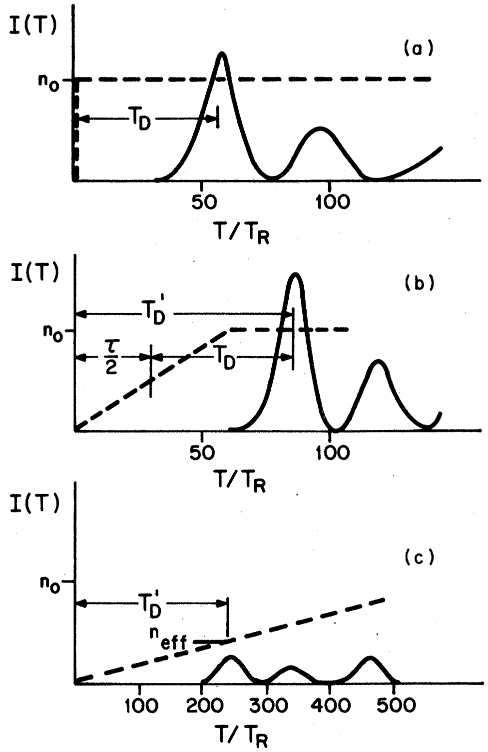

FIG. 1. Effect of changes in τ on $I(T)$. In all figures a system with parameters similar to those of Refs. 11–13 is used: unless otherwise indicated $n_0 = 5\times10^{10}$ cm^{-3}, $L = 2$ cm, $T_R = 0.5$ ns, and $\phi\approx 15$. The solid lines indicate output intensity in arbitrary units as a function of retarded time (same scale throughout). The dashed lines indicate $n^*(t)$, the total inversion density created up to that time: For purposes of illustration, a value of the polarization source term corresponding to that used to derive Eqs. (5h) and (5i) was used to generate all of these figures. (a) Instantaneous inversion ($\tau = 0$). (b) $\tau\approx T_D$ ($\tau = 60$ ns, $T_D\approx 57$ ns). Note that $T_D' = T_D + \tau/2$. (c) $\tau \gg T_D$ ($\tau = 600$ ns). Note the change in horizontal scale. Only a fraction $n_{eff}/n_0\approx 0.4$ of the atoms can contribute to the first lobe of radiation.

weighted over $n^*(t)$ for $t<\tau$:

$$f(\tau)\equiv 1-\frac{\langle n^*(t)\rangle}{n^*(\tau)}$$

$$=1-\left(\int_0^\tau n^*(t)dt\right)\Big/\tau n^*(\tau). \qquad (7)$$

If $n^*(t)=\Lambda t^{\beta-1}$, Λ constant and $\beta>1$, then $f=1-1/\beta$, and $0<f\leq 1$. Computer results confirm Eq. (6).

For sufficiently long systems, however, the observed delay time becomes shorter than the duration τ of the inversion pulse. Such a system will radiate before it has been fully inverted and will no longer be able to radiate all of its stored energy in one coherent series of bursts of radiation [Fig. 1(c)]. Here only $n^*(t)$ up to T_D' can affect the first lobe of output, so that $n_{eff}=n^*(t=T_D')$ must be used in place of n_0 in Eqs. (2) and (5). With τ replaced by T_D', Eq. (6) becomes

$$[1-f(T_D')]T_D'\approx(T_{sp}\phi^2/4)(8\pi/n_{eff}\lambda^2L). \qquad (8)$$

For any given $n^*(t)$ this equation can be solved for T_D' and then n_{eff} which can be used in Eqs. (5).

The maximum length of an efficient system, L_1, occurs when $T_D'=\tau$. If $n^*(t)=\Lambda t^{\beta-1}$, then for a given value of τ,

$$L_1=\frac{2\pi T_{sp}\phi^2\beta}{\lambda^2\Lambda\tau^\beta}=\frac{2\pi T_{sp}\phi^2}{n_0\lambda^2\tau[1-f(\tau)]}$$

$$=\frac{\phi^2(T_2'/\tau)}{4\alpha[1-f(\tau)]}. \qquad (9a)$$

In experiments up to now L_1 has typically been 10–100 times L (Table I). Equivalently, Eq. (9a) can be rewritten as a condition limiting the time τ during which an efficient system of length L is inverted:

$$\tau<\frac{\phi^2/4}{1-f(\tau)}\,\frac{T_2'}{\alpha L}. \qquad (9b)$$

For systems longer than L_1, solving Eq. (8) for $n_{eff}=n^*(T_D')=\Lambda T_D'^{(\beta-1)}$ gives

$$n_{eff}\approx\Lambda[n_x\tau/\Lambda]^{(1-1/\beta)}\propto\Lambda^{1/\beta}, \qquad (10)$$

and

$$T_D'=[n_{eff}/\Lambda]^{[1/(\beta-1)]}=[n_x\tau/\Lambda]^{1/\beta}\propto\Lambda^{-1/\beta},$$

where

$$n_x\equiv 2\pi T_{sp}\phi^2\beta/\lambda^2L\tau=n_{eff}T_D'/\tau.$$

Using this value of n_{eff} in Eq. (5a) gives

$$I_p\approx\beta\hbar\omega AL\Lambda(n_x\tau/\Lambda)^{(1-2/\beta)}. \qquad (11)$$

As an example, for $\beta=2$ [i.e., $n^*(t)=\Lambda t$] $I_p\approx 2\hbar\omega AL\Lambda$. Note that in this case $I_p\propto\Lambda^{2/\beta}$, not Λ^2, but N^2 radiation still occurs because the effective number of radiators $N=n_{eff}AL\propto\Lambda^{1/\beta}$. However, although I_p increases with increasing Λ, the efficiency E_p/E_{inc} (where $E_{inc}=\Lambda\tau^{(\beta-1)}\hbar\omega$ is the energy needed to invert the system) decreases: $E_p/E_{inc}=(4/\phi)(T_D'/\tau)^{(\beta-1)}$, instead of $4/\phi$ in the ideal case. This agrees with the computer results of Fig. 2. Equation (11) shows that for $L>L_1$, $I_p\propto L^{2/\beta}$, not L^2 (Fig. 2).

When the pumping process is very long, the system will only emit a few bursts of radiation of this type. As the system continues to radiate, the inversion density is replenished as fast as it is depleted, leading to a quasisteady state with $I\approx\hbar\omega AL[dn^*(t)/dt]$. Some evidence for the transition of such a system from its transient, n^2-dependent superradiant behavior to its steady state has been observed in Na by Gross *et al.*[8] This evolu-

TABLE I. Typical experimental parameters for several observations of superradiance, and calculated values of ϕ [Eq. (5h)], L_1 [Eq. (9)], and L_c [Eq. (24)]. The calculated value of L_1 assumes that Λ is constant over a time τ. It should be noted that the Na system is near the $\alpha L \sim \phi$ limit [Eq. (21)], that CH_3F is strongly affected by T_1 as discussed in Sec. IV A and Ref. 5, and that the CH_3F system is much closer to the L_1 and L_c limits than is any of the other systems.

	CH_3F	HF	Na	Cs	Tl
References	5–7	2–4	8	11–13	10
λ	496 μm	84 μm	3.4 μm	2.9 μm	1.3 μm
L	600 cm	100 cm	14 cm	2 cm	15 cm
T_2^*	(T_1= 60 ns)	220 ns	1.7 ns	32 ns	1 ns
n_0 (cm^{-3})	3×10^{12}	10^{12}	2×10^{10}	4×10^{10}	2×10^{15}
τ	65 ns	100 ns	2 ns	2 ns	5 ns
T_D	100 ns	400 ns	7 ns	12 ns	12 ns
T_R	0.3 ns	5 ns	0.2 ns	0.7 ns	0.05 ns
αL	200	45	9	50	20
ϕ (calculated)	25	17	13	14	24
L_1 (calculated)	900 cm	700 cm	120 cm	70 cm	40 cm
L_c (calculated)	1300 cm	1500 cm	120 cm	60 cm	80 cm

tion is an interesting problem which merits further study.

It should be noted that the results of this section apply to both uniform inversion, where the entire length of the sample is inverted simultaneously, and swept inversion. For a uniformly inverted system in which the inversion pulse crosses the medium transversely (as would be the case in some proposed x-ray laser schemes[46]), the transit time of this pulse could become comparable to T_D' and would then have to be taken into account when calculating $n^*(t)$.

B. Loss

The presence of loss can be important even in the case of infinite gain, $\alpha L = T_2'/T_R \gg 1$ [Eq. (2)].[3,22,26,28,47] This loss will be considered in two cases: (1) linear loss, where the loss coefficient $\kappa = (\partial \mathcal{E}/\partial x)_{\rm loss}$ [Eq. (11a), Ref. 22] is constant throughout the medium and (2) diffraction of a Gaussian beam, where[48]

$$\kappa(x) = x/(x^2 + L_0^2), \quad L_0 = A/\lambda. \tag{12}$$

Effects of large linear loss have been described previously.[26,28,47,49] Here we extend this analysis, discuss its region of applicability, and illustrate results in the intermediate regime.

(1) A large constant κ can be significant even for very high-gain systems (see Fig. 3). For large values of κL, $\mathcal{E}$ (as a function of retarded time T) becomes independent of x. References 26, 28, and 47 derive the asymptotic expression for this "steady-state" regime,

$$\mu_z \mathcal{E}(T)/\hbar = (1/\kappa L T_R)\operatorname{sech}[(T - T_0)/\kappa L T_R] \quad (T \geq 0) \tag{13}$$

which describes a single pulse of area

$$\theta = \mu_z \int_{-\infty}^{\infty} \mathcal{E}(T)dT/\hbar = \pi .$$

Here T_0 can be determined from the value of $\mathcal{E}$ at retarded time $T = 0^+$.

As shown in Ref. 22 an appropriate input condition to model spontaneous emission is a constant (step-function) input field of amplitude $\mathcal{E}_0 = \hbar\theta_0/\mu_z T_R$.[33] Using this boundary condition in Eq. (13) gives

$$(\mu_z \mathcal{E}_0/\hbar)\kappa L T_R = \operatorname{sech}(T_D/\kappa L T_R), \tag{14}$$

so that[50]

$$T_D = \kappa L T_R \operatorname{sech}^{-1}(\kappa L \theta_0), \tag{15}$$

where T_D has been used in place of T_0 since the peak output intensity occurs at T_0. When $\kappa L\theta_0 \ll 1$, as in the HF experiments where $\theta_0 \sim 10^{-8}$, $\operatorname{sech}^{-1}(\kappa L\theta_0) \approx \ln(2/\kappa L\theta_0) = \phi - \ln(\pi\kappa L)$, so that

$$T_D/T_R \approx \kappa L[\phi - \ln(\pi \kappa L)] \sim \kappa L \phi . \tag{16}$$

Equation (16) predicts a longer delay time than in the ideal limit [Eq. (5d)] unless

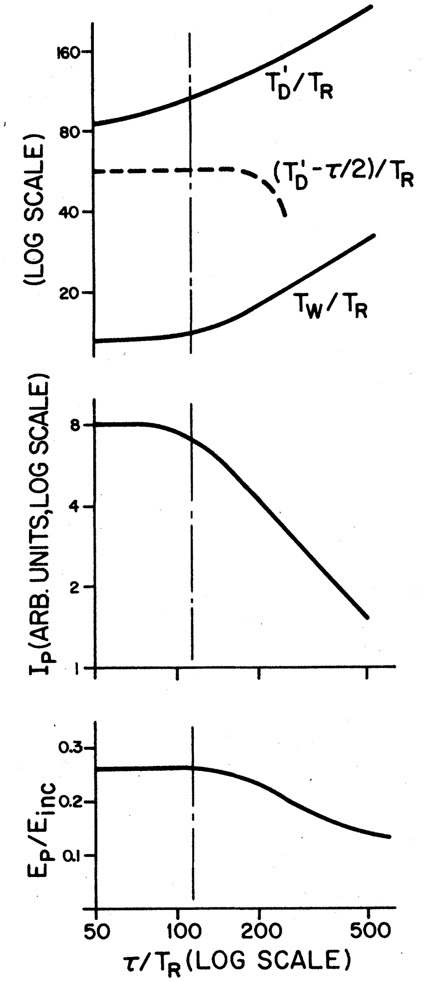

FIG. 2. Effect of changes in τ on T'_D/T_R, T_w/T_R, I_p, and $E_p/E_{\rm inc}$ for $n^*(t)=\Lambda t$, i.e., $\beta=2$. Theory predicts a breakpoint at $\tau=2T_D\approx 115\ T_R$ (vertical broken line), which is consistent with the data. Note that for large τ, $T'_D \propto T_R^{1/2}$, $T_w \propto T_R^{1/2}$, $I_p \propto T_R^{-1}$, and $E_p/E_{\rm inc} \propto T_R^{-1/2}$, in agreement with expressions given in the text.

$$\kappa L \lesssim \kappa L_2 = \phi/4. \tag{17}$$

As can be seen in Fig. 4, as κL increases from zero the ratio T_D/T_R immediately deviates from its ideal value. However, this behavior is best described as a very slow approach to two asymptotic expressions [Eqs. (5d) and (16)]. $L=L_2$, the crossover point between these two expressions, is therefore a reasonable approximation for the point at which the ideal limit breaks down and large κL effects start to dominate (Fig. 4).

Similar values for L_2 can be derived by examining I_p and T_W for the sech^2 pulse of Eq. (13) at $T=T_0$ $(=T_D)$:

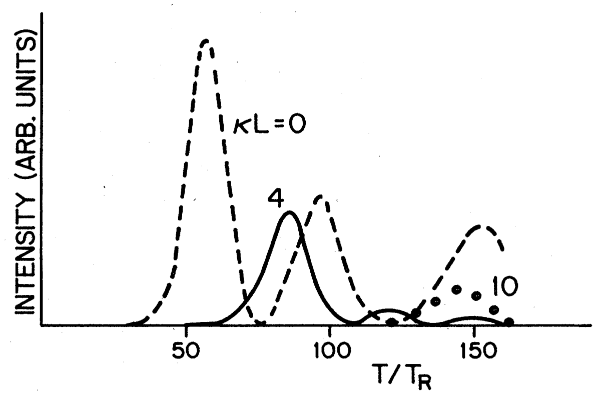

FIG. 3. Effect of changes in κL on output intensity. All parameters of the dashed curve (---) are as in Fig. 1(a). In the solid curve (——) $\kappa L=4$, and in the dotted curve (• • •) $\kappa L=10$.

$$I_p=(cA/8\pi)\mathcal{E}^2=(cA\hbar^2/8\pi\mu_z^2)(1/\kappa L T_R)^2$$
$$=N\hbar\omega/4T_R(\kappa L)^2, \tag{18}$$

which is independent of L, but still proportional to n_0^2. Comparison with Eq. (5a) shows that the crossover point is again $L_2=\phi/4\kappa$. The full width at half maximum (FWHM) of the pulse of Eq. (13) is $T_W\approx 1.8\ \kappa L T_R$. Comparison with Eq. (5b) gives the slightly different crossover point $0.55\ \phi/\kappa$, which has the same dependence on κ and ϕ.

We therefore see that the large-κL expressions start to become better approximations than the small-κL "ideal formulas" near $L_2=\phi/4\kappa$. This is illustrated by the computer results shown in Fig. 4 in which T_D/T_R is plotted as a function of κL. A similar conclusion is obtained from a computer plot of I_p vs κL.

Equation (17) shows that $\kappa L \gtrsim \phi/4$ is a necessary condition in order to reach the steady-state limit.

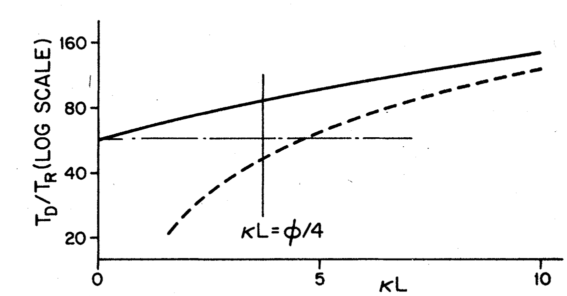

FIG. 4. Effect of changes in κL on T_D/T_R. All parameters of the system are as in Fig. 1(a) except that κL is allowed to vary from 0 to 10. The dotted line (---) is the large κL expression for T_D/T_R [Eq. (16)]. The broken line (•-•-•-) is the ideal limit ($\kappa L=0$) expression for T_D/T_R [Eq. (5c)]. T_D/T_R (solid line) slowly approaches the large κL limit for κL greater than $\phi/4$ (vertical line).

In the special case where $T_1 = T_2$ (i.e., T_1 is the dominant decay process, as discussed in Sec. IV A), Bonifacio *et al.*[47] have derived the additional condition that α/κ must be $\gtrsim \phi$ in order for superradiance to occur. It should be noted that combining these two conditions gives $\alpha L \gtrsim \phi^2/4$, a condition which is not only applicable to the case of large κL, but is also a general condition on T_1/T_R, as shown in Sec. IV A [Eq. (29c)].

Another condition on the length follows from the area theorem,[51] derived for systems with arbitrary T_2^* and negligible decay:

$$d\theta/dx = -\kappa\theta + \alpha\sin\theta, \tag{19}$$

where the area

$$\theta(x) = (\mu_z/\hbar)\int_{-\infty}^{\infty} \mathcal{E}(x,T)dT,$$

and where the gain coefficient α is defined in Eq. (1). To achieve full superradiance a small input pulse (e.g., $\theta_0 \sim 10^{-8}$) must grow to area $\sim\pi$. Since $\theta \ll 1$ throughout most of the pulse evolution $\sin\theta \sim \theta$ and integration of Eq. (19) gives

$$\ln(\theta/\theta_0) = -\mathfrak{L} + \alpha L, \tag{20}$$

where the total loss $\mathfrak{L} = \kappa L$ when κ is constant. Thus, in order for θ to grow to $\sim\pi$,

$$\alpha L - \mathfrak{L} > \phi. \tag{21}$$

This requirement for high gain is not sufficient, however, since in most cases Eq. (29a) below is a far more stringent condition than Eq. (21).

Equation (21) requires that $\alpha L > \phi$ for full superradiance, which implies that the duration of a superradiant pulse, $T_W \approx T_R\phi$, must always be shorter than the inverse bandwidth of the transition, $T_2' = T_R\alpha L$, i.e., $T_W < T_2'$. This relationship is important to the understanding of the transient nature of the superradiant process.[52]

(2) When the electric field is best modeled as a Gaussian beam, Eq. (12), computer results show that only the total loss $\mathfrak{L} = \int_0^L \kappa(x)dx$ is important, so that κ proportional to $x/(x^2 + L_0^2)$ and constant κ give virtually the same output for the same total loss. Equation (17) can therefore be generalized to

$$\mathfrak{L} = \int_0^L \kappa(x)dx \lesssim \phi/4. \tag{22}$$

For a Gaussian beam, $\mathfrak{L} = \frac{1}{2}\ln[1 + (L/L_0)^2]$, which will always be negligible for Fresnel number $2A/\lambda L \gtrsim 0.1$. For smaller Fresnel number, other considerations discussed in Sec. V B become important.

C. Uniform inversion: Cooperation length

In this section the case of uniform inversion is considered. As noted by Arecchi and Courtens,[29] in this case atoms near $x = L$ (the output end for the forward traveling wave) start to superradiate before radiation due to atoms near $x = 0$ can reach them. For short systems computer results show that the only effect is to increase the observed delay time by $L/2c$ (Fig. 5), the average transit time. For long systems, however, the output end can start to superradiate before radiation from $x = 0$ reaches the output face. The medium stops radiating as a whole when the transit time $T_{tr} = L/c$ equals the observed delay time $T_D + T_{tr}/2$. This defines a "cooperation length" L_c,

$$L_c/c \approx T_R\phi^2/4 + L_c/2c, \tag{23}$$

where T_R [Eq. (2)] is evaluated at $L = L_c$. Then

$$L_c \approx \phi(4\pi c T_{sp}/n_0\lambda^2)^{1/2}, \tag{24}$$

which is 5–30 times the system length in most experiments to date (Table I). As can be seen in Fig. 6, longer systems will break up into a num-

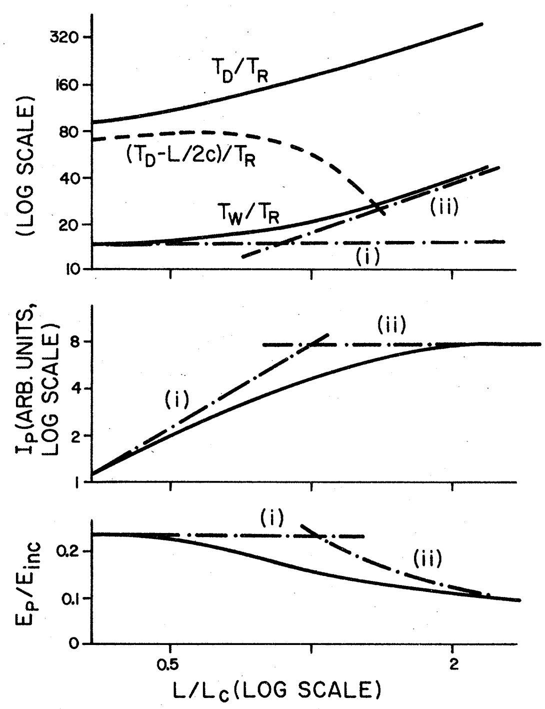

FIG. 5. Effect of changes in L on T_D/T_R, T_w/T_R, I_p, and E_p/E_{inc} in a *uniformly* inverted system otherwise the same as in Fig. 1. L_c is calculated to be $\approx$58 cm [Eq. (21)]. (i) Asymptotic limits for $L < L_c$ ("ideal formulas"). Note that $(T_D - L/2c)/T_R$ (dashed line) is approximately constant in this regime. (ii) Asymptotic limits for $L > L_c$.

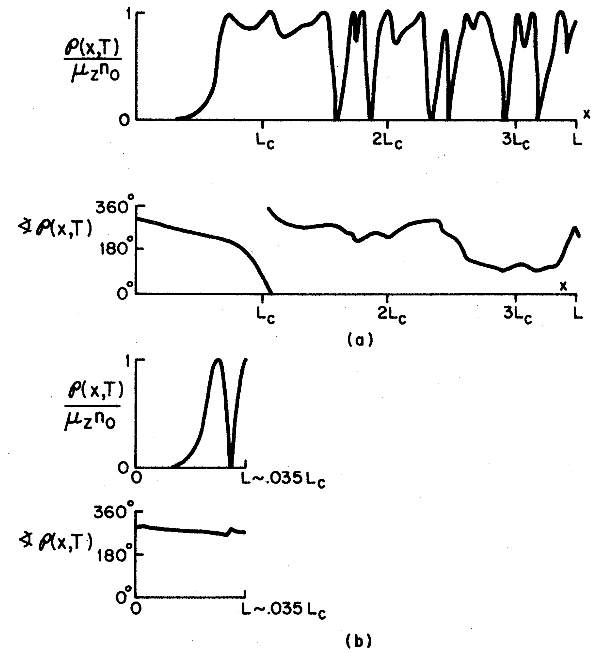

FIG. 6. Amplitude and phase of the polarization envelope density $\mathcal{P}(x,t)$ in *uniformly* inverted systems as a function of x, illustrating breakup of medium in long samples. The phase has been shifted in the diagram by 180° every time $|\mathcal{P}(x,t)|$ goes through zero in order to increase clarity. (a) $L = 200$ cm $\sim 3.5\, L_c$; otherwise as in Fig. 5. Here $t = 3.5$ ns, just after the first lobe of output radiation has been emitted [the time behavior of system (a) is shown in Fig. 7]. Note the rapid phase changes. (b) $L = 2$ cm $\sim 0.035\, L_c$; otherwise as in Fig. 5. Here $t = 70T_R = 35$ ns, again just after the first lobe of output radiation has been emitted.

ber of independently radiating segments in a manner described by Arecchi and Courtens,[29] although as discussed below, the cooperation length of Eq. (24) differs from theirs, especially for $\phi \gg 1$.

The results of Figs. 5–7 were obtained by computer simulation using the distributed polarization source described in Sec. II. When analyzing a long uniformly inverted system, a distributed source *must* be used, as opposed to an initial boundary condition, since polarization and the electric field are building up in various regions of the medium simultaneously.

Figure 6 illustrates the amplitude and phase of the polarization density envelope $\mathcal{P}$ of the forward wave as a function of distance into the medium. As can be seen, for samples much larger than L_c sharp variations in both amplitude and phase set in. Notice that in a long sample the phase remains constant only over regions of length $\sim L_c$.

In uniformly inverted systems a backward traveling wave of approximately equal intensity will also be present. However, as discussed in Sec.

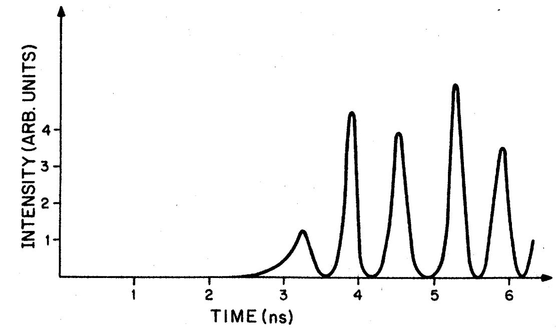

FIG. 7. Output intensity $I(T)$ of a uniformly inverted system of length $L = 200$ cm $\simeq 3.5\, L_c$ and otherwise the same as in Fig. 1. The transit time is 6.67 ns. The corresponding $I(T)$ for a much shorter system ($L \ll L_c$) is given in Fig. 3 of Ref. 22.

IV C, the interaction of the two traveling waves is negligible for cases of experimental interest.

Note that these arguments apply only to uniform inversion and not to the forward wave resulting from swept inversion, where the inversion pulse reaches all parts of the medium at the same retarded time. When swept inversion is used, as in all experiments up to now, transit-time effects are irrelevant, and all atoms can contribute coherently to the forward-wave output, regardless of the length of the sample. However, the backward wave can be affected by cooperation length effects. This has been observed by Ehrlich *et al.*[6] in the form of an increased ratio of forward-wave intensity to backward-wave intensity as length increases.[53]

For a uniformly inverted sample of length $L > L_c$ the output behavior deviates substantially from the ideal case. As shown in the computer results of Fig. 5, the percentage of input energy radiated in the first output lobe drops markedly, the ratio T_w/T_R increases, and I_p is no longer proportional to L^2. For $L = L_c$ the peak power of the first lobe is

$$I_p \approx \frac{4N\hbar\omega}{T_R\phi^2} = \frac{4n_0AL_c\hbar\omega}{T_{sp}(8\pi/n\lambda^2L_c)\phi^2} = 2n_0A\hbar\omega c\,. \qquad (25)$$

This expression also holds for $L > L_c$, since only the length L_c can contribute to the first lobe of the output radiation. However, the rest of the medium contributes to later output lobes, which as a result can be more intense than the first lobe (Fig. 7).

Our expression for L_c, Eq. (24), differs from that of Arecchi and Courtens,[29] $(4cT_{sp}/n_0\lambda^2)^{1/2}$, whenever $\theta_0 \ll 1$. They considered the large θ_0 regime ($\theta_0 \sim 1$), as would occur in a system prepared in a coherent superposition of states, and therefore assumed that the coherent decay process is completed after a time T_R rather than T_D. However, for an initially inverted medium $\theta_0 \ll 1$, so that $T_D \gg T_R$ and Eq. (24) applies.

IV. OTHER CONDITIONS NEEDED TO OBTAIN SUPERRADIANCE

A. Decay and dephasing times

In order for the ideal solutions of Sec. II to hold, the atomic decay and dephasing times must be long compared to the times which characterize the collective radiative decay. Specifically, two conditions must be met.

(1) The net gain, $\alpha L - \mathcal{L}$, must be large enough so that the total area of the output pulse can grow to $\sim\pi$. The condition $\alpha L - \mathcal{L} > \phi$ was derived above [Eq. (21)].

As mentioned previously, collective effects can occur in a limited sense even if Eq. (21) is not satisfied, as long as $T_R \ll T_{sp}$. In this regime of "limited superradiance",[22] defined by $T_R \ll T_{sp}$ and $\alpha L \ll 1$, only a small fraction of the stored energy is radiated coherently (since $T_2'/T_R = \alpha L \ll 1$). This regime includes such familiar effects as free induction decay[54, 55] and echoes.[56] In the intermediate regime $1 < (\alpha L - \mathcal{L}) < \phi$, the peak intensity will be much less than that given in Eq. (5a), and analytical results can be obtained from the linear theory of Crisp.[57] The output from an initially inverted system will become more directional as the gain is increased beyond one.[58]

(2) The homogeneous decay time T_1, due to spontaneous emission, collisions, and other mechanisms, must not be so short that incoherent decay becomes the dominant mode of deexcitation, which would prevent superradiance. The ideal limit holds when $T_1 \gg T_D$. If T_1 is comparable to T_D computer results show that Eqs. (5) can be made more accurate by using $n_0 e^{-T_D/T_1}$ in place of n_0 in Eqs. (2) and (5). In other words, a system with finite T_1 behaves approximately as if the population lost to T_1 decay up to time T_D had never been present. In the case of instantaneous inversion ($\tau = 0$) Eq. (5d) then becomes

$$T_D \approx \frac{8\pi T_{sp}}{n_0 e^{-T_D/T_1}\lambda^2 L}\,\frac{\phi^2}{4}\ , \tag{26}$$

or equivalently,

$$(T_D/T_1)e^{-T_D/T_1} \approx (T_{sp}/T_1)(8\pi/n_0\lambda^2 L)(\phi^2/4)\ . \tag{27}$$

In order for such a system to reach full superradiance, Eq. (27) must have a solution, which requires that the right-hand side be less than e^{-1}, the maximum of the function xe^{-x}. This gives

$$T_1 \gtrsim eT_{sp}(8\pi/n_0\lambda^2 L)(\phi^2/4)\ , \tag{28a}$$

which can also be written

$$T_1 \gtrsim eT_{D0}\ , \tag{28b}$$

where T_{D0} is the delay time which would exist if T_1 were infinite. This condition implies

$$\alpha L \gtrsim e(\phi^2/4)(T_2'/T_1)\ , \tag{29a}$$

$$n_0 L \gtrsim e(2\pi\phi^2/\lambda^2)(T_{sp}/T_1)\ . \tag{29b}$$

For a system where T_1 is the dominant mode of incoherent decay, Eq. (29a) becomes

$$\alpha L \gtrsim e\phi^2/4\ , \tag{29c}$$

a far more stringent requirement than Eq. (21) above. The effects of short T_1 are illustrated in the computer results of Fig. 8.

The effects of finite dephasing time T_2 and inhomogeneous broadening time T_2^* are different from the effect of finite T_1, in that high gain increases the effective T_2 and T_2^* to $T_2\alpha L$ and $T_2^*\alpha L$, respectively.[22] Since whenever $\alpha L > \phi$, both $T_2\alpha L$ and $T_2^*\alpha L$ are greater than T_D, there is no

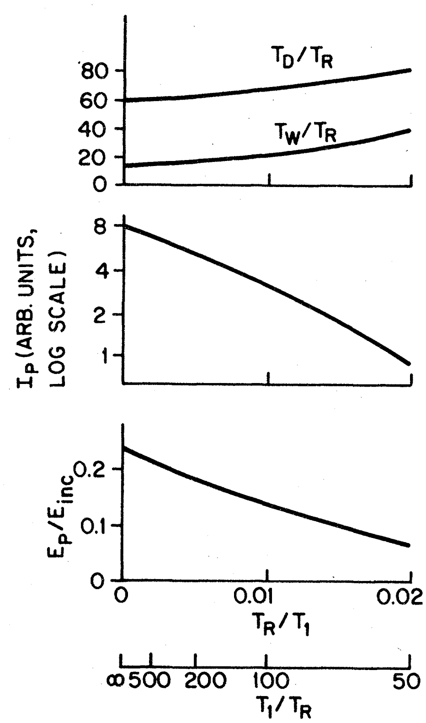

FIG. 8. Effect of changes in T_1 on T_D/T_R, T_w/T_R, I_p, and E_p/E_{inc}. All parameters are as in Fig. 1 (where $T_1 = 2\times10^9 T_R$) except T_1. The horizontal scale is linear in T_R/T_1. The data support the statement that n_0 should be replaced by $n_0 e^{-T_1/T_D}$ in Eqs. (2) and (5).

new condition on T_2 and T_2^* similar to that of Eq. (28b).

B. Feedback

The basic features of superradiance are not changed by the presence of feedback. However, the details of the radiation process may be modified. In particular, the delay time decreases and the amount of ringing increases (Fig. 9). This is because feedback tends to increase the effective length of the sample or, alternatively, to increase the initial tipping angle θ_0, a situation analogous to continuing to push an initially inverted pendulum after it has started to fall. An analogous problem, the radiative damping of an inverted NMR system in a resonant circuit, has been studied previously.[59]

In the HF experiments,[4,60] when feedback was deliberately introduced using mirrors superradiance was still observed, but with drastically reduced delay times. The pulse shapes and delays with feedback were similar to those observed in much higher density systems without feedback.

The effect of feedback on the output will be negligible as long as the output field $\mathcal{E}_1(t)$ due to the initializing spontaneous emission is significantly greater than the additional field $\mathcal{E}_F(t)$ which results from the feedback process. Consider a short (so that the transit time is negligible) high-gain system with a δ-function input electric field

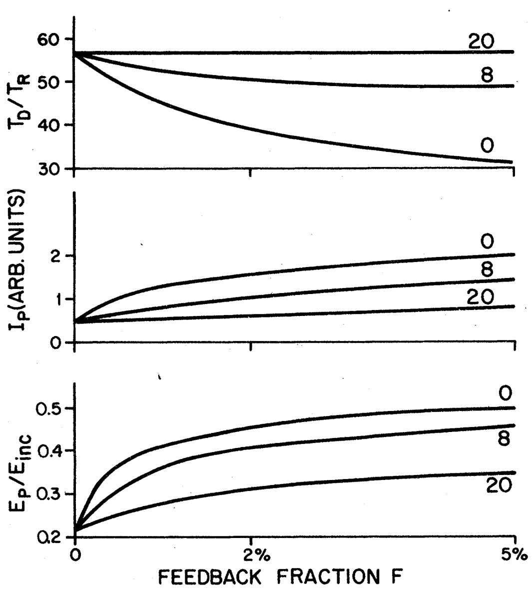

FIG. 9. Effect of feedback on T_D/T_R, I_p, E_p/E_{inc}. A fraction F of the output electric field is added, after a delay of 0, 8, or 20 T_R (compared to $T_D \approx 57\ T_R$ without feedback) to the input field of a system otherwise identical to that of Fig. 1.

of area $\theta_0 \ll 1$, in which a fraction F of the output at every time t is instantaneously added to the input. To obtain a simple upper limit of F for which feedback is negligible, consider the approximation in which a fraction F of only $\mathcal{E}_1(t)$, not the entire output, is added to the input. If $\mathcal{E}_2(t)$, the additional output due to this additional input $F\mathcal{E}_1(t)$, is much smaller than $\mathcal{E}_1(t)$, then the total additional field $\mathcal{E}_F(t)$ resulting from all feedback will also be negligible.

During most of the evolution process the pulse is weak, and the small area solution of Crisp[61] is valid:

$$\left(\frac{\mu_z}{\hbar}\right)\mathcal{E}_1(t)=\frac{\theta_0 I_1[2(t/T_R)^{1/2}]}{(tT_R)^{1/2}}\ , \tag{30}$$

where $I_1(x)$ is the modified Bessel function of order one. Similarly,[62]

$$\left(\frac{\mu_z}{\hbar}\right)\mathcal{E}_2(t)=\int_0^t dt'[(\mu_z/\hbar)F\mathcal{E}_1(t')] \times\left(\frac{I_1\{2[(t-t')/T_R]^{1/2}\}}{[(t-t')T_R]^{1/2}}\right)\ . \tag{31}$$

Inserting Eq. (30) into Eq. (31), evaluating[63] at $t=T_D$, and making the substitution $4t/T_R=\phi^2 \sin^2 u$, $4T_D/T_R=\phi^2$ gives

$$\left(\frac{\mu_z}{\hbar}\right)\mathcal{E}_2(t=T_D) \approx\left(\frac{2F\theta_0}{T_R}\right)\int_0^{\pi/2} I_1(\phi\cos u)I_1(\phi\sin u)du\ . \tag{32}$$

Using $I_1(x)\approx e^x/(2\pi x)^{1/2}$ for $x \gtrsim 1$, the integral is found to be approximately $e^{1.35\phi}/3\phi$, so that the condition for feedback to be negligible, $\mathcal{E}_2(t=T_D) \ll \mathcal{E}_1(t=T_D)$, becomes

$$(2F\theta_0/T_R)e^{1.35\phi}/3\phi \ll (2\theta_0/T_R\phi)e^{\phi}/(2\pi\phi)^{1/2}\ , \tag{33}$$

i.e.,

$$F \ll e^{-0.35\phi}\phi^{-1/2}. \tag{34}$$

Similar calculations can be made for systems in which the time required for radiation emerging from the output face of the sample to return to the input face via the feedback process is a significant fraction of T_D. In this case the additional delay reduces the influence of feedback (Fig. 9). Thus, the value of F for which feedback becomes important will be larger:

$$F \ll \frac{(1-T_x/T_D)^{1/2}}{e^{0.35\phi}\phi^{1/2}}\exp\{-1.35\phi[(1-T_x/T_D)^{1/2}-1]\}, \tag{35}$$

where T_x is the round trip transit time. For $\phi = 15$ and $T_x/T_D = 0.1$, this increases the maximum negligible feedback from $\sim 3\times10^{-3}$ to $\sim 10^{-2}$. Some computer results illustrating this point are given in Fig. 9.

As F gradually increases beyond the value allowed by Eq. (35), the delay time decreases and the peak power increases (Fig. 9). For F close to unity the feedback process in some ways resembles that of a laser amplifier, whose effective length, and consequently gain, is increased by a factor $\sim 1/(1-F)$. Thus, a single-pass system with $\alpha L=1$ and $F=0.95$ is to some extent similar to a system with $F=0$ and $\alpha L=20$. However, there are important differences. Owing to the small transmission $(1-F)$ associated with near-unity feedback, the system's energy recirculates and is released slowly, rather than in a few bursts. Furthermore, unlike a long single-pass system, the growth of the $\mathcal{E}$ field is influenced by the polarization which has built up in the medium during previous passes.

A superradiant system with feedback also differs from an ordinary laser in that the polarization cannot adjust instantaneously to the applied field. Consequently, the usual laser equations[64] do not apply.

C. Forward-backward wave interaction

In swept or uniform systems with small θ_0 and small transit time $(L/c \lesssim T_D)$, the interaction between forward and backward traveling waves is shown by computer analysis to be virtually negligible as far as affecting the first two lobes of output radiation. This is because for such systems the forward and backward waves only become sizable in the same region after much of the stored energy has been radiated. For short systems $(T_D \gtrsim L/c)$, as θ_0 becomes larger the fraction of stored energy which the system radiates in the first lobe increases, and the two waves begin to interact. Saunders, Hassan, and Bullough[31,32] have studied this interaction in the $\theta_0 \sim 0.1$ regime where the interaction tends to significantly increase delay times and decrease ringing.

To study this interaction, the forward-backward wave interaction was added to Eqs. (11) of Ref. 22 by writing separate $\partial\mathcal{E}/\partial z$ and $\partial\mathcal{P}/\partial T$ equations for the two waves and using a single $\partial n/\partial T$ equation combining $\mathcal{E}\mathcal{P}^*$ from both waves. Here $\mathcal{P}$ is the polarization density envelope as in Ref. 22. Separate solutions for the forward and backward waves were then iterated until convergence was reached. Computer results in short $(L \lesssim L_c)$ samples show that the interaction starts to become significant for the second lobe near $\theta_0 \sim 10^{-3}$, and for the first lobe near $\theta_0 \sim 10^{-2}$, values attainable only for relatively small inversion densities [Eq. (5e)]. As mentioned in Sec. II, all estimates of θ_0 for superradiance experiments reported to date give $\theta_0 \lesssim 10^{-2}$, so that the interaction of the two traveling waves should not be significant.

Longer systems could not be explored by computer due to the failure of this algorithm to converge. However, there is no reason to expect the forward-backward wave interaction to become significant in longer systems for small, experimentally relevant values of θ_0.

If θ_0 were much larger than in experiments to date, this interaction would become more important than in short systems. In particular, in long *uniformly inverted* systems the effect of depletion of inverted population by the oppositely traveling wave increases with *increasing* length. Similar considerations apply to the relatively unimportant backward traveling wave in long *swept* inversion systems. For the forward wave in such systems, however, the effect of this interaction *decreases* with increasing length, since both retardation effects of the type described in Sec. III C and this forward-backward interaction inhibit the growth of the backward wave. As discussed in Sec. III C Ehrlich *et al.*[6] have observed such a decrease in the intensity of the backward wave.

Unwanted feedback between forward and backward traveling waves can also influence the output. The effect of this feedback is similar to that occurring in the absence of the forward-backward wave interaction (Sec. IV B), except that the feedback is less important in the early stages of the evolution process, when the system is linear and the phases and polarization directions of the two waves are unrelated. Since sufficiently large feedback between forward and backward waves will cause their phases to become correlated, a lower limit on the acceptable amount of feedback can be estimated by using the one-way transit time in place of T_x in Eq. (35).

D. Initial polarization at the superradiant transition

Up to now, all observations of superradiance have used indirect inversion methods, such as three-level pumping,[2-7,11-13] two-photon excitation with a nonresonant intermediate state,[8] and three-wave mixing,[9,10] to achieve a complete population inversion. In principle, a complete inversion could be created directly, using a coherent pump pulse at the superradiant wavelength of area exactly π. However, direct inversion of a two-level system presents several difficulties. Transverse variations in the electric field associated with beam profile may cause different parts of the cross section to be subjected to different input areas, as described in Sec. VD below. The presence of level degeneracy would make it difficult to simultaneously invert all of the degenerate transitions, and also, propagation effects would prevent a pulse area near π from being maintained. Even in a nondegenerate system it would be difficult to generate a pulse of area exactly π. Deviations of the input area from exactly π would leave polarization at the superradiant transition after the inversion process is completed, the presence of which is equivalent to increasing θ_0. As shown in some early semiclassical studies by Burnham and Chiao,[65] such as increase in θ_0 would shorten the delay time and reduce the ringing, thus increasing the difficulty of completing the inversion process before coherent emission begins.

Specifically, the presence of polarization at the superradiant transition at $t=0$ will not significantly affect the output radiation provided that the effective tipping angle $\mathcal{P}(t=0)/\mu_z n_0$ is small compared to θ_0 due to spontaneous emission. This gives

$$\mathcal{P}(t=0) \lesssim \mu_z n_0 \sin\theta_0 \sim \mu_z n_0 \theta_0 \ , \tag{36}$$

where $\mathcal{P}(t=0)$ is the slowly varying envelope of the polarization at $t=0$. This condition implies that the area of the inversion pulse must be within θ_0 of π, typically a ratio of 10^{-6}.

Another problem with direct inversion is the change in the area of the inversion pulse as it travels through and interacts with the medium. To avoid this, the pulse energy $E_p = (\mathcal{E}^2/8\pi) Ac\tau$, where τ is the duration of the inversion pulse, must be larger than the energy which is needed to completely invert the medium $(n_0 AL\hbar\omega)$. For a pulse of area π $(\mu_z \mathcal{E}\tau/\hbar = \pi)$, this gives the condition

$$\tau \ll T_R \ . \tag{37}$$

This is a far more stringent requirement on τ that that of Sec. III A.

It should be noted that the other extreme, $\tau \gg T_R$, ensures complete absorption of the pulse by the medium, as occurs in self-induced transparency.[51] Thus, for direct inversion of a two-level system $\tau \ll T_R$ and $\tau \gg T_R$ serve to distinguish conditions under which superradiance and self-induced transparency, respectively, may be observed.

As mentioned earlier, all observations of superradiance have used indirect inversion methods. Even with indirect inversion, however, it is still possible that a large residual polarization could remain at the pump transition. This could lead to superradiant emission at *this* transition, which would deplete the population available for superradiance at the desired wavelength. This problem can be circumvented by using an incoherent pump pulse, or by choosing a three-level system in which the pump transition has a much shorter wavelength or much smaller matrix element (and therefore a much longer T_R) than the superradiant transition.

One should also note that in indirect excitation schemes the background emission which initiates the evolution of the system to a superradiant state can be modified by the presence of the pump field through Raman-type processes. In this case background spontaneous emission is effectively increased by the presence of the intense pump field, as discussed in Ref. 66. Although these processes tend to increase the effective initial tipping angle, estimates[67] show that this effect is small in all experiments performed to date.

V. DEVIATIONS FROM PLANE-WAVE BEHAVIOR

A. The plane-wave approximation

Our model assumes that the $\mathcal{E}$ field is a uniform one-dimensional plane wave. This model is a reasonable approximation as long as (a) the entire cross section radiates as one coherent plane wave, (b) transverse field components are small, and (c) in the coupled Maxwell-Schrödinger equations the $\mathcal{E}$ field envelope can be approximated by its average value over the cross section. These assumptions are not always valid, particularly in systems with small or large Fresnel number $\mathfrak{F} = 2A/\lambda L$.

The limitations of the plane-wave approximation can be avoided by including transverse variations and transverse fields in the wave equation. Such an approach has been used by Mattar and Newstein[68] and others[69] to study the effects of non-plane-wave behavior on the propagation of coherent pulses in absorbers. This section explores experimental circumstances for which non-plane-wave effects may become important.

B. Small Fresnel number

In systems with small Fresnel number, where the radiating volume is long and thin ("pencil"), the dependence of T_R on the system parameters differs from that of Eq. (2), which is valid for large Fresnel number ("disk"). This can be visualized by noting that for a long thin array of radiators, the solid angle over which phases sum constructively is very different from that of a short flat array. Rehler and Eberly[42a] found that a sharp change in the shape factor μ used in the formula for T_R occurs near $\mathfrak{F} \sim 0.1$ (Fig. 5, Ref.

42a). For $\mathfrak{F} \lesssim 0.1$, T_R is no longer given by Eq. (2), but instead becomes[30, 42a] approximately[70]

$$T_R = T_{sp}(8\pi/n_0 \lambda A) = T_{sp}(8\pi/n_0 \lambda^2 L)(\lambda L/2A). \quad (38)$$

The maximum length L_3 of a system which can be described by our Eqs. (2) and (5) is therefore given by $2A/\lambda L_3 \sim 0.1$, or

$$L_3 \sim 20A/\lambda. \quad (39)$$

For $L > L_3$, T_R remains constant as the length increases. Since T_R determines the time scale of the entire superradiant process, output pulses can no longer be shortened by increasing the length of the system. However, the output intensity should still be proportional to the square of the population inversion density.

The expressions for diffraction loss and for θ_0 must also be modified for samples with small Fresnel number. The diffraction loss $\mathfrak{L} = \frac{1}{2}\ln[1 + 4/\mathfrak{F}^2]$ (Sec. III B) is no longer negligible (as it is for large $\mathfrak{F}$) and must be taken into account. Also, the derivation of θ_0 in Ref. 22 uses a solid-angle factor which incorporates the shape factor μ as does T_R; for $\mathfrak{F} \lesssim 0.1$ this factor changes from $\lambda^2/4\pi A$ to $\lambda/4L$.[30, 42a] Since the expressions for the output parameters depend only logarithmically on θ_0, the effect of this change is small.

These changes in T_R and θ_0 and this choice of $\mathfrak{L}$ may be adequate to allow the use of the plane-wave solutions for small Fresnel numbers.

C. Large Fresnel number

A large-Fresnel-number system ($\mathfrak{F} \gtrsim 1$) may be unable to evolve to the coherent "plane" wave assumed in our model. In an initially inverted system spontaneous emission occurs independently in different sections of the cross section, and coupling between sections due to diffraction is needed to produce a single coherent phase front. In the $\mathfrak{F} \lesssim 1$ case the output face of the medium $(x = L)$ is in the far field zone of $x = 0$, where spontaneous emission is most important. Therefore, coupling between transverse sections should be sufficiently large for a single coherent wave to evolve. For large $\mathfrak{F}$, however, independent regions each with $F \gtrsim 1$ should evolve independently, since $x = L$ is in the near field zone of $x = 0$. Therefore, diffraction coupling may not be strong enough to create a single coherent wave front. Thus, different sections may have different phases which would not add constructively.

In addition, in high-gain media nonlinear effects such as self-focusing, self-defocusing, and beam trapping may place a limit on the largest coherent phase front which can be sustained. Similar limitations occur in pulse propagation in absorbers. For example, Gibbs *et al.*[71] have shown that self-induced transparency can be inhibited by transverse effects of this type.

D. Initial nonuniform cross section

A related experimental difficulty which could lead to non–plane-wave behavior is an initial inversion density which is nonuniform over the cross section of the system (e.g., near the perimeter of the cross section of the inverted region). Such nonuniformity could lead to independent radiation in distinct sections of the cross section, with different T_R's and different delays, and could also cause the effects of transverse fields to be significant.

As an example, consider the case of a *coherent* input pulse of Gaussian cross section which is used to indirectly populate the upper level of an initially unpopulated two-level system. If θ_0 at the center of the beam is several times π, the inversion density will vary greatly over the cross section, since variations in $\mathcal{E}$ will cause some sections to be completely populated (where θ_0, a linear function of $\mathcal{E}$, equals an odd multiple of π) and others to be completely unpopulated, forming a pattern of concentric rings. As a result, different sections of the cross section with different T_R's begin to evolve independently. This evolution may be further complicated by diffraction coupling between these rings. For large Fresnel number, small-diffraction coupling could result in independent sections of output radiation with different phase fronts and delays. A detector incapable of resolving these sections would then average ringing from different sections into a single asymmetric radiation lobe with a long tail.

As mentioned previously, the behavior of transverse variations and transverse fields during the evolution of a superradiant system merits further attention.

VI. SUMMARY

As discussed in Sec. II, useful approximate analytical solutions which are in close agreement with computer solutions of the full coupled Maxwell-Schrödinger equations [Eqs. (11), Ref. 22] can be derived in the "ideal superradiance" limit. Since experimental factors can cause the superradiant emission to deviate from that which would occur in the "ideal superradiance" limit, it is useful to summarize the conditions which are necessary for this limit to occur.

(a) The time τ during which the sample is inverted must not be too long:

$$\tau < \frac{\phi^2/4}{1-f}\,\frac{T_2'}{\alpha L}, \quad (9b')$$

or, equivalently,

$$L < \frac{\phi^2(T_2'/\tau)}{4\alpha(1-f)} , \quad (9a')$$

where $\frac{1}{2} \lesssim f < 1$ [Eq. (7)]. Otherwise, the efficiency is reduced [Eq. (11)].

(b) Loss places an upper limit on the length of the sample,

$$L < \phi/4\kappa , \quad (22')$$

where $\kappa = \mathcal{L}/L$ is the average loss/cm. Otherwise, the superradiant output will evolve into a steady state pulse [Eq. (13)].

(c) Loss also constrains the gain to exceed a minimum value:

$$\alpha L > \phi + \mathcal{L} . \quad (21')$$

Otherwise, limited superradiance[22] [Sec. IV A (1)] will occur.

(d) A high-gain condition is also imposed by the presence of incoherent decay:

$$\alpha L > \tfrac{1}{4}\, e\phi^2\, T_2'/T_1 . \quad (29a')$$

Otherwise, inversion is lost and the output decreases.

(e) The sample must be shorter than the cooperation length,

$$L_c = \phi(4\pi c T_{sp}/n_0 \lambda^2)^{1/2}, \quad (24')$$

in order that ideal superradiant pulses will be emitted from both ends of the medium. If $L > L_c$ and the sample is prepared via swept inversion, ideal superradiance will still occur in the forward direction, although backward emission will be reduced. However, if a sample of length $L > L_c$ is prepared via uniform inversion (Sec. III C), the medium will break up into independently radiating segments, preventing ideal behavior.

(f) Interference between forward and backward waves (Sec. IV C) will not prevent ideal behavior as long as $\theta_0 \lesssim 10^{-2}$ ($\phi \gtrsim 4.6$), which is generally the case in experiments performed to date.

(g) Excessive feedback can prevent ideal behavior (Sec. IV B). The fraction F of emerging radiation returned to the input face cannot be too large:

$$F \ll \frac{e^{-0.35\phi}}{\sqrt{\phi}} \left(\frac{(1 - T_x/T_D)^{1/2}}{\exp\{1.35\phi[(1-T_x/T_D)^{1/2} - 1]\}} \right) . \quad (35')$$

For $T_x \ll T_D$ the factor in large parentheses ~1, which places the most stringent requirement on F. Larger feedback shortens the pulse delay and increases the peak power.

(h) Polarization at the superradiant transition at $t = 0$ increases the effective tipping angle, and hence shortens the delay. This effect is unimportant if

$$\mathcal{P}(t=0) \lesssim \mu_x n_0 \theta_0 . \quad (36')$$

(i) Non-plane-wave behavior can prevent ideal superradiance (Sec. V). The conditions under which non-plane-wave features become significant are not fully understood at present.

In many experimental situations these requirements can be met, but in others they cannot be, leading to deviations from ideal behavior as discussed in Secs. III–V. Future work should be directed towards testing these predictions and utilizing them in practical applications.

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*This paper is based on material submitted by J. C. MacGillivray in partial fulfillment of the requirements for the degree of Doctor of Science at Massachusetts Institute of Technology. Present address: Optical Systems Division, Itek Corp., 10 Maguire Road, Lexington, Mass. 02173.

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