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Dispersive optical bistability with fluctuations

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Systems exhibiting dispersive optical bistability are considered as examples of nonequilibrium steady states lacking detailed balance. Methods for describing such states are developed. The Fokker-Planck equation for the steady-state distribution function of the transmitted electromagnetic field is derived. In the limit of small fluctuations the Fokker-Planck equation is reduced to the form of a Hamilton-Jacobi equation for a "nonequilibrium thermodynamic potential." This equation is solved in various approximations. The nonequilibrium thermodynamic potential acts like a free-energy for the first-order-type transition far from thermodynamic equilibrium lacking detailed balance. The potential entirely determines the steady-state distribution, the coexistence curve in the bistable domain, a Lyapunoff function of the deterministic equations of motion, and allows us to cast the deterministic equations of motion into the standard form of nonequilibrium thermodynamics.

I. INTRODUCTION

Systems in steady states far from thermodynamic equilibrium may exhibit instabilities which are analogous to phase transitions in thermodynamic equilibrium. In optics such analogies have been discussed first for the single-mode laser,[1] whose threshold may be compared to a second-order phase transition of mean-field type. Analogies to first-order phase transitions, again in the mean-field limit, are found in optical devices which exhibit regimes of bistable behavior with hysteresis. Examples are lasers with saturable absorbers[2] and dye lasers,[3] and systems where bistability is based on the nonlinearities of absorption or dispersion of passive media in Fabry-Perot étalons.[4,5]

For the comparison with phase transitions in equilibrium, a key role is played by the steady-state distribution function W of the relevant variables, e.g., the complex amplitude of the electric field strength. W is used to define a "nonequilibrium thermodynamic potential"[6,7]

$$\Phi = -\ln W$$

which may be compared with an equilibrium thermodynamic potential, e.g., the free energy.

The steady-state distribution W, and hence Φ, can only be found in a straightforward way, if the system under study has the property of detailed balance. This property is always present in thermodynamic equilibrium as a consequence of microscopic reversibility.[8] In this case Φ is identical with an equilibrium thermodynamic potential. In nonequilibrium steady states detailed balance is generally not satisfied, apart from special cases. (Detailed balance can always be restored by appropriately choosing the definition of time reversal, as is shown in a forthcoming paper by one of us (R.G.). Absence of detailed balance here means that the appropriate time-reversal transformation is not known before the steady-state distribution has been determined.) Important physical examples, where detailed balance is observed even in steady states far from thermodynamic equilibrium, are single-mode gas lasers near threshold,[9] and purely absorptive bistable devices on time scales long compared to the atomic relaxation times.[10]

In the present paper we are concerned with systems in steady states without detailed balance. A physically interesting optical example is provided by dispersive bistable devices. It is our aim here to develop and test practically methods for obtaining the potential Φ, at least approximately, in such cases.

Having determined Φ it becomes possible for the first time to make a real comparison of first-order phase transitions in thermodynamic equilibrium and corresponding transitions in steady states without detailed balance.

This comparison is made possible by the fact that the generalized thermodynamic potential Φ governs a nonequilibrium steady state in the same way as a thermostatic potential governs states of thermodynamic equilibrium: Thus Φ determines the steady-state probability distribution. In the deterministic limit Φ acts as a Lyapunoff function of the deterministic equations of motion. Its local or global minima correspond to locally or globally stable attractors. In multistable systems, as e.g., in optical bistability, the knowledge of the relative depth of the minima allows us to decide which branch is the more stable one. Therefore, the condition replacing the thermodynamic Maxwell construction for a first-order-type phase transition far from equilibrium can be determined. Finally, once Φ is known the deterministic equations of motion can be cast into the standard form of thermodynamics first given by Onsager.[13] In this standard form the drift is given as a superposition of two parts, one proportional to first-order derivatives of Φ, and the other having Φ as a constant of motion. The main difference to equilibrium thermodynamics, which remains after formulating the nonequilibrium theory in terms of Φ, is the absence of the simple time-reversal symmetries and detailed balance properties which characterize equilibrium thermodynamics.

The paper is organized as follows. In Sec. II we give a general discussion of steady states without detailed balance assuming that a Fokker-Planck description is valid. In Sec. III we consider optically bistable systems, briefly derive their equations of motion, and set up their Fokker-Planck equation.

In Sec. IV we discuss various limiting cases in which the steady-state distribution can be obtained exactly from the Fokker-Planck equation. In Sec. V we derive a Hamilton-Jacobi equation for Φ from the Fokker-Planck equation in the limit of small fluctuations and obtain various approximate solutions. Section VI contains a discussion of the results and the conclusions.

II. NONEQUILIBRIUM THERMODYNAMIC POTENTIAL FOR STEADY STATES WITHOUT DETAILED BALANCE

We consider systems in steady states subject to time-independent or periodic external fields. We assume that a Fokker-Planck description with a discrete set of variables is valid, i.e., a clear separation of time scales between macroscopic variables $q^\nu(\nu=1\ldots n)$ and microscopic variables exists and the macroscopic variables form a continuous Markoff process. The Fokker-Planck equation of this process is written in the form

$$\frac{\partial P}{\partial \tau}=-\frac{\partial}{\partial q^\nu}K^\nu(q)P+\tfrac{1}{2}\epsilon Q^{\nu\mu}\frac{\partial^2}{\partial q^\nu \partial q^\mu}P \ . \qquad (2.1)$$

$P(q|q_0,\tau)$ is the conditional probability density which reduces to an n-dimensional δ function for $\tau\to 0$, $K^\nu(q)$ is the drift, $\epsilon Q^{\nu\mu}$ is the diffusion matrix, which is assumed to be independent of q for simplicity and will later be considered as small. The summation convention is implied. Natural boundary conditions (i.e., P vanishing at infinity) are assumed. We also assume that the variables q are chosen in such a way, that the metric in q space[11] is Euclidean. The deterministic equations of motion corresponding to Eq. (2.1) in the limit $\epsilon\to 0+$ are then[11]

$$\dot q^\nu=K^\nu(q) \ . \qquad (2.2)$$

Let us suppose Eq. (2.1) can be solved for the steady-state distribution W satisfying

$$-\frac{\partial}{\partial q^\nu}K^\nu(q)\,W(q)+\tfrac{1}{2}\epsilon\, Q^{\nu\mu}\frac{\partial^2}{\partial q^\nu\partial q^\mu}W(q)=0 \qquad (2.3)$$

and define Φ by

$$\Phi(q)=-\epsilon\ln W(q) \ . \qquad (2.4)$$

In the absence of any constants of motion of the process q^ν the steady-state distribution is unique under certain general conditions,[12] which are assumed to hold.

If the externally applied forces allow the system to reach thermodynamic equilibrium, $\Phi(q)$ as defined by Eq. (2.4) is proportional to the corresponding thermodynamic potential. Furthermore, $K^\nu(q)$ then consists of two parts, the dissipative parts $d^\nu(q)$ and the reversible part $r^\nu(q)$:

$$K^\nu(q)=d^\nu(q)+r^\nu(q) \ . \qquad (2.5)$$

If we split the variables q^ν into even and odd under time reversal and denote the time reversed of q^ν by $\tilde q^\nu$

$$\tilde q^\nu=\epsilon^\nu q^\nu, \quad \epsilon^\nu=\pm 1, \qquad (2.6)$$

(no sum over ν in expressions with ϵ^ν), then

$$d^\nu(\tilde q)=\epsilon^\nu d^\nu(q) \ , \qquad (2.7)$$

$$r^\nu(\tilde q)=-\epsilon^\nu r^\nu(q) \ . \qquad (2.8)$$

The dissipative part is then of the form given by Onsager[13]:

$$d^\nu(q)=-\tfrac{1}{2}Q^{\nu\mu}\frac{\partial\Phi(q)}{\partial q^\mu} \ . \qquad (2.9)$$

The reversible part is not determined by Φ, but leaves the thermodynamic potential $\Phi(q)$ and the volume element in q space invariant:

$$r^\nu(q)\frac{\partial\Phi(q)}{\partial q^\nu}=0 \ , \qquad (2.10)$$

$$\frac{\partial r^\nu(q)}{\partial q^\nu}=0 \ . \qquad (2.11)$$

Examples of (2.10) are the invariance of entropy under reversible processes in closed systems and the invariance of free energy under reversible processes in isothermal systems with constant volume. Equation (2.11) follows from the Hamiltonian form of the reversible dynamics.

A more general version of (2.10), (2.11) which only makes use of the detailed balance property of the assumed equilibrium state is[6,7]

$$\frac{1}{\epsilon}r^\nu(q)\frac{\partial\Phi(q)}{\partial q^\nu}-\frac{\partial r^\nu}{\partial q^\nu}=0 \ , \qquad (2.12)$$

where $r^\nu(q)$ is defined by Eq. (2.8). Eq. (2.12) does not assume that the reversible dynamics is of Hamiltonian form.

Now we turn to steady states lacking detailed balance.[14] If we insert Eq. (2.4) into Eq. (2.3) we obtain

$$\tfrac{1}{2}Q^{\nu\mu}\frac{\partial\Phi}{\partial q^{\nu}}\frac{\partial\Phi}{\partial q^{\mu}}+K^{\nu}\frac{\partial\Phi}{\partial q^{\nu}} =\epsilon\left(\frac{\partial K^{\nu}}{\partial q^{\nu}}+\tfrac{1}{2}Q^{\nu\mu}\frac{\partial^2\Phi}{\partial q^{\nu}\partial q^{\mu}}\right) . \tag{2.13}$$

Similar to the case of thermodynamic equilibrium, Eqs. (2.5) and (2.9), we may define $r^{\nu}(q)$ by

$$K^{\nu}(q)=-\tfrac{1}{2}Q^{\nu\mu}\frac{\partial\Phi(q)}{\partial q^{\mu}}+r^{\nu}(q) . \tag{2.14}$$

Inserting (2.14) in (2.13) we then obtain the analog of (2.12)

$$r^{\nu}(q)\frac{\partial\Phi(q)}{\partial q^{\nu}}-\epsilon\frac{\partial r^{\nu}}{\partial q^{\nu}}=0 . \tag{2.15}$$

The difference between (2.15) and (2.12) is that no detailed balance property was assumed in (2.15), and that $r^{\nu}(q)$ is now defined by Eq. (2.14) and not identical with the reversible part of $K^{\nu}(q)$. In steady states lacking detailed balance, the time reversed $r^{\nu}(\tilde{q})$ is not related to $r^{\nu}(q)$ in any simple way. However, apart from the time-reversal properties, the formal structure of the equilibrium theory expressed in terms of a thermodynamic potential carries over to the nonequilibrium theory expressed in terms of a generalized thermodynamic potential. This gives us the key for a very direct comparison of equilibrium states and nonequilibrium steady states lacking detailed balance.

Unfortunately, the difficulty of finding $\Phi(q)$ explicitly is greatly increased in the absence of detailed balance. While in equilibrium states it is possible to determine d^{ν} and r^{ν} from Eqs. (2.5)–(2.8) without difficulty and then integrate Eq. (2.9) in order to obtain $\Phi(q)$, we have to turn directly to the partial differential equations (2.3) or (2.13) if detailed balance is lacking. These equations define elliptic, second-order, non-Hermitian boundary-value problems for which no general method of solution is known. Up to now only some simple models have been studied numerically, see, e.g., Ref. 15.

Fortunately, the mathematical difficulty of the problem can be reduced somewhat by taking the physically important limit of weak fluctuations, $\epsilon\to 0+$. In that limit the potential $\Phi(q)$ becomes independent of ϵ, provided the equation

$$\tfrac{1}{2}Q^{\nu\mu}\frac{\partial\Phi}{\partial q^{\nu}}\frac{\partial\Phi}{\partial q^{\mu}}+K^{\nu}\frac{\partial\Phi}{\partial q^{\nu}}=0 \tag{2.16}$$

obtained from (2.13) for $\epsilon\to 0+$ still has a meaningful solution. On the other hand, the asymptotic dependence of the probability density W on ϵ, which has an essential singularity for $\epsilon\to 0+$, is still taken into account by Eq. (2.4).

Equation (2.16) was originally obtained by independent phenomenlogical arguments by one of us.[16] Since these arguments shed further light on the analogy to systems in thermodynamic equilibrium they are briefly repeated here. Suppose the deterministic dynamics of a system is given in the form of Eq. (2.2) and a positive matrix of transport parameters $Q^{\nu\mu}$ is also given on physical grounds, e.g., by a knowledge of the equilibrium properties of the system. Then it is possible to construct a potential $\Phi(q)$ which will only decrease or be stationary under the dynamics (2.2)

$$\frac{d\Phi(q)}{dt}=K^{\nu}\frac{\partial\Phi}{\partial q^{\nu}}\leq 0 \tag{2.17}$$

in the following manner: Define Φ essentially as the solution of the differential geometrical problem of splitting the vector field $K^{\nu}(q)$ into two orthogonal fields, one of which is a gradient field. More precisely, split

$$K^{\nu}(q)=-\tfrac{1}{2}Q^{\nu\mu}\frac{\partial\Phi(q)}{\partial q^{\mu}}+r^{\nu}(q) \tag{2.18}$$

with

$$r^{\nu}(q)\frac{\partial\Phi(q)}{\partial q^{\nu}}=0 . \tag{2.19}$$

The property (2.17) then follows from the positivity of the matrix $Q^{\nu\mu}$. Equations (2.18) and (2.19) can be combined to yield Eq. (2.16).

The relation of $\Phi(q)$ to the probability density W in the limit of weak fluctuations is not apparent in this second derivation of (2.16). However, it points to another important feature: If $\Phi(q)$ can be determined from Eq. (2.16) it is a Lyapunoff function of the deterministic equation (2.2) which allows us to discuss the local and global stability of their solutions. Natural boundary conditions for (2.3) imply by (2.4) that $\Phi(q)$ approaches $+\infty$ at the boundaries. It is therefore clear that for globally unstable systems (2.2), whose trajectories can reach infinity, a solution $\Phi(q)$ with the required boundary condition cannot be found. Other properties of $\Phi(q)$ follow from its relation to $W(q)$: It must be a single-valued function of q bounded from below in order to be meaningful. Any attractors of the deterministic Eqs. (2.2), which may be fixed points, limit cycles, or strange attractors[17] have to correspond to minima $\Phi(q)$. Local minima are only locally stable. Global minima are globally stable.

Equation (2.16) is a first-order partial differential equation of the form of a time-independent Hamilton-Jacobi equation in classical mechanics, for vanishing energy. The corresponding time-dependent Hamilton-Jacobi equation is obtained from Eq. (2.1) by introducing a time-dependent $\Phi(q,t)$ via the time-dependent probability density $W(q,t)$ as in (2.4). The zero-energy condition in the time-independent equation is then recognized as being related to the eigenvalue zero of the Fokker-Planck operator for the steady-state distribution.

Mathematical literature exists[18] in which the initial-value problem of the Hamilton Jacobi equation is related to a second-order initial-value problem with a small parameter ϵ multiplying the second-order derivatives. Under certain technical conditions, which seem to be satisfied in physical applications, the solution of the second-order problem within a given fixed time interval approaches the solution of the first-order problem in the limit $\epsilon \to 0+$. However, we have not found any theorem proven in the literature, by which the solutions of the time-independent Hamilton-Jacobi equation would be correspondingly related to the solution of a second-order problem. Mathematically, the two kinds of problems are very different. While the second-order problem is a non-Hermitian boundary-value problem with $W \to 0$ at infinity, the time-independent first-order problem must be considered as a different boundary-value problem where $\Phi(q)$ is given on a hypersurface intersecting the field of characteristics. Uniqueness of the solution for the first-order problem may only result from the requirement of single valuedness.

In the absence of a well-developed mathematical framework for our procedure we have adopted a pragmatic attitude: If from a general solution of Eq. (2.16) one can find a unique single-valued one with the desired behavior $\Phi \to \infty$ at infinity, it will be taken as an approximation of a solution of Eq. (2.13). Unfortunately, a general solution of Eq. (2.16), or even a complete integral, cannot be obtained generally in a systematic way. Further approximations based on small parameters contained in $K^\nu(q)$ are therefore necessary. Such approximate solutions of Eq. (2.16), for the case of the dispersive optical bistability, will be explicitly constructed in Sec. V.

Here we summarize what actually is achieved, once the single-valued solution of Eq. (2.16) has been obtained, one way or other.

(i) The steady-state probability density is known asymptotically for $\epsilon \to 0+$.

(ii) A Lyapunoff function of the deterministic system with local or global minima on its locally or globally stable attractors is available.

(iii) A potential analogous to a thermodynamic potential is known. The condition replacing the Maxwell construction for a first-order-type phase transition far from equilibrium can then be given.

(iv) The deterministic drift $K^\nu(q)$ can be decomposed into a gradient part which stabilizes the attractors of $K^\nu(q)$, and a remaining part r^ν with the potential as a constant of the motion. These two parts of K^ν are the analogs of the dissipative and the reversible currents in thermodynamic equilibrium. A comparison of the dynamics of steady states and thermodynamic equilibrium states can then be made.

III. EQUATIONS OF MOTION OF A MODEL OF OPTICAL BISTABILITY

Intrinsic optical bistability[4] of a Fabry-Perot étalon driven by an external monochromatic optical field is achieved by filling the étalon with media with a sufficiently strong and fast nonlinear response to the applied field. Usually two extreme cases are distinguished.

For absorptive bistability[19] one uses a driving field in resonance with the Fabry-Perot and an absorption line of the medium. Due to the resonance of the Fabry-Perot the medium is suddenly bleached beyond a critical driving field strength. This transition occurs with hysteresis, which leads to bistability.

For dispersive bistability,[20] one detunes the Fabry-Perot and drives the filling medium off resonance. The nonlinearity of the refractive index is then responsible for a sudden tuning of the Fabry-Perot beyond a critical field strength, which again leads to hysteresis and a bistable regime of operation. Usually, both absorptive and dispersive nonlinearities occur together.

As a simple model[5] we will consider a homogeneously broadened two-level system in dipole coupling to a single mode of the étalon. The equations of this model are the coupled Maxwell-Bloch equations in rotating-wave approximation[21]

$$\dot{E}^* = i(\omega_0 - \omega)E^* - \kappa E^* + igP^* + \kappa E_0^* + F^*(t) , \tag{3.1}$$

$$\dot{P}^* = i(\nu - \omega)P^* - \gamma_\perp P^* - ig\sigma E^* + \Gamma^*(t) , \tag{3.2}$$

$$\dot{\sigma} = -\gamma_\parallel(\sigma - \sigma_0) - 2ig(P^*E - E^*P) + \Gamma_\sigma(t) . \tag{3.3}$$

The notation is as follows. ω_0, ν, ω are the frequencies of the empty resonator, the two-level system and the driving field, respectively. $\gamma_\perp = 1/T_2$, $\gamma_\parallel = 1/T_1$ are the transverse and the longitudinal relaxation rates, respectively, $\kappa = (C/L)(1-R)$ is the inverse resonator lifetime, determined by the length L and the reflectivity R of the resonator, $g = e_0 d\,(2\pi\nu^2/\hbar\omega_0)^{1/2}$ is the dipole coupling constant of the two-level transition and the resonator mode. We neglect the spatial variation of the field amplitudes. This approximation is known in the theory of the optical bistability under the somewhat unfortunate name of "mean-field approximation", a term which must not be confused with the mean-field approximation in the theory of phase transitions, which is also generally used in this field.

The atomic dipole matrix element is $e_0 d$. The number N of two-level atoms interacting with the electric field is taken as a constant. The variables of the model are the amplitudes of the internal electric field strength within the beam cross section, which is given by

$$(2\pi\hbar\omega_0)^{1/2}[E^*(t)e^{i\omega t}+\text{c.c.}]$$

the polarization of the medium, given by

$$e_0 d[P^*(t)e^{i\omega t}+\text{c.c.}]$$

and the inversion density $\sigma(t)$. The parameter σ_0 gives the value of $\sigma(t)$ in thermal equilibrium and is negative for passive media. The system is externally driven by random forces $F^*(t)$, $\Gamma^*(t)$, $\Gamma_\sigma(t)$, which describe the random influences of heat reservoirs, quantum fluctuations, and external noise, and an incident coherent electric field ($E_0^* e^{i\omega t}+\text{c.c.}$). The fluctuations of the external laser field will be approximated by a Gaussian, δ-correlated process $F_0^*(t)$ which can be lumped into $F^*(t)$. A more realistic model of the laser fluctuations would have to include the slow phase diffusion which is neglected here because its time scale is long compared to the time scales of E, P, and σ.

It is now assumed, that the atomic response to the electric field is very fast. On the time scale $1/\kappa$ of the electric field the atomic variables are then given by the adiabatic approximation[21]

$$\sigma(t)=\left(\sigma_0+\frac{1}{\gamma_\parallel}\Gamma_\sigma(t)\right)\Big/\left(1+\frac{4g^2}{\gamma_\parallel}\frac{\gamma_\perp|E(t)|^2}{(\nu-\omega)^2+\gamma_\perp^2}\right), \tag{3.4}$$

$$P^*(t)=\frac{-ig[\gamma_\perp+i(\nu-\omega)][\sigma_0+(1/\gamma_\parallel)\Gamma_\sigma]}{(\nu-\omega)^2+\gamma_\perp^2+4(g^2\gamma_\perp/\gamma_\parallel)|E|^2}E^*(t)+\frac{\Gamma^*(t)}{\gamma_\perp-i(\nu-\omega)}. \tag{3.5}$$

The equation for $E^*(t)$ is then

$$\dot{E}^*=\kappa(i\delta-1)E^*+\frac{g^2(1+i\Delta)[\gamma_\parallel\sigma_0+\Gamma_\sigma(t)]E^*(t)}{\gamma_\parallel\gamma_\perp(\Delta^2+1)+4g^2|E|^2}+F^*(t)+\kappa E_0^*, \tag{3.6}$$

where

$$\delta=\frac{1}{\kappa}(\omega_0-\omega),\quad \Delta=\frac{1}{\gamma_\perp}(\omega-\nu) \tag{3.7}$$

and

$$\hat{F}^*(t)=F^*(t)+\frac{\Gamma^*(t)}{\gamma_\perp(1-i\Delta)}. \tag{3.8}$$

We henceforth neglect the "multiplicative" noise source $\Gamma_\sigma(t)$ compared to the additive sources which are lumped in $\hat{F}^*(t)$. Their influence has been studied in.[22,10] Furthermore $\hat{F}^*(t)$ is assumed to represent Gaussian white noise

$$\langle\hat{F}^*(t)\rangle=0, \tag{3.9}$$

$$\langle\hat{F}^*(t)\hat{F}^*(t+\tau)\rangle=0=\langle\hat{F}(t)\hat{F}(t+\tau)\rangle, \tag{3.10}$$

$$\langle\hat{F}^*(t)\hat{F}(t+\tau)\rangle=\hat{Q}\,\delta(\tau). \tag{3.11}$$

$\hat{Q}$ contains three physically distinct contributions:

$$\hat{Q}=Q_0+Q_{\text{th}}+Q_{\text{sp}}, \tag{3.12}$$

Q_0 is the strength of an external noise source (which will usually be the largest contribution),

$$Q_{\text{th}}=\frac{2\kappa}{V}n_{\text{th}} \tag{3.13}$$

represents thermal noise,[21] which is negligible at optical frequencies (n_{th} = thermal quantum number, V = volume of the medium which interacts with the field).

$$Q_{\text{sp}}=2\gamma_\perp n_2/V \tag{3.14}$$

gives the strength of the quantum noise due to spontaneous emission,[21] which is proportional to the average number density n_2 of excited two-level atoms. For large interaction volumes Q_{th} and Q_{sp} approach zero, because this part of the noise is averaged over the volume V. We measure times in units of the cavity lifetime $\kappa^{-1}(\kappa t\to t)$ and rescale the field amplitude

$$E^*=\left(\frac{\gamma_\parallel\gamma_\perp(1+\Delta^2)}{4g^2}\right)^{1/2}\tilde{E}^*,\qquad E_0^*=\left(\frac{\gamma_\parallel\gamma_\perp(1+\Delta^2)}{4g^2}\right)^{1/2}\tilde{E}_0^*, \tag{3.15}$$

but drop the tilde henceforth. Without loss of generality we may take E_0 as a real quantity. Equation (3.6) then assumes the form

$$\dot{E}^*=(i\delta-1)E^*+E_0-\Gamma^2\frac{(1-i\Delta)}{1+|E|^2}E^*+R^*(t) \tag{3.16}$$

with

$$\Gamma^2=-\frac{g\sigma_0}{2\kappa\sqrt{1+\Delta^2}}\left(\frac{\gamma_\parallel}{\gamma_\perp}\right)^{1/2}, \tag{3.17}$$

$$\langle R^*(t+\tau)R(t)\rangle=Q\delta(\tau), \tag{3.18}$$

$$Q=\frac{4g^2\hat{Q}}{\gamma_\perp\gamma_\parallel\kappa(1+\Delta^2)}. \tag{3.19}$$

Γ^2 is positive for $\sigma_0<0$, i.e., for passive media. The Fokker-Planck equation corresponding to Eq. (3.9) can now be written down. We use Cartesian coordinates

$$E=x+iy \tag{3.20}$$

and obtain

$$\frac{\partial P}{\partial t} = +\frac{\partial}{\partial x}\left(x - \delta y - E_0 + \Gamma^2 \frac{x - \Delta y}{1 + x^2 + y^2}\right)P + \frac{\partial}{\partial y}\left(y + \delta x + \Gamma^2 \frac{y + \Delta x}{1 + x^2 + y^2}\right)P + \frac{1}{2} Q \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right). \tag{3.21}$$

Natural boundary conditions ($P \to 0$ for $x^2 + y^2 \to \infty$) are physically appropriate.

Equation (3.21) will be considered in the following section as an example of the general theory presented in Sec. II. A description of dispersive optical bistability with a cubic nonlinearity under the influence of quantum noise has been given by P. D. Drummond *et al.*[26]

IV. SOLUTIONS OF THE EQUATIONS OF MOTION

A. Attractors of the deterministic equations

Neglecting fluctuations we obtain the following system of first-order differential equations

$$\dot{x} = -x + \delta y + E_0 - \Gamma^2 \frac{x - \Delta y}{1 + x^2 + y^2}, \quad \dot{y} = -y - \delta x - \Gamma^2 \frac{y + \Delta x}{1 + x^2 + y^2}. \tag{4.1}$$

We assume $\Gamma^2 > 0$, i.e., a passive medium $\sigma_0 < 0$. The model is globally stable, since the positive function

$$F = x^2 + y^2 \tag{4.2}$$

can only decrease for $xE_0 \le x^2 + y^2$:

$$\dot{F} < 0 \quad \text{for } xE_0 < x^2 + y^2. \tag{4.3}$$

Since Eqs. (4.1) are two autonomous first-order equations, its attractors can only be fixed points and limit cycles. In view of Eq. (4.3) all attractors must be located within a circle of radius E_0 around the origin (cf. Fig. 1). All attracting fixed points must even lie inside the circle $(x - \frac{1}{2}E_0)^2 + y^2 = \frac{1}{4}E_0^2$. Therefore, for $E_0 = 0$, the only attractor is the origin itself, which is a stable fixed point, $\dot{x} = \dot{y} = 0$ for $x = y = 0$, $E_0 = 0$.

A complete discussion of the attractors of Eq. (4.1) can be given for the special case $\Delta = \delta$. The reason for this is the fact that we have determined the nonequilibrium thermodynamic potential Φ for this case exactly (cf. Sec. IV B 2 and Sec. V A). Hence, we know an exact Lyapunoff function of Eqs. (4.1) for $\delta = \Delta$. It is given by [cf. Eq. (4.27) and Sec. V A]:

$$\Phi = \frac{1}{Q}\left[\left(x - \frac{E_0}{1 + \delta^2}\right)^2 + \left(y + \delta \frac{E_0}{1 + \delta^2}\right)^2 + \Gamma^2 \ln(1 + x^2 + y^2)\right]. \tag{4.4}$$

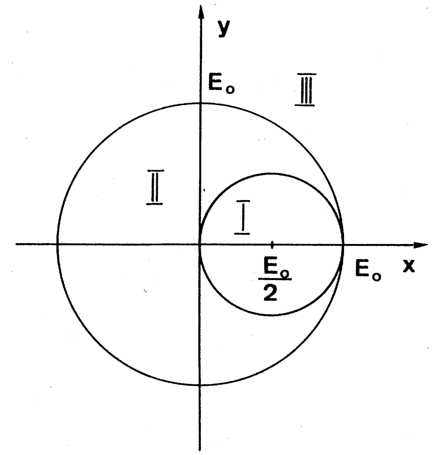

FIG. 1. Regions in the phase space of Eq. (4.1) where $F = x^2 + y^2$ may increase (region I) can only decrease (regions II, III), and contains attractors (regions I, II). All attracting or nonattracting fixed points lie in region I.

It is clear that $\Phi > 0$ in the entire x, y plane and one easily checks, that

$$\frac{d\Phi}{dt} \le 0 \tag{4.5}$$

under the time evolution of Eqs. (4.1).

The attractors of Eqs. (4.1) are therefore minima of Φ. These minima $\hat{x}$, $\hat{y}$ are isolated points which lie along the direction

$$\frac{\hat{y}}{\hat{x}} = \tan\psi = -\delta. \tag{4.6}$$

Thus, the only attractors are fixed points. Bistability may occur for $\Gamma^2 > 8$. Φ has one minimum for

$$E_0 < E_0^{(1)}, \tag{4.7}$$

two minima and one maximum for

$$E_0^{(1)} < E_0 < E_0^{(2)}, \tag{4.8}$$

and again one minimum for

$$E_0 > E_0^{(2)}, \tag{4.9}$$

where

$$E_0^{(1,2)} = \left(\frac{\Gamma^2}{2} - 1 \mp \frac{\Gamma}{2}(\Gamma^2 - 8)^{1/2}\right)^{1/2}\left(1 + \frac{2\Gamma}{\Gamma \mp (\Gamma^2 - 8)^{1/2}}\right). \tag{4.10}$$

In the bistable regime global stability is exchanged between the two locally stable branches where the two minima of Φ have equal depth. Let us derive the rule which replaces the Maxwell construction of thermodynamic equilibrium for this case. We may rewrite the potential (4.4) in the form

$$\Phi = \Phi_0 - \frac{2}{Q}\frac{E_0 x}{1+\delta^2} + \frac{2}{Q}\frac{\delta E_0 y}{1+\delta^2} + \text{const}, \qquad (4.11)$$

where

$$\Phi_0 = \frac{1}{Q}\left[x^2+y^2+\Gamma^2\ln(1+x^2+y^2)\right] \qquad (4.12)$$

is the potential for $E_0 = 0$. The extrema $\hat{x}$, $\hat{y}$ of Φ satisfy

$$E_0 = \frac{Q}{2}(1+\delta^2)\frac{\partial\Phi_0}{\partial\hat{x}} = -\frac{Q}{2\delta}(1+\delta^2)\frac{\partial\Phi_0}{\partial\hat{y}} \qquad (4.13)$$

and lie on the curve

$$\hat{y} = -\delta\hat{x}. \qquad (4.14)$$

Introducing the distance from the origin as a parameter t along that curve by

$$d\hat{x} = \frac{dt}{(1+\delta^2)^{1/2}}, \quad d\hat{y} = -\frac{\delta\, dt}{(1+\delta^2)^{1/2}}, \qquad (4.15)$$

we obtain the differential

$$d\Phi_0 = \frac{2}{Q}E_0(\hat{x}(t), \hat{y}(t))\frac{dt}{(1+\delta^2)^{1/2}}. \qquad (4.16)$$

Integrating Eq. (4.16) for that value $E_0 = E_0^c$ where exchange of global stability occurs between two states, which we call $\hat{x}_1$, $\hat{y}_1$ and $\hat{x}_2$, $\hat{y}_2$,

$$E_0^c = E_0(\hat{x}_1, \hat{y}_1) = E_0(\hat{x}_2, \hat{y}_2), \qquad (4.17)$$

we obtain

$$\int_{\hat{x}_1,\hat{y}_1}^{\hat{x}_2,\hat{y}_2} E_0(\hat{x}(t),\hat{y}(t))\frac{dt}{(1+\delta^2)^{1/2}} = \frac{Q}{2}\left[\Phi_0(\hat{x}_2,\hat{y}_2) - \Phi_0(\hat{x}_1,\hat{y}_1)\right]. \qquad (4.18)$$

Using Eq. (4.11) with the condition

$$\Phi(\hat{x}_1,\hat{y}_1) = \Phi(\hat{x}_2,\hat{y}_2), \qquad (4.19)$$

we obtain

$$\int_{\hat{x}_1,\hat{y}_1}^{\hat{x}_2,\hat{y}_2} E_0(\hat{x}(t),\hat{y}(t))\,dt = E_0^c[(\hat{x}_2-\hat{x}_1)^2 + (\hat{y}_2-\hat{y}_1)^2]^{1/2}. \qquad (4.20)$$

This is, of course, just the Maxwell rule of thermodynamic equilibrium, which is thereby proven also for our nonequilibrium steady state lacking detailed balance with $\delta = \Delta$. The fact that the Maxwell rule applies unchanged is merely an accident and may be traced to the fact that Φ differs from Φ_0 merely by a form linear in E_0 and in x, y. For the general case $\delta \neq \Delta$ this will no longer be true. The general condition (4.17) then still applies, but a simple geometric interpretation similar to the Maxwell rule is no longer possible.

It is not known rigorously, whether for $\delta \neq \Delta$ the system also has limit cycles for some values of E_0. However, numerical solutions and our later approximate results for Φ make this seem unlikely. The fixed points of Eqs. (4.1) are located in the circle I of Fig. 1 and satisfy a cubic equation, which has three roots if the condition[20,23]

$$[\tfrac{1}{2}\Gamma^2(1+\Delta^2) - \delta\Delta - 1]^3 > \tfrac{27}{8}\Gamma^2(\Delta^2+1)^2(\delta^2+1) \qquad (4.21)$$

is satisfied. For the special case $\delta = \Delta$ the condition (4.21) reduces to $\Gamma^2 > 8$, the condition obtained from the Lyapunoff function (4.4).

B. Solvable limits of the time-independent Fokker-Planck equation

1. Special cases with detailed balance

i. The case $E_0 = 0$. In this case, the two-level atoms for $t \to \infty$ are able to come into complete thermodynamic equilibrium with the electromagnetic field. Microscopic reversibility then requires that detailed balance is obeyed. The model which we consider here indeed satisfies this general requirement. Time reversal in Eqs. (3.17) is defined by

$$t \to -t, \quad x \to x, \quad y \to -y, \quad E \to E^*, \quad E_0 \to E_0. \qquad (4.22)$$

We readily obtain d^ν defined by (2.5), and (2.7)

$$d_x = -x - \Gamma^2\frac{x}{1+x^2+y^2}, \qquad (4.23)$$
$$d_y = -y - \Gamma^2\frac{y}{1+x^2+y^2},$$

and, integrating Eq. (2.9) with $Q^{\nu\mu} = \begin{pmatrix} Q & 0 \\ 0 & Q\end{pmatrix}$

$$\phi = \frac{1}{Q}\left[x^2+y^2+\Gamma^2\ln(1+x^2+y^2)\right]. \qquad (4.24)$$

According to the general theory of Sec. II, Φ is a Lyapunoff function, and its minima are stable states. Hence, Eq. (4.24) shows that for $E_0 = 0$ the origin $x = 0 = y$ is indeed the only stable state in the entire x, y plane in agreement with our earlier conclusion. The reversible drift r^ν, defined by (2.5) and (2.8) is obtained as

$$r_x = \delta y + \Gamma^2\Delta\frac{y}{1+x^2+y^2},$$
$$r_y = -\delta x - \Gamma^2\Delta\frac{x}{1+x^2+y^2}. \qquad (4.25)$$

It satisfies Eqs. (2.11) and (2.10) with Φ given by (4.24).

ii. The case $E_0 \neq 0$, $\Delta = 0 = \delta$. This is the special case of the purely absorptive optical bistability, whose probability distribution was obtained in Ref. 10. Despite the fact that this is a genuine example of a nonequilibrium steady state, it still satisfies the condition of detailed balance with respect to the time-reversal transformation (4.22). However, the physical reason for detailed balance is here not the reversibility of the underlying microscopic processes but rather the special phase-matching condition of the coupling between the system and the external driving field, which ensures that the reversible drift r^ν in this case vanishes identically. The irreversible drift d^ν is therefore identical with the total drift K^ν. The potential Φ is given by[10]

$$\Phi = \frac{1}{Q}\left[(x-E_0)^2 + y^2 + \Gamma^2 \ln(1+x^2+y^2)\right]. \qquad (4.26)$$

Since Φ is a Lyapunoff function, we can conclude from Eq. (4.26) that for $E_0 < E_0^{(1)}$ there is only one fixed point which is attracting for the entire x, y plane, that for $E_0^{(1)} < E_0 < E_0^{(2)}$ there are two locally attracting fixed points and one repelling fixed point (bistable regime), and that for $E_0^{(2)} < E_0$ there is again only one attracting fixed point. Here $E_0^{(1,2)}$ are given by Eq. (4.10) which is independent of the value of δ, as long as $\delta = \Delta$, and thus applies to $\delta = \Delta = 0$. The form of the potential (4.26) also shows, that other attractors (i.e., other fixed points and limit cycles) do not exist.

2. *A special case without detailed balance:* $E_0 \neq 0$, $\Delta = \delta$

This is an interesting special case of dispersive bistability which has no detailed balance, but whose stationary distribution function can be determined exactly nevertheless. It is given by the potential

$$\Phi = \frac{1}{Q}\left[\left(x - \frac{E_0}{1+\delta^2}\right)^2 + \left(y + \frac{\delta E_0}{1+\delta^2}\right)^2 + \Gamma^2 \ln(1+x^2+y^2)\right] \qquad (4.27)$$

as may be verified by substitution into the Fokker-Planck equation.

The drift r^ν in the steady state, defined by Eqs. (2.14) and (2.15), is given by

$$r_x = \delta\left(y + \frac{\delta E_0}{1+\delta^2}\right) + \Gamma^2 \Delta \frac{y}{1+x^2+y^2} ,$$
$$r_y = -\delta\left(x - \frac{E_0}{1+\delta^2}\right) - \Gamma^2 \Delta \frac{x}{1+x^2+y^2} . \qquad (4.28)$$

The absence of detailed balance is manifest by the presence of terms in Φ and r^ν, which are irreversible under the time-reversal transformation (4.22). The solution we have obtained here contains the result of *ii* of Sec. IV B 1 as a special case. It also reduces to the potential of *i* of Sec. IV B 1 if $E_0 \to 0$, but generalizes that result to $E_0 \neq 0$ only for $\Delta = \delta$. The generalization as compared to *ii* is that E_0 now couples directly to x and y. The extrema of the potential are therefore tilted away from the real axis, i.e., the most probable phase shift between the driving field and the field amplitude is neither zero nor $-\pi/2$, but intermediate to pure dissipation and pure dispersion at $\tan\varphi = -\delta$. Physically, the special matching $\delta = \Delta$ imposes the constraint, that this phase shift is the same as in the empty Fabry-Perot and independent of the magnitude of the field amplitude. For $\delta \neq \Delta$ the phase shift between the driving field and the driven field will depend on the magnitude of the driven field. This effect is hard to take into account and presents the real crux of the problem which we will take up in Sec. V. The stability properties following from the potential Φ as a Lyapunoff function have already been discussed in Sec. IV A.

3. *Special regions of phase space*

i. Small amplitudes $x^2+y^2<1$, $E_0 \ll 1$. In this case saturation of the interaction between atoms and electromagnetic field is negligible and Eq. (3.22) reduces to a linear Ornstein-Uhlenbeck process. Detailed balance is not satisfied for this process, which determines the potential Φ near the origin

$$\Phi = \frac{1+\Gamma^2}{Q}\left[\left(x - \frac{(1+\Gamma^2)E_0}{(1+\Gamma^2)^2 + (\delta+\Gamma^2\Delta)^2}\right)^2 + \left(y + \frac{(\delta+\Gamma^2\Delta)E_0}{(1+\Gamma^2)^2+(\delta+\Gamma^2\Delta)^2}\right)^2\right]. \qquad (4.29)$$

We can split K^ν according to Eq. (2.14) and obtain for r^ν

$$r_x = \left(y + \frac{(\delta+\Gamma^2\Delta)E_0}{(1+\Gamma^2)^2+(\delta+\Gamma^2\Delta)^2}\right)(\delta+\Gamma^2\Delta) ,$$
$$r_y = -\left(x - \frac{(1+\Gamma^2)E_0}{(1+\Gamma^2)^2+(\delta+\Gamma^2\Delta)^2}\right)(\delta+\Gamma^2\Delta) , \qquad (4.30)$$

which contains reversible and irreversible parts and explicitly exhibits the absence of detailed balance. It may be worthwhile to point out that the irreversible part of r^ν is even in the detuning parameters δ and Δ. This is physically satisfactory, since the dissipative part of the drift r^ν should be independent of the sign of the detuning. For $E_0 \to 0$ the irreversible parts of r^ν vanish.

The potential (4.29) and the drift r^ν (4.30) reduces, in this limit, to the first terms of an expansion in x, y of Eqs. (4.24) and (4.25), respectively, as they must. For $\delta=\Delta$, Eqs. (4.29) and (4.30) reduce to the first terms of an expansion of (4.27) and (4.28) in x, y.

ii. Large amplitudes, $x^2+y^2 \gg \max(1, \Gamma^2)$. In this case the saturation is so strong, that the electromagnetic field is decoupled from the atoms and we obtain

$$\Phi = \frac{1}{Q}\left[\left(x-\frac{E_0}{1+\delta^2}\right)^2+\left(y+\frac{\delta E_0}{1+\delta^2}\right)^2\right] \tag{4.31}$$

and

$$r_x = \delta\left(y+\frac{\delta E_0}{1+\delta^2}\right), \tag{4.32}$$

$$r_y = -\delta\left(x-\frac{E_0}{1+\delta^2}\right).$$

Again, the irreversible part of r^ν is independent of the sign of the detuning parameter δ. The result (4.31) and (4.32) reduces to the result obtained from (4.27) and (4.28) for $(x^2+y^2)\gg \max(1, \Gamma^2)$, as it should.

V. SOLUTIONS FOR WEAK FLUCTUATIONS

A. The Hamilton-Jacobi equation

We now want to consider the case where $Q \ll 1$. We proceed as in the general case in Sec. II and take

$$W \sim \exp\left(-\frac{\varphi}{Q}\right) \tag{5.1}$$

and assume that φ becomes independent of Q for $Q\to 0$. The Fokker-Planck equation (3.22) for $Q\to 0$ then reduces to the Hamilton-Jacobi equation for φ:

$$\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\varphi}{\partial y}\right)^2-\left(x-E_0-\delta y+\Gamma^2\frac{x-\Delta y}{1+x^2+y^2}\right)\frac{\partial\varphi}{\partial x} -\left(y+\delta x+\Gamma^2\frac{y+\Delta x}{1+x^2+y^2}\right)\frac{\partial\varphi}{\partial y}=0\,. \tag{5.2}$$

We will now seek solutions of this nonlinear first-order partial differential equation instead of solving the original linear second-order differential equation. Before doing so in the next subsection, we note that Eq. (5.2) still includes the same exactly solvable special cases as the original time-independent Fokker-Planck equation:

(i) $E_0=0$,
(ii) $\Delta=\delta$.

The potential φ in these cases is related to Φ obtained from the Fokker-Planck equation by $\Phi=\varphi/Q$.

We can therefore conclude that Eq. (5.1) with φ independent of Q holds *exactly* in the two limits $E_0=0$, $\Delta=0=\delta$ where detailed balance is satisfied, and in the case $\Delta=\delta$ without detailed balance which can be solved exactly. We also note that for very small and very large amplitudes the Hamilton-Jacobi equation is solved by $\varphi=Q\Phi$, where Φ is given by (4.29) for $x^2+y^2\ll 1$, $E_0 \ll 1$, and by (4.31) for $x^2+y^2 \gg \max(1,\Gamma^2)$.

B. Approximate solutions

It seems that exact solutions of the Hamilton-Jacobi equation (5.2) can only be given in the two cases $E_0=0$ and $\Delta=\delta$. In all other cases one has to resort to approximations. These can be generated by considering either E_0 or $(\Delta-\delta)$ or both as small parameters. Of course, these small parameters could already have been introduced in the Fokker-Planck equation, and a perturbative solution of that equation could have been generated.[24] However, it would then be necessary to compute all eigenfunctions of the Fokker-Planck problem in zero order, a task which for many cases is impossible to perform in practice. The immense practical advantage of considering the Hamilton-Jacobi equation instead is that we only need the "zero-energy" solution of the unperturbed problem. The disadvantage is the restriction to the special case of weak fluctuations. Fortunately, in many practical cases noise is indeed a small perturbation and this restriction is then not serious at all. We begin to consider perturbative solutions, by further investigating Eq. (5.2) for small and for large amplitudes $(x^2+y^2)^{1/2}$.

1. Limit of small amplitudes: $x^2+y^2 \ll 1$, $E_0 \ll 1$

In this limit the drift K^ν reduces to

$$K_x = -(1+\Gamma^2)x+E_0+(\delta+\Delta\Gamma^2)y+\Gamma^2(x^2+y^2)(x-\Delta y)\,,$$

$$K_y = -(1+\Gamma^2)y-(\delta+\Delta\Gamma^2)x+\Gamma^2(x^2+y^2)(y-\Delta x)\,. \tag{5.3}$$

We determine the asymptotic form of φ by considering the cubic terms in the drift as small perturbation. The unperturbed problem has the solution $\varphi_0=Q\Phi$ where Φ is given by Eq. (4.29). Writing $\varphi=\varphi_0+\varphi_1+\cdots$

we obtain in first order the linear inhomogeneous differential equation of first order:

$$\frac{\partial\varphi_1}{\partial x}\left(\frac{\partial\varphi_0}{\partial x}-x(1+\Gamma^2)+E_0+y(\delta+\Delta\Gamma^2)\right)+\frac{\partial\varphi_1}{\partial y}\left(\frac{\partial\varphi_0}{\partial y}-y(1+\Gamma^2)-x(\delta+\Delta\Gamma^2)\right)$$
$$=-\Gamma^2(x^2+y^2)(x-\Delta y)\frac{\partial\varphi_0}{\partial x}-\Gamma^2(x^2+y^2)(y+\Delta x)\frac{\partial\varphi_0}{\partial y}\ .$$

Since φ_0 is a known quadratic form, the general solution of this partial differential equation is easily worked out by the method of characteristics (cf. Appendix A).

We obtain

$$\begin{aligned}\varphi&=\varphi_0+\varphi_1+\cdots\\&=\gamma\left(x-\frac{\gamma E_0}{\gamma^2+\eta^2}\right)^2+\gamma\left(y+\frac{\eta E_0}{\gamma^2+\eta^2}\right)^2\\&+(\delta-\Delta)\Gamma^2\frac{E_0^2}{(\gamma^2+\eta^2)^3(9\gamma^2+\eta^2)}\left[\eta\cdot(\gamma^2-3\eta^2)(9\gamma^2+\eta^2)x^2\right.\\&\left.+2\gamma(\eta^2-3\gamma^2)(7\eta^2-\gamma^2)xy-\eta(17\gamma^4-10\eta^2\gamma^2+5\eta^4)y^2\right]\\&+(\delta-\Delta)\Gamma^2\frac{6E_0^3}{(\gamma^2+\eta^2)^2(9\gamma^2+\eta^2)}\left[2\gamma\eta x+(\gamma^2-\eta^2)y\right]\\&+(\delta-\Delta)\Gamma^2\frac{2E_0(x^2+y^2)}{(\gamma^2+\eta^2)(9\gamma^2+\eta^2)}\left[4\gamma\eta x+(3\gamma^2-\eta^2)y\right]-\tfrac{1}{2}\Gamma^2(x^2+y^2)^2+\cdots+\text{const},\end{aligned}\tag{5.4}$$

where we used the abbreviations

$$\gamma=1+\Gamma^2,\quad \eta=\delta+\Delta\Gamma^2\ .\tag{5.5}$$

As is shown in Appendix A, the requirement that φ_1 is a regular, single-valued function in the x, y plane determines the solution uniquely up to an arbitrary additive constant. Thus the hope expressed in Sec. II is indeed fulfilled in this example. The same is true for all other examples considered below. In the special cases $\Delta=\delta$ and $E_0=0$ the solution we have obtained here for small amplitudes reduces to the power series up to quartic terms generated from the exact solutions (4.27) and (4.24), respectively, as, of course, it must. However, the solution φ_1 found here can *only* make sense for small amplitudes. This becomes manifest by the fact that the probability density W derived from φ_1 is not normalizable at large amplitudes. Indeed, already the expanded drift (5.3) is meaningless for large amplitudes since it describes a globally unstable system.

Sometimes, Eq. (5.3) with negative $\Gamma^2<0$ is used as a very simple model of optical bistability, which is globally stable. A realization of $\Gamma^2<0$ is possible by inverting the two levels of the atoms, $\sigma_0>0$. In this case Eq. (5.4) leads to a normalizable probability distribution also at large amplitudes. For $\delta=\Delta$ that distribution function is again an exact solution of the Fokker-Planck equation. More generally, Eq. (5.3) may be expected to represent a reasonable approximation for a model with $\Gamma^2<0$ as long as $|(\delta-\Delta)\Gamma^2|$ remains small.

For our more realistic model with $\Gamma^2>0$, Eq. (5.3) is restricted to small amplitudes x^2+y^2 and small E_0, but it allows us to discuss an interesting feature in that region. The shape of the potential near the origin is influenced by somewhat larger values of E_0 than could be allowed in Eq. (4.29). We see that even those terms of φ which are linear and quadratic in x and y receive corrections which are nonlinear in E_0. Thus, even arbitrarily close to the origin the potential $\Phi_1=\varphi/Q$ ceases to be a solution of the Ornstein-Uhlenbeck process obtained by linearizing around the origin, because the potential near the origin is influenced by the form of the potential at larger amplitudes which become important with increasing E_0. Stated differently, even though $W\sim\exp(-\varphi/Q)$ satisfies the Fokker-Planck equation (for small Q) linearized around the origin, it is not identical with the solution of the corresponding Ornstein-Uhlenbeck process, because it does not satisfy natural boundary conditions at large amplitudes. Instead, its behavior at large amplitudes is governed by the non-negligible drift at large amplitudes.

2. *Limit of small* Γ^2

For very large amplitudes $x^2+y^2 \gg \max(1,\Gamma^2)$ the interaction between atoms and field becomes very weak. It therefore appears reasonable to treat in this region the whole term $\sim \Gamma^2$ in Eq. (5.2) as a perturbation. The potential $\varphi_0 = Q\Phi$ in zeroth order is given by Eq. (4.3). Writing $\varphi = \varphi_0 + \varphi_1 \cdots$, we obtain in first order in Γ^2 a linear inhomogeneous partial differential equation of first order for φ_1:

$$\frac{\partial\varphi_1}{\partial x}\left(x+\delta y+\frac{(\delta^2-1)}{\delta^2+1}E_0\right)+\frac{\partial\varphi_1}{\partial y}\left(y-\delta x+\frac{2\delta}{1+\delta^2}E_0\right)=\frac{2\Gamma^2}{1+x^2+y^2}\left(x^2+y^2-\frac{(x-\Delta y)}{1+\delta^2}E_0+\frac{(y+\Delta x)}{1+\delta^2}\delta E_0\right). \quad (5.6)$$

This equation is solved by the method of characteristics in Appendix B. We obtain there with $\tilde{E}_0 = (E_0/1+i\delta)$, $\beta = x+iy$

$$\varphi_1 = \Gamma^2\ln(1+|\beta|^2)+i(\Delta-\delta)\Gamma^2\int_0^\infty d\tau\,\frac{\tilde{E}_0(\beta^*-\tilde{E}_0^*)e^{-(1+i\delta)\tau}-\text{c.c.}}{|\tilde{E}_0|^2+|\beta-\tilde{E}_0|^2e^{-2\tau}+[\tilde{E}_0(\beta^*-\tilde{E}_0^*)e^{-i(1+i\delta)\tau}+\text{c.c.}]}. \quad (5.7)$$

It is shown there, that the requirement of single valuedness and regularity of φ, in the entire x,y plane makes the solution of Eq. (5.6) unique up to a constant. Unfortunately, the definite integral in Eq. (5.7) cannot be generally expressed in terms of tabulated functions. It is clear, however, that the integral defines a single-valued and regular function in the entire x,y plane. For $\Delta=\delta$ and for $E_0=0$ the exact results are recovered. We now consider three limiting cases, in which the result (5.7) can be evaluated further.

i. Driving field in resonance with the cavity, $\delta=0$. In this case one may neglect the oscillatory behavior of the integrand in (5.7). After substituting $e^{-\tau}=U$, the integral in Eq. (5.7) is elementary and we obtain

$$\varphi=\varphi_0+\varphi_1+\cdots=(x-E_0)^2+y^2+\Gamma^2\ln(1+x^2+y^2)+2\Gamma^2\Delta E_0\frac{y}{[(x-E_0)^2+y^2(1+E_0^2)]^{1/2}}\arctan\left(\frac{[(x-E_0)^2+y^2(1+E_0^2)]^{1/2}}{1+E_0x}\right)+\cdots+\text{const}. \quad (5.8)$$

We note that the argument of the arctan has a pole at $x=-1/E_0$ for all y. The regularity of φ then requires that two neighboring branches of the arctan are pieced together at $x=-1/E_0$ in a continuous way.

ii. Driving field far from resonance with the cavity, $1/\delta\to 0$. Introducing $U=\delta\tau$ and $\epsilon=1/\delta$ one may evaluate the integral in Eq. (5.7) asymptotically for $\epsilon\to 0$. We obtain

$$\varphi_1=\Gamma^2\ln(1+|\beta|^2)+\Gamma^2\frac{(\delta-\Delta)}{\delta}\lim_{\epsilon\to 0+}\left(\ln\{1+|\tilde{E}_0|^2+|\beta-\tilde{E}_0|^2e^{-2\epsilon u}+[\tilde{E}_0^*(\beta-\tilde{E}_0)e^{iu-\epsilon u}+\text{c.c.}]\}\right)\Big|_{u=0}^{u=\infty}. \quad (5.9)$$

The upper boundary only contributes a constant and we obtain

$$\varphi_1=\Gamma^2\ln(1+|\beta|^2)+\Gamma^2\left(\frac{\Delta-\delta}{\delta}\right)\ln(1-|\tilde{E}_0|^2+|\beta-\tilde{E}_0|^2+2\,\text{Re}\,\tilde{E}_0^*\beta)+\text{const}. \quad (5.10)$$

In Cartesian coordinates we arrive at the expression

$$\varphi=\left(x-\frac{E_0}{\delta^2}\right)^2+\left(y+\frac{E_0}{\delta}\right)^2+\Gamma^2\frac{\Delta}{\delta}\ln(1+x^2+y^2)+\cdots+\text{const}. \quad (5.11)$$

iii. The limit of small E_0. Equation (5.6) can be evaluated in closed form if we restrict ourselves to the terms of first order in E_0. We obtain

$$\varphi_1=\Gamma^2\ln(1+|\beta|^2)+i(\Delta-\delta)\Gamma^2\int_0^\infty d\tau\left(\frac{\tilde{E}_0\beta^*e^{-(1+i\delta)\tau}}{1+|\beta|^2e^{-2\tau}}-\text{c.c.}\right). \quad (5.12)$$

Substituting $e^{-2\tau}=U$ and using a formula of Ryzhik-Gradstein[25] we obtain

$$\varphi_1=\Gamma^2\ln(1+|\beta|^2)+i\Gamma^2(\Delta-\delta)\left[\frac{\tilde{E}_0\beta^*}{1+i\delta}F\left(1,\tfrac{1}{2}+\frac{i\delta}{2};\tfrac{3}{2}+\frac{i\delta}{2};-|\beta|^2\right)-\text{c.c.}\right], \quad (5.13)$$

where F is a hypergeometric function. This result will be useful for a comparison with other more general results obtained in the next section. We note that because of the relation

$$F(1,\tfrac{1}{2};\tfrac{3}{2};-|\beta|^2)=\frac{1}{|\beta|}\arctan|\beta|, \quad (5.14)$$

the solutions (5.13) and (5.18) coincide in the region where both δ and E_0 are small.

C. Expansion in E_0

We now want to construct a solution of the Hamilton-Jacobi equation in first order in the amplitude E_0 of the external driving field. This seems to be a promising approximation, since it must contain the two exact solutions for $E_0=0$ and $\delta=\Delta$ as special cases. Furthermore, this approximation must be able to reproduce the result (5.4) at very small amplitudes and the result (5.13) for small Γ^2, which applies to the region of large amplitudes, and one may therefore expect that the solution is also a good approximation in the intermediate regime. We expand

$$\varphi=\varphi_0+\varphi_1+\cdots\,, \tag{5.15}$$

where $\varphi_0=Q\Phi$ is the solution (4.24) of Eq. (5.2) for $E_0=0$ and φ_1 is of order E_0. For φ_1 we obtain the linear inhomogeneous partial differential equation of first order:

$$\left(x+\delta y+\Gamma^2\frac{x+\Delta y}{1+x^2+y^2}\right)\frac{\partial\varphi_1}{\partial x}+\left(y-\delta x+\Gamma^2\frac{y-\Delta x}{1+x^2+y^2}\right)\frac{\partial\varphi_1}{\partial y}=-2E_0\left(x+\Gamma^2\frac{x}{1+x^2+y^2}\right). \tag{5.16}$$

This equation is solved in Appendix C. We obtain in polar coordinates $x=r\cos\psi$, $y=r\sin\psi$

$$\varphi=\varphi_0+\varphi_1=r^2-2\frac{E_0}{(1+\delta^2)^{1/2}}r\cos\tilde{\psi}+\Gamma^2\ln(1+r^2)$$

$$-2E_0\Gamma^2\frac{\delta-\Delta}{(1+\delta^2)^{1/2}}r\,\mathrm{Im}\left(\frac{F\left(1,\tfrac{1}{2}-i\dfrac{\delta}{2};\tfrac{3}{2}-\dfrac{i}{2}\dfrac{\delta+\Delta\Gamma^2}{1+\Gamma^2};-\dfrac{r^2}{1+\Gamma^2}\right)e^{i\tilde{\psi}}}{1+\Gamma^2-i(\delta+\Delta\Gamma^2)}\right), \tag{5.17}$$

where we introduced the rotated phase angle

$$\tilde{\psi}=\psi+\arctan\delta\,. \tag{5.18}$$

F is a hypergeometric function in the notation of Ref. 25. Equation (5.17) determines the potential φ exactly to first order in E_0 and to zero order in Q, the strength of the fluctuations. It is the main result of this paper, and illustrates the scope and power of the general methods described in Sec. II. Since the result (5.17) is rather opaque in its most general form, we discuss a number of special cases and make contact with the results obtained in the preceding sections.

i. Special cases. The exact solutions with detailed balance obtained in Sec. IV A for $E_0=0$ and for $\delta=\Delta=0$ are correctly contained in Eq. (5.17). The exact solution without detailed balance which we obtained for $\delta=\Delta$ is also contained exactly. This is, of course, expected since all these exact solutions are rigorously of first order in E_0, and zero order in Q where (5.17) is exact.

ii. Asymptotic behavior for small and large amplitudes. Next, we check the behavior of (5.17) for small r against our results in Secs. IV B and V B. To compare with Sec. IV B it is sufficient to replace the hypergeometric function by its value at $r=0$, $F=1$. For comparison with Sec. V B the first-order term in the power-series expansion of the hypergeometric function has to be taken into account. The result of Sec. IV B is obtained exactly, whereas only the correction term linear in E_0 in the expansion (5.4) of Sec. V A is reproduced, as one expects. Thus, to first order in E_0 the departure of the steady-state probability density from the solution of the Ornstein-Uhlenbeck process near the origin is correctly given.

Next we turn to the behavior at large amplitudes. To first order in Γ^2 the result (5.17) correctly reduces to the result (5.13) of Sec. V B. For a more detailed discussion of the behavior at large amplitudes we make use of the asymptotic representation of F for large r^2 (Ref. 25):

$$F\left(1,\tfrac{1}{2}-i\frac{\delta}{2};\tfrac{3}{2}-\frac{i\alpha}{2};-\frac{r^2}{1+\Gamma^2}\right)\to\frac{(1+\Gamma^2)^{1/2}}{2r}(1-i\alpha)\left(\frac{r^2}{1+\Gamma^2}\right)^{i\delta/2}\frac{\Gamma\left(\tfrac{1}{2}+i\dfrac{\delta}{2}\right)\Gamma\left(\tfrac{1}{2}-i\dfrac{\alpha}{2}\right)}{\Gamma\left(1+\dfrac{i}{2}(\delta-\alpha)\right)}, \tag{5.19}$$

where $\Gamma(x)$ is the gamma function, and must not be confused with the parameter Γ. We then obtain the asymptotic result

$$\varphi=r^2-\frac{2E_0r\cos\tilde{\psi}}{(1+\delta^2)^{1/2}}+\Gamma^2\ln(1+r^2)-AE_0\sin\left(\tilde{\psi}+\delta\ln\frac{r}{1+\Gamma^2}-\psi_0\right), \tag{5.20}$$

with

$$A=\left[\frac{2\pi\Gamma^2(\delta-\Delta)}{1+\delta^2}\left(\sinh\frac{\pi\Gamma^2(\delta-\Delta)}{2(1+\Gamma^2)}\right)\Big/\left(\cosh\frac{\pi\delta}{2}\cosh\frac{\pi(\delta+\Gamma^2\Delta)}{2(1+\Gamma^2)}\right)\right]^{1/2} \tag{5.21}$$

and

$$\psi_0=\mathrm{Im}\left[\ln\Gamma^2\left(1+\frac{i(\delta-\Delta)\Gamma^2}{2(1+\Gamma^2)}\right)+\ln\Gamma^2\left(\tfrac{1}{2}-i\frac{\delta}{2}\right)+\ln\Gamma^2\left(\tfrac{1}{2}+\frac{i}{2}\frac{\delta+\Gamma^2\Delta}{1+\Gamma^2}\right)\right]. \tag{5.22}$$

In order to obtain the amplitude A we have made use of the formulas

$$|\Gamma(\tfrac{1}{2}+ix)|^2=\frac{\pi}{\cosh\pi x},\qquad |\Gamma(1+ix)|^2=\frac{\pi x}{\sinh\pi x}. \tag{5.23}$$

The result (5.20) shows in a very transparent way the twofold influence of the detuning δ on the potential φ: It shifts the preferred direction away from the real axis by the $\cos\tilde{\psi}$ term and it introduces a dependence of the preferred angle on the amplitude r by the $\sin[\tilde{\psi}+\delta\ln(r/r_0)]$ term. At very large amplitudes the second effect is negligible. At $r=0$ the function (5.20) has a singularity, which is spurious, however, since the asymptotic formula applies only for large r and since we have already checked that the full result (5.17) behaves correctly at small amplitudes.

iii. The case $\delta=0, \Delta\neq 0$. In this case the result (5.17) reduces to

$$\varphi=(r^2-2rE_0\cos\psi)+\Gamma^2\ln(1+r^2)+2E_0\Gamma^2\Delta r\,\mathrm{Im}\left(\frac{F\left(1,\tfrac{1}{2};\tfrac{3}{2}-i\frac{\Delta\Gamma^2}{2(1+\Gamma^2)};-\frac{r^2}{1+\Gamma^2}\right)e^{i\psi}}{1+\Gamma^2-i\Delta\Gamma^2}\right). \tag{5.24}$$

For further more specific discussion and graphical representation of some results we now restrict ourselves to the special case $\delta=0$, i.e., to the case of perfect resonance of the driving field with the cavity mode. The detuning of both fields from the atomic transition Δ is taken into account and gives rise to departure from detailed balance. If we keep only the term of first order in Δ in Eq. (5.24) we may take $\Delta=0$ in the curly bracket and and use the formula (5.14) to obtain

$$\varphi=(r^2-2rE_0\cos\psi)+\Gamma^2\ln(1+r^2)+2E_0\Gamma^2\Delta\frac{\sin\psi}{(1+\Gamma^2)^{1/2}}\arctan\frac{r}{(1+\Gamma^2)^{1/2}}. \tag{5.25}$$

This result is much simpler than Eq. (5.24), but restricted to small Δ. For large amplitudes but arbitrary Δ we obtain from Eqs. (5.20) for $\delta=0$

$$\varphi=r^2-2rE_0\cos\psi+\Gamma^2\ln(1+r^2)+E_0\frac{\pi\Gamma^2\Delta}{(1+\Gamma^2)^{1/2}}\sin\psi \tag{5.26}$$

which coincides with the asymptotic form of (5.25) for large amplitudes, despite the fact that it is valid for arbitrary Δ. At intermediate amplitudes and arbitrary Δ, the hypergeometric function has to be evaluated numerically. For this purpose it is useful, to employ the transformation

$$F(a,a+\tfrac{1}{2};c;-z)=(1+z)^{-a}F(2a,2c-2a-1;c;\zeta) \tag{5.27}$$

with

$$\zeta=\frac{(1+z)^{1/2}-1}{2(1+z)^{1/2}}, \tag{5.28}$$

since $0\leq\zeta\leq\frac{1}{2}$ in the domain $0\leq z\leq\infty$, and one may use the power-series expansion of F in ζ, which converges for all z. Some numerical results are discussed in the concluding section.

VI. DISCUSSION AND CONCLUSION

In the preceding sections we have described a general method for obtaining the steady-state probability distribution of Fokker-Planck models without detailed balance in the limit of small fluctuations. We have applied this method to a model of dispersive optical bistability. The steady-state distribution function of the transmitted field amplitude $x+iy$ is given by

$$W(x,y)\sim\exp\left(-\frac{\varphi(x,y)}{Q}\right).$$

The main result obtained in this paper is an expression for $\varphi(x,y)$ which is exact to first order in the external driving field E_0 and to zero order in the fluctuation intensity Q. This result is given by Eq. (5.17) and may be written in the equivalent form

$$\varphi(x,y)=\left(x-\frac{E_0}{1+\delta^2}\right)^2+\left(y+\frac{\delta E_0}{1+\delta^2}\right)^2+\Gamma^2\ln(1+x^2+y^2)$$
$$-2E_0\Gamma^2(\delta-\Delta)\,\mathrm{Im}\left((x+iy)\frac{F\left(1,\frac{1}{2}-i\frac{\delta}{2};\frac{3}{2}-\frac{i}{2}\frac{\delta+\Delta\Gamma^2}{1+\Gamma^2};-\frac{x^2+y^2}{1+\Gamma^2}\right)}{(1-i\delta)(1+\Gamma^2-i(\Delta\Gamma^2+\delta)}\right). \tag{6.1}$$

This result has been compared with exact results for $E_0=0$ and for $\delta=\Delta$ and it has been simplified and compared with other results for small and for large amplitudes x,y. The potential $\varphi(x,y)$ exact to first order in E_0 and zero order in Q acts as a Lyapunoff function of the deterministic equations and is a minimum on a stable attractor of these equations. We may therefore check the quality of our expansion of φ to first order in E_0 by comparing the minima of the approximate $\varphi(x,y)$ with the exact attractors of the deterministic equations of motion. For small driving fields E_0 the minimum of $\varphi(x,y)$ is located at

$$x=E_0\frac{1+\Gamma^2}{(1+\Gamma^2)^2+(\delta+\Delta\Gamma^2)^2},\qquad y=-\frac{(\Gamma^2\Delta+\delta)E_0}{(1+\Gamma^2)^2+(\delta+\Delta\Gamma^2)^2}, \tag{6.2}$$

which agrees exactly with the stable fixed-point attractor of the deterministic equations for small E_0. For large driving fields E_0 such agreement cannot be expected. The deterministic equations have the attracting fixed point for $E_0\to\infty$:

$$r=\frac{E_0}{(1+\delta^2)^{1/2}}, \tag{6.3}$$

$$\tan\psi=-\delta+\frac{1}{E_0^2}(\delta-\Delta)\Gamma^2(1+\delta^2). \tag{6.4}$$

The potential (6.1), for $E_0\to\infty$, has a single minimum, which is located at

$$r=\frac{E_0}{(1+\delta^2)^{1/2}}, \tag{6.5}$$

$$\tan\psi=-\delta+\frac{1}{E_0}\frac{A}{2}(1+\delta^2)^2\cos\alpha, \tag{6.6}$$

where A is given by Eq. (5.21). The asymptotic values of r and ψ are therefore given exactly. However, as expected, the approach of these asymptotic values is not reproduced correctly by the minima of the approximated potential and comes out as too slow.

For the present purposes, the quality of the comparison between the deterministic attractors and the minima of φ is most crucial at intermediate driving fields E_0 where bistability occurs. For the special case $\delta=0$ the deterministic attractors ψ and r are plotted in Figs. 2(a) and 2(b) as a function of E_0 for various values of Δ. The corresponding minima of φ, given by Eq. (5.24) are plotted in Figs. 3(a) and 3(b). For small E_0 there is excellent quantitative agreement. For intermediate E_0 there is still good qualitative agreement. In particular, the size of the bistable regime is reproduced very well by the potential with quantitative agreement on the unbleached branch and qualitative agreement on the unstable intermediate branch (where φ has a maximum) and the stable bleached branch. For large E_0 the approach to the asymptotic values $r=E_0$, $\psi=0$ is too slow.

The approximation deterioriates, if we use $\Gamma^2(\delta-\Delta)$, or, for $\delta=0$, $\Gamma^2\Delta$ as additional small parameter. The advantage of this approximation is that we obtain φ in terms of elementary functions:

$$\varphi(r,\psi)=(r^2-2rE_0\cos\psi)+\Gamma^2\ln(1+r^2)$$
$$+2E_0\frac{\Delta\Gamma^2}{(1+\Gamma^2)^{1/2}}\sin\psi\arctan\frac{r}{(1+\Gamma^2)^{1/2}}. \tag{6.7}$$

There is still qualitative agreement with the exact attractors shown in Figs. 2(a) and 2(b) but quantitative agreement is poor, except on the unbleached branch.

The relative depth of the two minima of φ is shown in Fig. 4 for the potential (5.24). The absolute stability of the two minima is exchanged where they have equal depth. This defines the analog of the Maxwell rule of thermodynamic equilibrium for our nonequilibrium steady state. In Fig. 5 we compare for the potential (5.24) the two cases $\Delta=0$, where the Maxwell rule applies unchanged (cf. Sec. IV A) and $\Delta=0.5$, $\Delta=1$, where departures from the Maxwell rule are obtained. In Fig. 6 we present a plot of the two-dimensional probability density $W\sim\exp(-\varphi/Q)$ obtained from the potential (6.7).

The potential φ can be used to split the deterministic drift in the manner of Eqs. (2.14) and (2.15). We obtain

$$r_x=K_y+\frac{1}{2}\frac{\partial\varphi}{\partial x},$$

$$r_y=K_y+\frac{1}{2}\frac{\partial\varphi}{\partial y}.$$

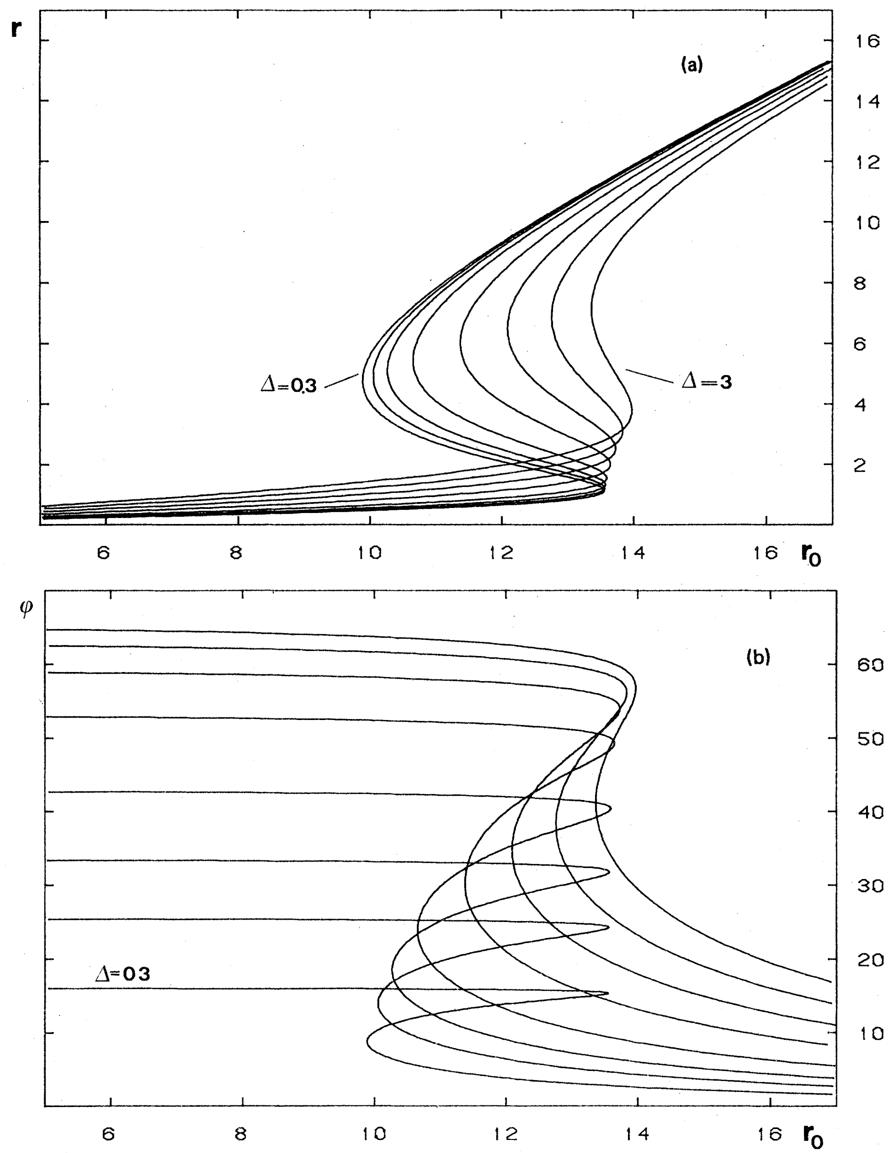

Fig. 2. Deterministic stationary points in polar coordinates for $\Gamma^2=25$, $\Delta=0.3$, 0.5, 0.7, 1.0, 1.5, 2.0, 2.5, 3.0.

For the potential (5.24), which we rewrite in the form

$$\varphi=+r^2+\Gamma^2\ln(1+r^2)-2rr_0\frac{\cos\tilde{\psi}}{(1+\delta^2)^{1/2}}+i(\delta-\Delta)E_0[C(\Delta^2)e^{i\psi}F(r)-C^*(\Delta^2)e^{-i\psi}F^*(r)]\,,$$

the resulting expression for r^ν is obtained as

$$r^\nu=r^\nu_{\rm rev}+r^\nu_{\rm irrev}$$

with the components, in polar coordinates,

$$r_{\rm rev}=r_{\rm rev}(\Delta=\delta)-\tfrac{1}{2}(\delta-\Delta)E_0\begin{pmatrix}(CF^1+C^*F^{*1})\sin\psi\\(CF+C^*F^*)\cos\psi\end{pmatrix}, \tag{6.8}$$

$$r_{\rm irrev}=r_{\rm irrev}(\Delta=\delta)+\tfrac{1}{2}(\delta-\Delta)E_0\begin{pmatrix}(CF^1-C^*F^{*1})\cos\psi\\-(CF-C^*F^*)\sin\psi\end{pmatrix}, \tag{6.9}$$

where $F^1=dF/dr$ and $r^{\rm irr,rev}(\delta=\Delta)$ is

$$r_{\rm rev}(\Delta=\delta)=\frac{\Delta E_0}{1+\Delta^2}\begin{pmatrix}\sin\psi\\\cos\psi\end{pmatrix}-\Delta r\begin{pmatrix}0\\\dfrac{1+\Gamma^2+r^2}{1+r^2}\end{pmatrix}, \tag{6.10}$$

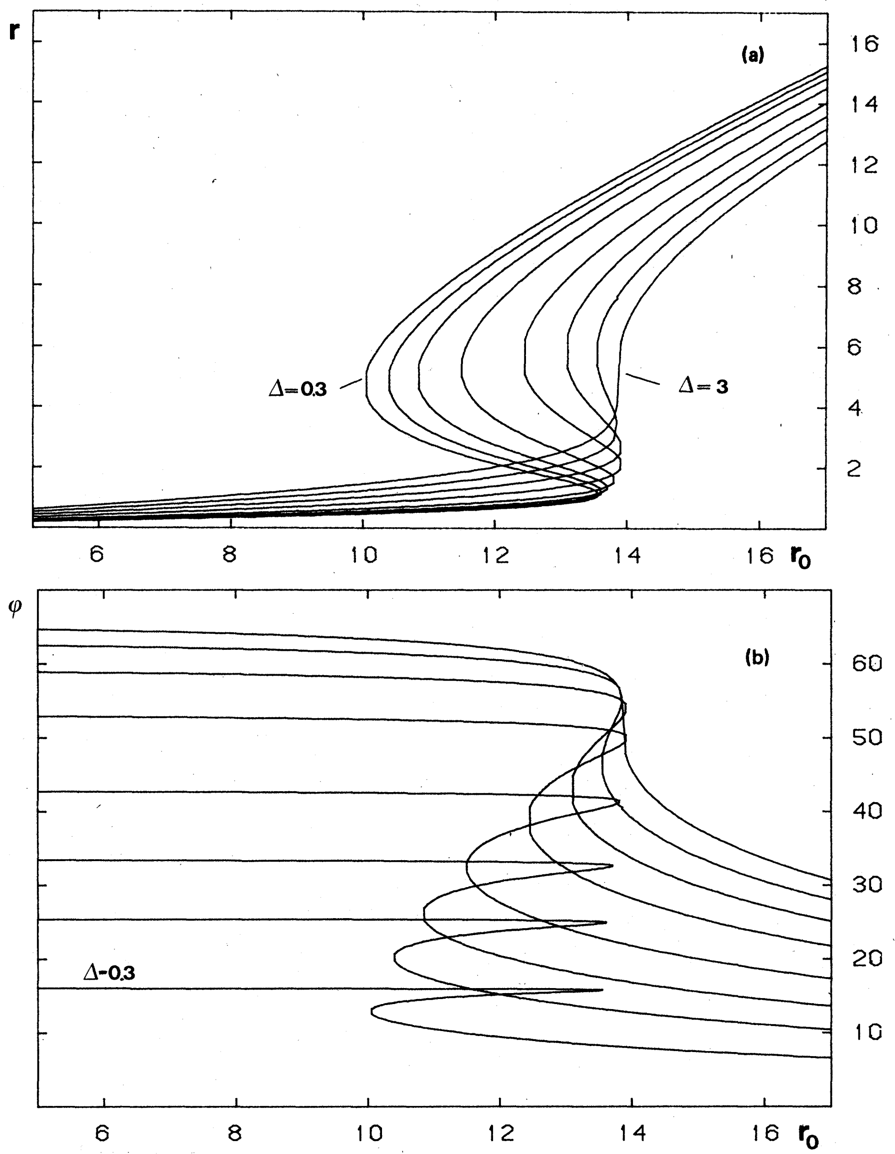

FIG. 3. Most probable values of the stationary distribution (5.24) in polar coordinates for $\Gamma^2=25$, $\Delta=3.0$, 0.5, 0.7, 1.0, 1.5, 2.0, 2.5, 3.0.

$$r_{\text{irrev}}(\Delta=\delta)=-\frac{\Delta^2 E_0}{1+\Delta^2}\begin{pmatrix}-\cos\psi\\ +\sin\psi\end{pmatrix}. \tag{6.11}$$

The result we obtain here generalizes obviously the previously derived result, which was restricted to the case $\delta=\Delta$.

APPENDIX A: SOLUTION OF EQ. (5.3)

It will be useful to introduce the abbreviations

$$\gamma=1+\Gamma^2,\quad \eta=\delta+\Delta\Gamma^2. \tag{A1}$$

Substituting φ_0 from Eq. (4.29) into (5.3) we obtain the inhomogeneous partial differential equation:

$$\frac{\partial\varphi_1}{\partial x}\left(\gamma x+\eta y+\frac{\eta^2-\gamma^2}{\eta^2+\gamma^2}E_0\right)+\frac{\partial\varphi_1}{\partial y}\left(\gamma y-\eta x+\frac{2\gamma\eta}{\gamma^2+\eta^2}E_0\right)=-2\gamma\Gamma^2(x^2+y^2)\left((x^2+y^2)\frac{E_0}{\gamma^2+\eta^2}\left[(-\gamma+\Delta\eta)x+(\gamma\Delta+\eta)y\right]\right). \tag{A2}$$

The characteristics satisfy, with $\beta \equiv x+iy$, the complex equation

$$\frac{d\beta}{dt}=\beta(\gamma-i\eta)-E_0\,\frac{(\gamma-i\eta)^2}{\gamma^2+\eta^2}\,, \tag{A3}$$

which is integrated immediately to yield

$$\beta(t)=\beta_0 e^{(\gamma-i\eta)t}+\frac{\gamma-i\eta}{\gamma^2+\eta^2}E_0\,. \tag{A4}$$

φ_1 is now given by the total differential

$$\frac{d\varphi_1}{dt}=-2\gamma\Gamma^2|\beta(t)|^4-\frac{\gamma\Gamma^2E_0}{\gamma^2+\eta^2}|\beta(t)|^2\{\beta(t)[-\gamma+\eta\Delta-i(\gamma\Delta+\eta)]+\beta^*(t)[-\gamma+\eta\Delta+i(\gamma\Delta+\eta)]\}\,. \tag{A5}$$

Upon integration, using (A4) we obtain

$$\varphi_1=\Bigg(-\Gamma^2\gamma\,\frac{(\gamma+i\eta)(\gamma+3i\eta)(1-i\Delta)E_0^3}{(\gamma^2+\eta^2)^3}\,\tilde\beta-\Gamma^2\gamma\,\frac{(\gamma+i\eta)(1-i\Delta)E_0^2}{2(\gamma^2+\eta^2)^3}\,\tilde\beta^2$$
$$-\Gamma^2\frac{E_0^2}{\gamma^2+\eta^2}|\tilde\beta|^2-\gamma\Gamma^2\frac{(\gamma+i\eta)(3\gamma+i\eta)(3-i\Delta)E_0}{(9\gamma^2+\eta^2)(\gamma^2+\eta^2)}|\tilde\beta|^2\tilde\beta-\frac{\Gamma^2}{4}|\tilde\beta|^4\Bigg)+(\text{c.c.})+f(\beta,\beta^*) \tag{A6}$$

with

$$\tilde\beta=\beta-\frac{\gamma-i\eta}{\gamma^2+\eta^2}E_0\,. \tag{A7}$$

f is the general solution of the homogeneous part of Eq. (A2). It is therefore an arbitrary function of the integration constant of the characteristics (A4) expressed as function of β,β^*. In order to find the integration constant we solve (A4) for t:

$$(\gamma-i\eta)t=\ln\frac{\tilde\beta}{\beta_0}\,. \tag{A8}$$

Eliminating t we find

$$\frac{1}{\gamma}\ln\left|\frac{\tilde\beta}{\beta_0}\right|=\frac{i}{2\eta}\ln\frac{\beta\beta_0^*}{\beta^*\beta_0}\,. \tag{A9}$$

With $\beta_0=r_0e^{i\psi_0}$, $\beta=re^{i\psi}$ we obtain the constant of integration

$$\frac{\psi_0}{\eta}+\frac{1}{\gamma}\ln r_0=\frac{\psi}{\eta}+\frac{1}{\gamma}\ln r\,. \tag{A10}$$

Thus f is an arbitrary function of the form

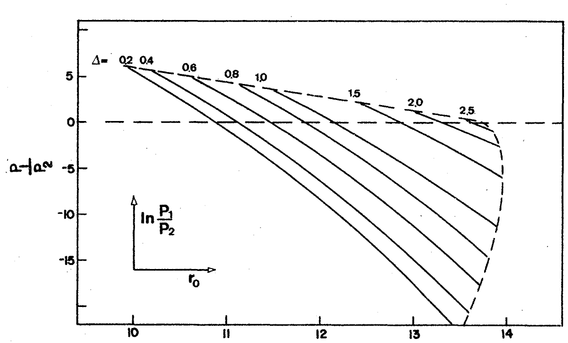

FIG. 4. Ratio of the probabilities of the two branches in the bistable domain.

$$f=f\left(\psi+\frac{\eta}{\gamma}\ln r\right). \tag{A11}$$

We now impose on φ_1 the condition that it is a regular single-valued function in the x,y plane. The latter requirement restricts f to 2π-periodic functions. However, all 2π-periodic functions of $\psi+(\eta/\gamma)\ln r$ except the constant function have a singularity at $r=0$. Therefore, f must be a constant

$$f=\text{const}$$

and φ_1 is unique except for an additive constant. In Eq. (5.4) the solution (A6) has been written as a quartic polynomial in x,y. We also made use of the relation $\eta-\Delta\gamma=\delta-\Delta$.

APPENDIX B: SOLUTION OF EQ. (5.6)

The characteristics of Eq. (5.6) satisfy the complex equation

$$\dot\beta=(1-i\delta)\beta+\frac{(\delta+i)^2}{1+\delta^2}E_0\,, \tag{B1}$$

where we have put $\beta=x+iy$. The solution of (B1)

$$\beta=\tilde E_0+\beta_0e^{(1-i\delta)t} \tag{B2}$$

with

$$\tilde E_0=\frac{1-i\delta}{1+\delta^2}E_0 \tag{B3}$$

is inserted in Eq. (5.6) to obtain the characteristic equation for φ_1:

$$\frac{d\varphi_1}{d\tau}=2\Gamma^2\frac{e^{2\tau}|\beta_0|^2+\frac{1}{2}(\tilde E_0\beta_0^*e^{(1+i\delta)\tau}(1+i\Delta)+\text{c.c.})}{1+|\tilde E_0|^2+|\beta_0|^2e^{2\tau}+(\tilde E_0\beta^*e^{(1+i\delta)\tau}+\text{c.c.})}. \tag{B4}$$

Upon integration and after eliminating β_0 by using (B2), Eq. (B4) yields a particular solution for φ_1:

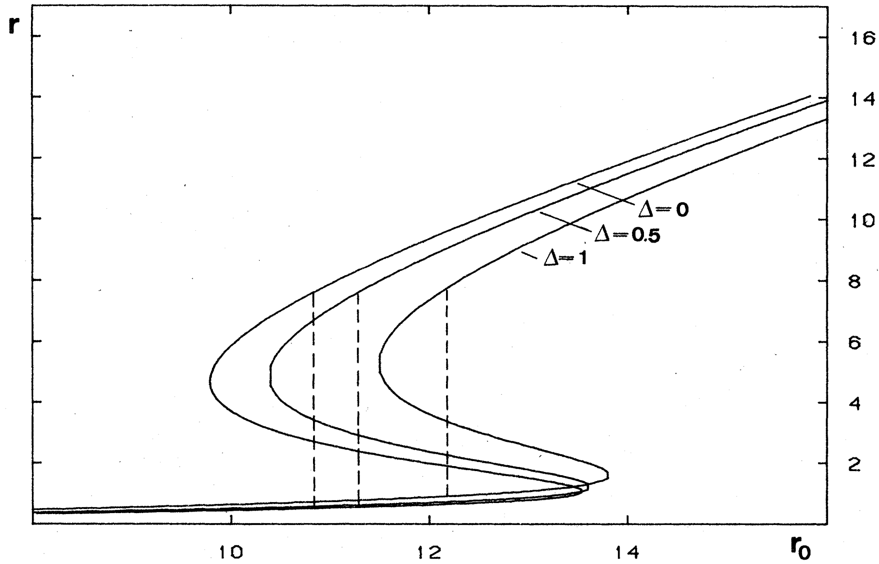

FIG. 5. Generalization of the Maxwell construction.

$$\varphi_1 = \Gamma^2 \ln(1+|\beta|^2) + i\Gamma^2(\Delta-\delta)\int_0^\infty d\tau \frac{(\tilde{E}_0(\beta^* - \tilde{E}_0^*)e^{-(1+i\delta)\tau} - \text{c.c.})}{|\tilde{E}_0|^2 + 1 + |\beta - \tilde{E}_0|^2 e^{-2\tau} + [\tilde{E}_0(\beta^* - \tilde{E}_0^*)e^{-(1+i\delta)\tau} + \text{c.c.}]} \,. \tag{B5}$$

We note that the particular solution (B5) is regular and single valued in the entire x, y plane.

A general solution is obtained by adding to (B5) a general solution of the homogeneous part of Eq. (5.6). The latter may be taken as an arbitrary function f of the constant of integration, which is obtained by eliminating t between Eq. (B2) and its complex conjugate. We obtain the constant of integration

$$|\beta_0^{\delta-i}| = |(\beta - \tilde{E}_0)^{\delta-i}| \,. \tag{B6}$$

Equivalently we may write $\psi_0 + \delta \ln r_0 = \tilde{\psi} + \delta \ln \tilde{r}$, with $\beta_0 = r_0 e^{i\psi_0}$, $\beta - \tilde{E}_0 = \tilde{r} e^{i\tilde{\psi}}$.

The requirement of single valuedness in the entire complex β plane restricts f to 2π-periodic functions of $\tilde{\psi} + \delta \ln \tilde{r}$. Then the only way to avoid a singularity at $\tilde{r} = 0$ is to restrict f further to the constant function. In this way, the solution φ_1 given by Eq. (B5) becomes unique up to a constant.

APPENDIX C: SOLUTION OF EQ. (5.16)

We define $\tilde{\varphi}_1$ by

$$\varphi_1 = -\frac{2E_0 x}{1+\delta^2} + \frac{2\delta E_0 y}{1+\delta^2} + \tilde{\varphi}_1 \tag{C1}$$

and obtain

$$\frac{\partial \tilde{\varphi}_1}{\partial x}\left(x + \delta y + \Gamma^2 \frac{x + \Delta y}{1 + x^2 + y^2}\right) + \frac{\partial \tilde{\varphi}_1}{\partial y}\left(y - \delta x + \Gamma^2 \frac{y - \Delta x}{1 + x^2 + y^2}\right) = -\frac{2E_0\Gamma^2}{1+\delta^2}\,\frac{(\delta - \Delta)}{1 + x^2 + y^2}\,(x\delta + y) \,. \tag{C2}$$

Introducing polar coordinates by

$$x = r\cos\psi, \quad y = r\sin\psi \tag{C3}$$

the differential equation of the characteristics of (C2) may be written as

$$\frac{dr}{d\psi} = -\frac{r(1+\Gamma^2+r^2)}{\delta r^2+\delta+\Delta\Gamma^2}. \quad \text{(C4)}$$

Upon integration we find

$$\psi = \psi_0 + \frac{(\delta-\Delta)\Gamma^2}{2(1+\Gamma^2)} \ln\frac{r^2}{1+\Gamma^2+r^2} - \delta\ln r. \quad \text{(C5)}$$

$\tilde{\varphi}_1$ varies along the characteristics according to

$$\frac{d\tilde{\varphi}_1}{dr} = -\frac{2E_0\Gamma^2(\delta-\Delta)}{(1+\Gamma^2+r^2)(1+\delta^2)^{1/2}} \sin[\psi(r)+\arctan\delta]. \quad \text{(C6)}$$

A particular integral of (C2) is therefore given by

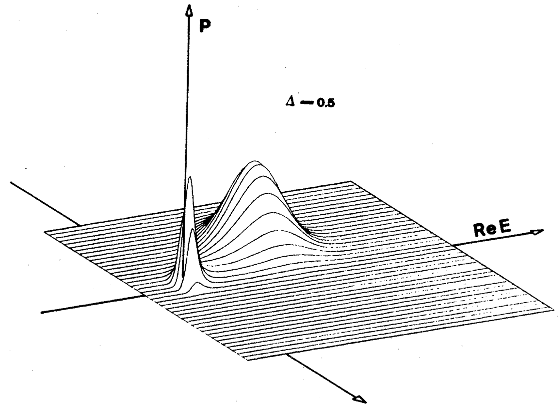

FIG. 6. Probability distribution for the dispersive optical bistability ($\Delta = 0.5$, $\Gamma^2 = 25$, $r_0 = 10.1$).

$$\tilde{\varphi}_1(r,\psi) = \frac{2E_0\Gamma^2(\delta-\Delta)}{(1+\delta^2)^{1/2}} \int_0^r dr' \sin\left[\tilde{\psi} + \frac{\delta-\Delta}{2(1+\Gamma^2)}\Gamma^2 \ln\left(\frac{r'^2}{1+\Gamma^2+r'^2}\frac{1+\Gamma^2+r^2}{r^2}\right) - \delta\ln\frac{r'}{r}\right] \Big/ (1+\Gamma^2+r'^2), \quad \text{(C7)}$$

where $\tilde{\psi} = \psi + \arctan\delta$ and where we have used Eq. (C5) twice in order to express $\psi(r')$ by r' and in order to eliminate ψ_0 in favor of ψ and r.

Introducing $z = r'/r$, $g^2 = (1+\Gamma^2)/r^2$, $a = \{(\Delta-\delta)/[2(1+\Gamma^2)]\}\Gamma^2$ we arrive at

$$\tilde{\varphi}_1 = -\frac{E_0\Gamma^2(\delta-\Delta)}{(1+\delta^2)^{1/2} ri} \left(e^{i\tilde{\psi}}(1+g^2)^{-ia} \int_0^1 dz \frac{(z)^{-i2a-i\delta}}{(g^2+z^2)^{1-ia}} - \text{c.c.}\right). \quad \text{(C8)}$$

The integral can be performed, using a formula of Ryzhik-Gradstein,[25] and we obtain

$$\tilde{\varphi}_1 = +E_0 \frac{\Gamma^2(\delta-\Delta)}{(1+\delta^2)^{1/2}} ir \left(\frac{F\left(1, \frac{1}{2} - i\frac{\delta}{2}; \frac{3}{2} - \frac{i}{2}\frac{\delta+\Gamma^2\Delta}{1+\Gamma^2}; -\frac{r^2}{1+\Gamma^2}\right)e^{i\tilde{\psi}}}{1+\Gamma^2-i(\Delta\Gamma^2+\delta)} - \text{c.c.}\right). \quad \text{(C9)}$$

F is a hypergeometric function, which is regular and single valued for $0 \le r < \infty$. A general solution of Eq. (C2) is obtained by adding an arbitrary function

$$f = f\left(\psi + \frac{\Delta-\delta}{2(1+\Gamma^2)}\Gamma^2 \ln\frac{r^2}{1+\Gamma^2+r^2} + \delta\ln r\right). \quad \text{(C10)}$$

However, only by choosing f as a 2π-periodic function we obtain a single-valued solution. If $\delta \neq -\Gamma\Delta$, all 2π-periodic functions (C10) necessarily have a singularity at $r = 0$, except for the constant function, $f = \text{const}$. In the special case $\delta = -\Gamma\Delta$ the singularity for $r = 0$ is avoided and the 2π-periodic function f remains arbitrary in this order in E_0. It is only restricted by requiring that single valuedness and regularity of the solution persist to higher order in the expansion. However, requiring that the solution $\tilde{\varphi}_1$ be a continuous function of δ and $\Gamma\Delta$ we may take $f = \text{const}$ also for $\delta = -\Gamma\Delta$. The solution of Eq. (5.12) is therefore given by Eqs. (C1) and (C9).

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