Self-phase modulation in long-geometry optical waveguides

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We calculate the effect of self-phase modulation of a pulse propagating in a long-geometry waveguide. Our calculations go beyond the usual theory, which does not take into account the envelope time variation in the nonlinear term of the wave equation. We show that for long waveguides with relatively small group-velocity dispersion but finite nonlinear coefficient n_2, the pulse will develop a sizable asymmetric frequency and temporal spectra.

Self-phase modulation was first observed by Shimuzu[1] when a modulated spectrum appeared after self-focusing had taken place in a liquid-filled cell and was explained as phase modulation due to the intensity-dependent refractive index. It has since been observed[2] in the absence of self-focusing or self-trapping and at low powers by using liquid-filled glass fibers.

Recently, some measurements of frequency broadening of mode-locked laser pulses due to self-phase modulation in single-mode silicacore fibers have been reported.[3] A study of the development of the broadened spectrum in these results showed the output spectrum to be asymmetric. In Ref. 3 it was assumed that an asymmetric incoming pulse shape was responsible for the observed asymmetric spectrum. The authors were able to generate, for short samples, in their computer calculations, an asymmetric spectrum similar to the observed one. However, for long samples the computed spectra and the observed one are in discrepancy. We believe that corrections to the commonly used theory of self-phase modulation may be responsible for this discrepancy. In this paper we intend to address ourselves to this point.

We present calculations for the frequency broadening due to self-phase modulation which show an asymmetry in the output spectrum for symmetric initial pulse. Comparison with results of Ref. 3 is difficult to make at present, firstly due to large asymmetry of the incoming pulse and secondly due to dispersion effects which will be discussed later in this paper. We thus limit ourselves to the development of the theory and determine the realistic conditions under which comparison between theory and observation is easy to make.

We limit ourselves to a one-dimensional wave propagation of an optical pulse propagating in a glass characterised by a nonlinear refractive index n. The electric field $\vec{E}(z,t)$ is given by the wave equation[4,5]

$$\frac{\partial^2\vec{E}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\vec{D}_L}{\partial t^2}=\frac{2n_2n_0}{c^2}\frac{\partial^2}{\partial t^2}(|\vec{E}|^2\vec{E}), \tag{1}$$

where n_2 represents the nonlinear part of the refractive index

$$n(\omega,\vec{E})=n(\omega)+n_2|\vec{E}|^2,$$

with $n_0=n(\omega_0)$, where ω_0 is the frequency of the electric field and $\vec{D}_L$ is the linear displacement.

We write the electric field in terms of a slowly varying envelope

$$\vec{E}(z,t)=\hat{e}A(z,t)\exp i(qz-\omega_0 t), \tag{2}$$

where q is the propagation constant and find[5] that $A(z,t)$ obeys the equation

$$\left[\frac{\partial^2}{\partial z^2}+2iq\frac{\partial}{\partial z}-q^2+\left(k_0^2+2ik_0k_0'\frac{\partial}{\partial t}-(k_0'^2+k_0k_0'')\frac{\partial^2}{\partial t^2}\right)\right]A(z,t)=\frac{2n_2n_0}{c^2}e^{i\omega_0 t}\frac{\partial^2}{\partial t^2}(|A|^2Ae^{-i\omega_0 t}), \tag{3}$$

where

$$k^2(\omega)=\frac{\omega^2n^2(\omega)}{c^2},\quad k_0^2=\frac{\omega_0^2n_0^2}{c^2},\quad k_0'=\left.\frac{\partial k}{\partial\omega}\right|_{\omega_0},\quad\text{and } k_0''=\left.\frac{\partial^2k}{\partial\omega^2}\right|_{\omega_0}. \tag{4}$$

In Eq. (3) we retain only the first and second time derivatives of A. Here we make the usual assumption that our dielectric is weakly dispersive. This is an extremely good approximation for SiO_2 glasses for our operating frequency which is much smaller than the electronic resonance frequency and much larger than the ionic resonance frequency of SiO_2.

To first order we take $q=k_0$ as for the plane-wave situation, and identify k_0' to be the reciprocal of the group velocity v_g. Thus we are left with the following equation:

$$\left(\frac{\partial^2}{\partial z^2}+2ik_0\frac{\partial}{\partial z}+2ik_0k_0'\frac{\partial}{\partial t}-(k_0'^2+k_0k_0'')\frac{\partial^2}{\partial t^2}\right)A(z,t)=\frac{2n_2n_0}{c^2}e^{i\omega_0t}\frac{\partial^2}{\partial t^2}(|A|^2Ae^{-i\omega_0t}). \quad (5)$$

In treating self-phase modulation one usually replaces the right-hand side of Eq. (5) by $-2(n_2/n_0)k_0^2|A|^2A$ as the dominant contribution, assuming that $\omega_0T\gg 1$. Here T represents the width of the pulse. However, as we shall see shortly, this assumption is correct only for short enough samples. For long samples, as for the case of self-phase modulation in optical waveguides, we have to consider the corrections arising from the time derivatives of $|A|^2A$. We will treat this effect perturbatively and thus will retain only the first-order term, i.e., the first derivative of $|A|^2A$. Equation (5) is therefore approximated by

$$\left(\frac{\partial^2}{\partial z^2}+2ik_0\frac{\partial}{\partial z}+2ik_0k_0'\frac{\partial}{\partial t}-(k_0'^2+k_0k_0'')\frac{\partial^2}{\partial t^2}\right)A(z,t)=-\frac{2n_2}{n_0}k_0^2|A|^2A-\frac{4in_2}{c}k_0\frac{\partial}{\partial t}(|A|^2A). \quad (6)$$

We now make a coordinate transformation to a moving coordinate system defined by

$$\xi=z,\quad \tau=t-z/v_g.$$

Equation (6) transforms to yield

$$\left\{\left[\left(\frac{\partial}{\partial\xi}-2k_0'\frac{\partial}{\partial\tau}\right)+2ik_0\right]\frac{\partial}{\partial\xi}-k_0k_0''\frac{\partial^2}{\partial\tau^2}\right\}A(\xi,\tau)=-\frac{2n_2}{n_0}k_0^2|A|^2A-\frac{4in_2}{c}k_0\frac{\partial}{\partial\tau}(|A|^2A). \quad (7)$$

To further simplify Eq. (7) we realized that without the dispersion (k_0'') and the nonlinearity (n_2) the solution for A is given by an arbitrary function of τ, say, $F((z-v_gt)/\bar{z})$, where $\bar{z}$ is the length of the pulse. It then becomes obvious that while $\partial/\partial\xi$ and $k_0'\partial/\partial\tau$ are of the order of $\bar{z}^{-1}$, the term k_0 is proportional to λ_0^{-1}. We can now use the slowly changing envelope approximation which required $\bar{z}$ to be much larger than λ_0 so that $(\partial/\partial\xi-2k_0'\partial/\partial\tau)A$ is neglected relative to k_0A, i.e., the effects of backscattered radiation are neglected. This implies that the changes in A per wavelength are extremely small. This condition is compatible with our suggested experimental situation, where changes in A are only observed after the pulse propagates hundreds of meters in the guide. We next define the constants

$$\alpha=-\tfrac{1}{2}k_0'',\quad \lambda=\frac{n_2}{n_0}k_0,\quad \gamma=\frac{2n_2}{c} \quad (8)$$

and obtain our final equation

$$i\frac{\partial A}{\partial\xi}+\alpha\frac{\partial^2A}{\partial\tau^2}+\lambda|A|^2A+i\gamma\frac{\partial}{\partial\tau}(|A|^2A)=0. \quad (9)$$

In order to solve the above equation, we separate the real and imaginary parts of A by writing

$$A=\psi e^{i\theta} \quad (10)$$

and find that ψ and θ are the solutions of the coupled equations

$$\psi\frac{\partial\theta}{\partial\xi}=\alpha\frac{\partial^2\psi}{\partial\tau^2}-\alpha\psi\left(\frac{\partial\theta}{\partial\tau}\right)^2+\lambda\psi^3\frac{\partial\theta}{\partial\tau} \quad (11a)$$

and

$$-\frac{\partial\psi}{\partial\xi}=2\alpha\frac{\partial\psi}{\partial\tau}\frac{\partial\theta}{\partial\tau}+\alpha\psi\frac{\partial^2\theta}{\partial\tau^2}+3\gamma\psi^3\frac{\partial\psi}{\partial\tau}. \quad (11b)$$

It was not found possible to solve Eqs. (11) exactly and a perturbative method was utilized to obtain approximate expressions for ψ and θ, exact in λ but in power series in α and γ. Let

$$\theta=\theta_0+\gamma\theta_1+\alpha\theta'+\cdots, \quad (12a)$$

$$\psi=\psi_0+\gamma\psi_1+\alpha\psi'+\cdots. \quad (12b)$$

Substituting in the Eqs. (11), we obtain

$$(\psi_0+\gamma\psi_1+\alpha\psi')\left(\frac{\partial\theta_0}{\partial\xi}+\gamma\frac{\partial\theta_1}{\partial\xi}+\alpha\frac{\partial\theta'}{\partial\xi}\right)=\alpha\frac{\partial^2\psi_0}{\partial\tau^2}-\alpha\psi_0\left(\frac{\partial\theta_0}{\partial\tau}\right)^2+\lambda(\psi_0^3+3\gamma\psi_0^2\psi_1+3\alpha\psi_0^2\psi')-\gamma\psi_0^3\frac{\partial\theta_0}{\partial\tau} \quad (13a)$$

and

$$-\frac{\partial}{\partial\xi}(\psi_0+\gamma\psi_1+\alpha\psi')$$

$$=2\alpha\frac{\partial\psi_0}{\partial\tau}\frac{\partial\theta_0}{\partial\tau}+\alpha\psi_0\frac{\partial^2\theta_0}{\partial\tau^2}+3\gamma\psi_0^2\frac{\partial\psi_0}{\partial\tau}. \quad (13b)$$

Comparing successively increasing powers of γ and α in Eqs. (13) one gets

$$\psi_0=\psi_0(\tau),$$

$$\theta_0 = \lambda \xi \psi_0^2 ,$$

$$\psi_1 = -\xi \frac{\partial}{\partial \tau} \psi_0^3 ,$$

$$\theta_1 = -4\lambda \xi^2 \psi_0^3 \frac{\partial \psi_0}{\partial \tau} ,$$

$$\psi' = -\tfrac{1}{2} \lambda \xi^2 \psi_0 \left[4 \left(\frac{\partial \psi_0}{\partial \tau} \right)^2 + \frac{\partial^2 \psi_0^2}{\partial \tau^2} \right] ,$$

$$\theta' = \xi \frac{1}{\psi_0} \frac{\partial^2 \psi_0}{\partial \tau^2} - \tfrac{2}{3} \lambda^2 \xi^2 \psi_0^2 \left[5 \left(\frac{\partial \psi_0}{\partial \tau} \right)^2 + \psi_0 \frac{\partial^2 \psi_0}{\partial \tau^2} \right] .$$

Combining the above results with Eqs. (10) and (12), the solution for $A(\xi, \tau)$ is obtained as

$$A(\xi, \tau) = \left\{ \psi_0 - 3\xi\gamma\, \psi_0^2 \frac{\partial \psi_0}{\partial \tau} - \xi^2 \lambda \alpha \psi_0 \left[3 \left(\frac{\partial \psi_0}{\partial \tau} \right)^2 + \psi_0 \frac{\partial^2 \psi_0}{\partial \tau^2} \right] \right\}$$

$$\times \exp i \left\{ \xi \lambda \psi_0^2 - 4\xi^2 \lambda \gamma \psi_0^3 \frac{\partial \psi_0}{\partial \tau} + \xi \alpha \frac{1}{\psi_0} \frac{\partial^2 \psi_0}{\partial \tau^2} - \tfrac{2}{3} \xi^2 \lambda^2 \alpha \psi_0^2 \left[5 \left(\frac{\partial \psi_0}{\partial \tau} \right)^2 + \psi_0 \frac{\partial^2 \psi_0}{\partial \tau^2} \right] \right\} , \qquad (14)$$

where $\psi_0 = \psi_0(\tau)$.

Eq. (14) represents the solution to Eq. (9), correct to the lowest order in γ and α. The function ψ_0 depends on τ only and represents the initial amplitude of A, i.e., when the pulse enters the waveguide. For simplicity, we take the initial pulse to be a Gaussian wave form represented by

$$\psi_0 = A_0 \exp(-\tau^2/T^2) . \qquad (15)$$

Substituting Eq. (15) in Eq. (14), the expression for $A(\xi, \tau)$ takes the form

$$A(\xi, \tau) = A_0 \left[1 + \frac{6\xi\gamma\tau}{T^2} A_0^2 e^{-2(\tau^2/T^2)} + \frac{2\xi^2 \lambda \alpha}{T^2} \left(1 - 8 \frac{\tau^2}{T^2} \right) A_0^2 e^{-2(\tau^2/T^2)} \right] e^{-(\tau^2/T^2)}$$

$$\times \exp i \left\{ \xi A_0^2 \lambda e^{-2(\tau^2/T^2)} \left[1 + \frac{8\xi\gamma\tau A_0^2}{T^2} e^{-2(\tau^2/T^2)} + \frac{4}{3} \frac{\xi^2 \lambda \alpha}{T^2} \left(1 - 12 \frac{\tau^2}{T^2} \right) A_0^2 e^{-2(\tau^2/T^2)} \right] - \frac{2\xi\alpha}{T^2} \left(1 - \frac{2\tau^2}{T^2} \right) \right\} . \qquad (16)$$

Using Eq. (2), the expression for the electric field can be rewritten as

$$\vec{\mathrm{E}}(\xi, \omega) = \hat{e} e^{i(k_0 \xi - (\omega_0 \xi / v_g))} \int d\tau\, A(\xi, \tau) e^{i(\omega - \omega_0)\tau} ,$$

where $A(\xi, \tau)$ is given by Eq. (16). It is apparent that most of the contribution to the frequency spectrum in the expression for the electric field comes from the self-phase modulation term in Eq. (16). Thus, except for a factor in the amplitude, the expression for $\vec{\mathrm{E}}(\xi, \omega)$ with the frequency spread is given by

$$\vec{\mathrm{E}}(\xi, \omega) = \hat{e} A_0 e^{i(k_0 \xi - (\omega_0 \xi / v_g))} \int F d\tau , \qquad (17)$$

where

$$F = \exp i \left\{ (\omega - \omega_0)\tau + \xi \lambda A_0^2 e^{-2(\tau^2/T^2)} \left[1 + \frac{8\xi\gamma\tau}{T^2} A_0^2 e^{-2(\tau^2/T^2)} + \tfrac{4}{3} \xi^2 \frac{\lambda \alpha}{T^2} \left(1 - 12 \frac{\tau^2}{T^2} \right) A_0^2 e^{-2(\tau^2/T^2)} \right] - \frac{2\xi\alpha}{T^2} \left(1 - \frac{2\tau^2}{T^2} \right) \right\} . \qquad (18)$$

In the above expansion, terms proportional to α arise from the dispersion effects and are seen to give rise to a symmetric modulation of the phase. The term proportional to γ, however, is proportional to τ and thus results in asymmetric phase modulation.

To analyze the expression in Eq. (18) we note that in our problem we have two fundamental lengths, the nonlinear length $\xi_0 = (\lambda A_0^2)^{-1}$ and the dispersive length $\xi_1 = T^2/|\alpha|$, and also the dimensionless parameter $\omega_0 T$. We rewrite Eq. (18) in terms of these parameters as

$$F=\exp i\left\{(\omega-\omega_0)\tau+\frac{\xi}{\xi_0}e^{-2(\tau^2/T^2)}\left[1+\frac{\xi}{L_1}\frac{\tau}{T}e^{-2(\tau^2/T^2)}+\frac{4}{3}\left(\frac{\xi}{L_2}\right)^2\left(1-12\frac{\tau^2}{T^2}\right)e^{-2(\tau^2/T^2)}\right]-\frac{2\xi}{\xi_1}\left(1-2\frac{\tau^2}{T^2}\right)\right\}. \tag{18}$$

Here $L_1=\omega_0 T\xi_0/16$ and $L_2=(\xi_1\xi_0)^{1/2}$. The effect of the linear dispersion can be omitted in many cases since it is easy to make $\xi_1 \gg \xi_0$. The correction terms depend on L_1 and L_2. The L_1 term is the correction arising from the finite duration of the pulse and is the only term that contributes in our theory to asymmetric spectrum. To observe the asymmetric effects we want the L_1 term to be the dominant correction. We thus need L_2 to be larger than L_1. We found it not to be the case in the situation of the experiment in Ref. 3, which makes comparison between theory and experiment rather difficult at present.

We note that the ratio $\rho=L_1/L_2$ should be, ideally, smaller than unity for our purposes. Here

$$\rho=\frac{1}{16}\left(\frac{c^2k_0k_0''}{2n_0n_2A_0^2}\right)^{1/2}$$

is independent of T and becomes small either at high beam intensity or for vanishingly small dispersion, i.e., $k_0''\simeq 0$.

It must be remembered that the expansion obtained above is a perturbation expansion and is expected to be valid only when the parameters of the problem are appropriately chosen to ensure that the higher-order terms are successively smaller than the leading terms. Also, in order that the asymmetric effects are measurable, the dispersion effects should be small enough so that the shape of the pulse is not greatly distorted.

The dispersion effects can be reduced by operating at a frequency where the dispersion is negligible[6] as has been recently demonstrated. Thus, in Eq. (17), the integrand reduces to

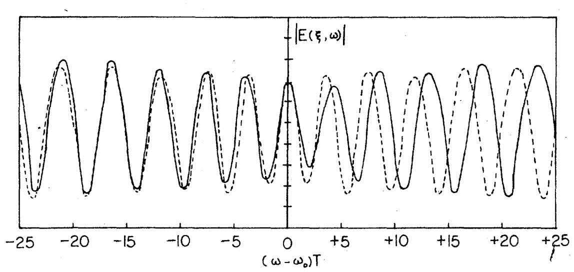

FIG. 1. $|E(\xi,\omega)|$ is given in arbitrary units, as a function of $(\omega-\omega_0)T$. The dashed curve represents the spectrum obtained by dropping the term proportional to γ in Eq. (19) while the solid curve represents the actual spectrum using the parameters given in the text.

$$F=\exp i\left[(\omega-\omega_0)\tau+\xi\lambda A_0^2e^{-2(\tau^2/T^2)}\left(1+\frac{8\xi\gamma\tau}{T^2}A_0^2e^{-2(\tau^2/T^2)}\right)\right]. \tag{19}$$

In order to make an estimate of the asymmetric effect, we calculate the value of the coefficient $8\xi\gamma\tau A_0^2/T^2$, using values that can be attained in the laboratory. Taking $\xi=1$ km, $\tau\sim T=5$ ps, $n_2=1.4\times10^{13}$ esu, and $A_0=500$ S V/cm,[7] we calculate $8\xi\gamma\tau A_0^2/T^2\sim 0.25$. Thus the effect of the asymmetric term compared to 1 is about 25% for the parameters chosen above and should be experimentally measurable.

In Fig. 1 we show a plot of $|E(\xi,\omega)|$ as a function of $(\omega-\omega_0)T$ around $\omega=\omega_0$, for the above parameters. For comparison purposes, the symmetric spectrum, obtained by dropping the γ-dependent term in Eq. (19), is also plotted. The effect of the γ term in causing asymmetry in the spectrum is quite evident.

We point out, finally, that not only is the spectrum defined by $|E(\xi,\omega)|^2$ asymmetrical, but the intensity spectrum given by

$$I(\xi,\omega)=\mathrm{FT}\,|E(\xi,t)|^2=\mathrm{FT}\,|A(\xi,t)|^2$$

is also asymmetric. (FT stands for the Fourier transform.) In the case for which $\alpha=0$, we obtain for the intensity spectrum

$$I(\xi,\omega)=\mathrm{FT}\left[A_0^2e^{-2(\tau^2/T^2)}\left(1+\frac{12\xi\gamma\tau}{T^2}A_0^2e^{-2(\tau^2/T^2)}\right)\right]. \tag{20}$$

The asymmetrical part of the integrand is proportional to γ in Eq. (20) and contributes to the asymmetrical intensity spectrum.

In conclusion, we have shown that self-phase modulation in optical waveguides, under realistic conditions, will result in asymmetrical power and intensity spectra.

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