Statistical properties of coherent radiation in a nonlinear optical amplifier

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The differential equation for the photon conditional probability distribution in a nonlinear amplifier is numerically solved, using the continuous approximation. Saturation effects on the development of fluctuations in the amplification of coherent light are studied. They are compared with the results obtained by solving the moment equations.

I. INTRODUCTION

Some years ago[1] statistical properties of the fluctuations in the amplified light were studied, but only for the linear case. Recently, more realistic studies have been published which take into account the nonlinearity of the amplifier. Some papers treat the amplification of coherent light.[2] Others, in treating the amplification, take into account the spontaneous emission[3,4,5] and at least a recent paper gives a more complete description of the light-amplification processes.[6] Recent progress in integrated devices and optical fibers has enhanced the interest of studying light amplifiers in view of their use in the optical communications system.

In this paper, we present numerical results of the photon conditional probability distribution for an amplified coherent light. The probability distribution, and first and second moments of the distribution are displayed. We use a model of nonlinear amplification which is an appropriate modification of the equation of Scully and Lamb.[7] Our results show that the normalized second- and third-order moments of the distribution first increase and then are continuously decreasing whereas the first moment (i.e., the mean photon number) is continuously increasing. The combination of the two facts may be of some interest in the theory of optical communication.

II. PHYSICAL HYPOTHESIS OF THE MATHEMATICAL APPROACH

We consider a gaseous optical amplifier consisting of two-level systems which are made of two excited states of an atom or a molecule. They are held in an inverted population state by an external pump. The Bohr frequency of these systems is nearly resonant with the frequency of the amplified light. This light is assumed to be monochromatic. The photon-number distribution at the input of the amplifier $P_n(0)$ is given. The photon-number distribution at the abscissa z into the amplifier $P_n(z|n_0)$ given the mean photon number n_0 at $z=0$, contains the information which is necessary to calculate the main properties of the amplified light. The fundamental equations which govern the evolution of $P_n(z|n_0)$, which we call $P_n(z)$ in the following, were first extensively studied by Shimoda, Takahasi, and Townes[1] for the linear amplification case. Actually an optical amplifier can be considered as linear only for a very small number of photons. It is an unrealistic case since the purpose of an amplifier is to give an output number of photons sufficiently great to drive a photon detector (a counter or another system). Equations (3) of Ref. 1 should be improved in order to take into account the saturation process which unavoidably occurs.[8] Scully and Lamb[7] have given an equation which takes into account the saturation phenomenon, but only for the terms which govern the photon emission. These terms are multiplied by a saturation function

$$f(n)=\frac{1}{1+\chi n}\,, \tag{1}$$

where n is a phonon number and χ a coefficient which is depending on the nature of the amplifier medium. In our model we take into account the saturation of the absorption by multiplying the terms which govern the absorption by the appropriate form of the function $f(n)$.

Recently, a similar expression for the function of saturation has been found.[6,10] The solution of these nonlinear equations cannot be easily expressed in a closed form. One common way to solve the problem is to solve the set of k equations for the $\langle n^k\rangle$ moments of the photon number or the related coefficients

$$g_k=(\langle n^k\rangle/\langle n\rangle^k)-1\quad(k>1)\,. \tag{2}$$

Usually one restricts the study to $k=1$ and 2. We then have the mean value of n and the g_2 coefficient which is the most commonly used ratio to describe the second-order statistical properties of a stationary random field. Unfortunately, the approximations which are mandatory in order to have tractable partial differential equations for the evolution of $\langle n\rangle$ and g_2, are very drastic.

One can replace the expression $1/(1+\chi n)$ by $1-\chi n$ or a Taylor expansion at the second or third order. Then the mean value $\langle n^3\rangle$ and eventually $\langle n^4\rangle$ are replaced by the decorrelated expressions

$$\langle n^3\rangle = \langle n^2\rangle\langle n\rangle \tag{3a}$$

and

$$\langle n^4\rangle = \langle n^2\rangle^2 . \tag{3b}$$

Another way consists in making the following approximation:

$$\left\langle \frac{n^k}{1+\chi n}\right\rangle = \frac{\langle n^k\rangle}{1+\chi\langle n\rangle} . \tag{4}$$

The first method has a major drawback. The Taylor development is not valid for $\chi\langle n\rangle \geq 1$, i.e., for the very large values of $\langle n\rangle$ for which the saturation phenomena become significant.

The second method is a more dangerous approach. The approximation contained in (4) is equivalent to an all order decorrelation. Such a decorrelation is valid only for a Glauber pure coherent field. So we are computing the g_2 coefficient which depends on statistical nature of the field and we make simultaneously an approximation which assumes that the field is coherent. A decorrelation process at any order is only valid if the field is coherent or nearly coherent. So we have to use as the input a Glauber coherent field and we have to check the $g_2(\tau)$ coefficient to ensure that it remains very small compared to one. So it is not advisable to use the results of such calculation if they have not been controlled by an exact calculation [i.e., a numerical integration of the system of differential equations which govern the evolution of $P_n(z)$].

In this paper we present some new results obtained by an exact numerical integration. We compare them with the results obtained by the approximate method and which are in only a qualitative agreement.

Using the Eqs. (17) and (59) of Ref. 7 and introducing the modification described above we find that we have to integrate the system of nonlinear differential equations

$$\frac{dP_n(\tau)}{d\tau} = -[(n+1)f(n+1)+\beta n f(n)]P_n(\tau) + n f(n)P_{n-1}(\tau) + \beta(n+1)f(n+1)P_{n+1}(\tau) , \tag{5}$$

where

$$\tau = Az/c , \tag{6a}$$

$$\beta = B/A , \tag{6b}$$

c is the speed of light, z is the abscissa into the amplifier, and A and B are, respectively, the coefficients which govern the emission and the absorption of the photons. These equations take into account the spontaneous emission of photons in the one normal mode which is considered in this model. We made the following approximations in order to solve the system (5) more effectively.

(a) The discrete function $P_n(\tau)$ is replaced by a continuous function $P(n,\tau)$. This approximation is usual[9] and is justified by the great number of photons which are always involved in the process.

$$\text{(b)}\quad P(n\pm 1,\tau) = P(n,\tau) \pm \frac{\partial P(n,\tau)}{\partial n} + \frac{1}{2}\frac{\partial^2 P(n,\tau)}{\partial n^2} . \tag{7}$$

$$\text{(c)}\quad f(n\pm 1) = f(n) \mp \chi f^2(n) . \tag{8}$$

It is important to note that these approximations are valid if $\langle n\rangle \gg 1$ and if $P(n,\tau)$ is a smooth function of n.

If the conditions for validity of these approximations are satisfied for $\tau=0$ they will remain satisfied for any value of τ if $P(n,\tau)$ is always a smooth function of n. So the smoothness condition will remain valid. However, the validity of the above approximations can be easily controlled during the computation.

As suggested in Ref. 9, to solve the equations more effectively we have considered the transformation

$$P(n,\tau) = \exp[R(n,\tau)] , \tag{9}$$

and we have integrated the partial differential equation of the form

$$\frac{\partial R(n,\tau)}{\partial \tau} = F\left(R(n,\tau), \frac{\partial R}{\partial n}, \frac{\partial^2 R}{\partial n^2}, n\right) . \tag{10}$$

This last transformation does not involve any supplementary approximation. The detailed expression for $\partial R/\partial\tau$ is given in the Appendix. We give also some details concerning the method used to compute Eq. (10).

III. MOMENT EQUATIONS

Solving the "exact" equation (10) directly is difficult and requires a long computation time. The solution $P(n,\tau)$ is obtained for one set of parameters (β,χ,n_0). So the study of the dependence of the solution against the variation of the parameters is a cumbersome task. These practical considerations make the solution of the moment equations an interesting approach despite the severe shortcomings of this method, which have been pointed out in Sec. II.

The time evolution of the mean photon number $\langle n\rangle$ and the second moment $\langle n^2\rangle$ may be derived from Eq. (5) and are

$$\frac{d\langle n\rangle_\chi}{d\tau} \approx \frac{\Delta\langle n\rangle_\chi}{1+\chi\langle n\rangle_\chi} + \frac{1+\chi\Delta\langle n\rangle_\chi}{(1+\chi\langle n\rangle_\chi)^2}, \tag{11a}$$

$$\frac{d\langle n^2\rangle_\chi}{d\tau} \simeq \frac{2\Delta\langle n^2\rangle_\chi + \Omega\langle n\rangle_\chi + 1}{(1+\chi\langle n\rangle_\chi)^2} + \chi\,\frac{(3-\beta)\langle n^2\rangle_\chi + (1+\beta)\langle n\rangle_\chi + 2\Delta\langle n^3\rangle_\chi}{(1+\chi\langle n\rangle_\chi)^2}, \tag{11b}$$

where we set

$$\Delta = 1-\beta \tag{12a}$$

and

$$\Omega = 3+\beta\,. \tag{12b}$$

We assume that $\langle n\rangle \gg 1$ and we use the decorrelation method mentioned above. For $\chi=0$, the solutions to Eqs. (11a) and (11b) are easy to find. We obtain

$$\langle n\rangle_0(\tau) = (n_0 + 1/\Delta)e^{\Delta\tau} - 1/\Delta\,, \tag{13}$$

and

$$\langle n^2\rangle_0(\tau) = \left[n_0^2 + n_0 + \frac{\Omega}{\Delta}\left(n_0 + \frac{1}{\Delta}\right) - \frac{1+\beta}{\Delta^2}\right]e^{2\Delta\tau} - \frac{\Omega}{\Delta}\left(n_0 + \frac{1}{\Delta}\right)e^{\Delta\tau} + \frac{1+\beta}{\Delta^2}, \tag{14}$$

where n_0 is the given mean photon number at the origin of the time axis. The symbol $g_2^0(\tau)$ stands for the $g_k(\tau)$ coefficient defined by Eq. (2) in the particular case $k=2$, $\chi=0$, and $g_2^\chi(\tau)$ is the same coefficient for $\chi\neq 0$. Then $g_2^0(\tau)$ is very well approximated by the relation

$$n_0 g_2^0(\tau) = \frac{2}{\Delta}\left(1 - \frac{1+\beta}{2}e^{-\Delta\tau}\right). \tag{15}$$

For an infinite value of τ it reaches the asymptotic value $2/\Delta$. To solve Eq. (11a) in the nonlinear case ($\chi\neq 0$) we find that a reasonably good method consists in using a perturbation method starting with

$$\frac{d\langle n\rangle_\chi}{d\tau} = \frac{\Delta\langle n\rangle_0(\tau)}{1+\chi\langle n\rangle_0(\tau)}, \tag{16}$$

where $\langle n\rangle_0(\tau)$ is given by Eq. (13). We then obtain

$$\langle n\rangle_\chi(\tau) = n_0 + \frac{1}{\chi}\ln\left(1 + \frac{\chi n_0}{1+\chi n_0}(e^{\Delta\tau}-1)\right). \tag{17}$$

Of course, the exact solution of Eq. (11a) can be calculated but only in an implicit form. This has actually been done by Litvak[5] who also gave an explicit approximate expression of $\langle n\rangle_\chi(\tau)$ for large τ. But the interest of Eq. (17) is first to give a good approximation to $\langle n\rangle_\chi(\tau)$ in the full range of τ, and also to allow us to compute analytically the second moment via Eq. (11b). Here again starting with $\langle n^2\rangle_0(\tau)$ given by Eq. (14), we use a perturbation method to seek a solution to $\langle n^2\rangle_\chi(\tau)$. We consider two cases: $\tau<\tau_s$ and $\tau>\tau_s$ where

$$\tau_s = \frac{1}{\Delta}\ln\left(\frac{1}{\chi n_0}\right) \tag{18}$$

(with the parameters chosen here $\tau_s \simeq 23$). Note that

$$\langle n\rangle_0(\tau_s) = 1/\chi \tag{19a}$$

and

$$\langle n\rangle_\chi(\tau_s) \simeq n_0 + \frac{1}{\chi}\ln\left(\frac{2}{1+\chi n_0}\right). \tag{19b}$$

For $\tau<\tau_s$ solving Eq. (11b) leads to

$$n_0 g_2^\chi(\tau) = n_0 g_2^0(\tau) + \frac{\Omega\chi n_0}{2\Delta}\ln\left(\frac{(1+\chi n_0 e^{\Delta\tau})^2}{(1+\chi n_0)e^{\Delta\tau}}\right), \tag{20}$$

where $g_2^0(\tau)$ is given by Eq. (15). For $\tau>\tau_s$ the same method is unfortunately inappropriate because it requires having the boundary value which is in our case unknown.

We find that, if we replace in Eq. (11b) the second term of the right-hand side by its Taylor development at first order we obtain results which are in a qualitative agreement with the exact results. Under these assumptions Eq. (11b) becomes

$$\frac{d\langle n^2\rangle_\chi}{d\tau} = \Omega\langle n\rangle_\chi(1-\chi\langle n\rangle_\chi) + 2\Delta\,\frac{\langle n^2\rangle_\chi}{1+\chi\langle n\rangle_\chi} \tag{21}$$

whose solution, valid only for $\tau>\tau_s$, leads to

$$g_2^\chi(\tau) = g_2^0(\tau) - \Omega\chi\tau\,. \tag{22}$$

We need to keep $n_0 g_2^\chi(\tau) > 1$. The upper limit of τ for this expression will be $\tau_l \simeq 1/\Delta\chi n_0 = 100$, with the parameters chosen here.

Note that these solutions (19b), (20), and (22) have been obtained from oversimplified moment equations. Their simple analytical expressions make them very useful for an approximate study of the variation of the statistical properties versus the parameters of the problem. Nevertheless, they give at least a correct qualitative behavior of the moments versus the time, and they are also reasonably good estimates to the exact solutions which we calculate now.

IV. RESULTS AND DISCUSSION

We have computed the first three moments of the fluctuating amplified light intensity using the exact equation method (EEM) and the moment equation method (MEM). The calculations have been performed in the linear case ($\chi=0$) and in the nonlinear case ($\chi=10^{-4}$). In the two cases we have chosen $\beta=0.9$. The input field was a coherent Glauber field with a mean photon number of 1000.

In the linear case the moment equation method is an exact method since decorrelations are not required. The exact calculations for the same conditions, have led to severe computational difficulties (probably connected with some instabilities of the differential equation system) so we have only used the moment equation method to compute the first and second moments of the distribution.

Figure 1 displays the photon number distribution $P_n(\tau)$ in the nonlinear case for the values of $\tau=5$, 15, 30, 40, and 50. In Fig. 2 we display the first and the second reduced moments in both the linear case and the nonlinear case.

In the linear case, we note that the first moment is exponential and the reduced second moment is constant for long times. In the nonlinear case the amplification process is nearly exponential when the amplifier is weakly saturated ($\chi\langle n\rangle \ll 1$) and tends to become linear when the amplifier is strongly saturated. The first moment calculated by Eq. (19a) becomes linear for $\tau > \tau_s$. It is important to note that the limiting form of the exact equation shows that the amplification is truly linear for a very strong saturation case.

More interesting is the behavior of $g_2^\chi(\tau)$. In the weak saturation region it is increasing. It reaches its maximum value in the moderate saturation region ($\tau \sim \tau_s$); then it is continuously decreasing. The g_2^χ calculated by the moment equation has the same behavior but the coordinates of the maximum of g_2^χ are only in a qualitative agreement with the values given by the exact calculation. The computation of $g_3^\chi(\tau)$ as defined by Eq. (2) for $k=3$ shows similar behavior for $g_3^\chi(\tau)$ as for $g_2^\chi(\tau)$. For $\tau \ll 1$, a lack of accuracy in the computation of $g_2^\chi(\tau)$ should be noted. The curves $n_0 g_2^\chi(\tau)$ (d) and $n_0 g_2^0(\tau)$ (b) should more clearly coincide. Also for $\tau \to 0$ all the curves plotted in Fig. 2 approach 1.

The decreasing part of the $n_0 g_2^\chi(\tau)$ curve that is the region for which there exist reductions of relative fluctuations, would be very interesting to transmit and detect optical information after nonlinear coherent amplification.

The use of two decorrelation processes in the integration of Eq. (11b) is unorthodox. In fact, it is not easy to justify physically nor mathematically the use of two different approximations in the same integration process. We can only assert that these approximations are the ones which the better fit the exact curve. In these conditions the moment equations cannot be used to predict the behavior of a nonlinear amplifier if an exact calculation has not been yet performed since we have no sure way to choose the best approximation if we are not guided by the exact calculation. Its only practical use is to study the effects of small variations of the parameters around the values which have been used in the exact computation. Therefore, our calculations are another warning against the unconsidered use of the decorrelations in quantum optics.

V. CONCLUSION

In summary the results presented here show that nonlinear amplification of laser light tends to reduce the relative fluctuations of the intensity. The simple model we considered does not require any limitation for the gain. We also established approximate and locally valid formulas for both

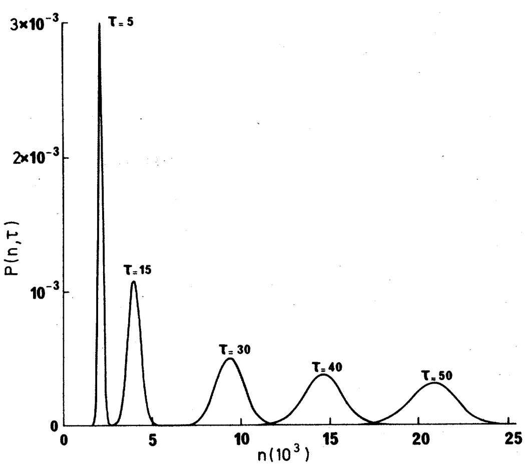

FIG. 1. Photon-number distribution for the nonlinear case ($\chi=10^{-4}$) with the values of $\tau=5$, 15, 30, 40, and 50. The distributions are normalized so that $\sum_n P(n,\tau) = 1$.

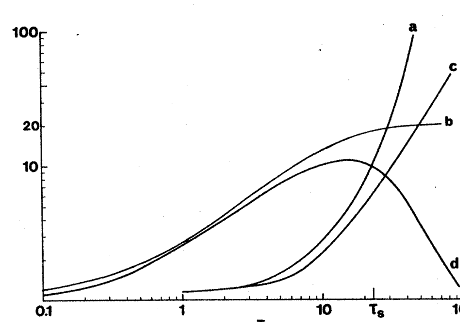

FIG. 2. Variation of the first two reduced moments versus τ for the linear case ($\chi=0$) and the nonlinear case ($\chi=10^{-4}$). $\langle n\rangle_0(\tau)/n_0$ is displayed by curve a. $\langle n\rangle_\chi(\tau)/n_0$ is displayed by curve c. $g_2^0(\tau)n_0$ and $g_2^\chi(\tau)n_0$ are displayed, respectively, by curves b and d.

gain and second-order photon-number fluctuations. In the intermediate region from linear amplification to saturation, no change in the photon distribution shape have been observed because of the high level of the photon number involved. Calculations for a low level of photon number will be done in a future publication.

APPENDIX

We give here some details on the computational methods used to obtain the numerical solution of Eq. (5). Using the approximations (7) and (8) we obtain, after a straightforward calculation, the differential equation

$$\frac{\partial P(n,\tau)}{\partial \tau}=\begin{cases} -\Delta\left(\dfrac{1}{1+\chi n}-\chi(1+\chi n)^{-2}(n+1)\right)P(n,\tau) \\ +\left(\dfrac{\beta-\Delta n}{1+\chi n}-\chi(1+\chi n)^{-2}\beta(n+1)\right)\dfrac{\partial P(n,\tau)}{\partial n} \\ +\dfrac{1}{2}\left(\dfrac{\sigma n+\beta}{1+\chi n}-\chi(1+\chi n)^{-2}\beta(n+1)\right)\dfrac{\partial^2 P(n,\tau)}{\partial n^2}\,, \end{cases} \tag{A1}$$

where

$$\Delta = 1-\beta\,, \tag{A2a}$$

$$\sigma = 1+\beta\,. \tag{A2b}$$

Using the relation (9) we obtain the more simple differential equation

$$\frac{\partial R(n,\tau)}{\partial\tau}=A\,\frac{\partial R}{\partial n}+B\left[\left(\frac{\partial R}{\partial n}\right)^2+\frac{\partial^2 R}{\partial n^2}\right]+C\,, \tag{A3a}$$

where

$$A=f(n)[\beta(n+1)-1]-f^2(n)\chi(n+1)\beta\,, \tag{A3b}$$

$$B=\tfrac{1}{2}\{f(n)[\beta+n(1+\beta)]-f^2(n)\beta(n+1)\}\,, \tag{A3c}$$

$$C=f(n)(\beta-1)+f^2(n)\chi[n(1-\beta)+1-\beta]\,. \tag{A3d}$$

Equation (A3a) has been numerically integrated using a grid method. In other words, given the values $R(n,\tau)$, $\partial R(n,\tau)/\partial n$, $\partial^2 R(n,\tau)/\partial n$ for a value of τ and for p equally spaced values of n, we integrate equation (A3a), using a Runge-Kutta method, to obtain these functions for the same value of n but for $\tau'=\tau+\delta\tau$, and we repeat the process. But the value n_{max} of n corresponding to the maximum P_{max} of $P(n,\tau)$ and the width of the distribution $P(n,\tau)$ are varying during the integration. As our program assumes that $P(n,\tau)$ is computed for a fixed number p of values of n, we have to extrapolate $P(n,\tau)$ to take into account the evolution of the shape of $P(n,\tau)$. We use the following process. For $n>n_{max}$ we use the extrapolation function

$$P^+(n)=P_{max}\{\cos^2(\alpha)\exp[-\varphi(n-n_{max})^\lambda] + \sin^2(\alpha)\exp[-\delta(n-n_{max})^\epsilon]\}\,. \tag{A4}$$

For $n<n_{max}$ we use the similar extrapolation function

$$P^-(n)=P_{max}\{\cos^2(\alpha')\exp[-\varphi'(n_{max}-n)^{\lambda'}] + \sin^2(\alpha')\exp[-\delta'(n_{max}-n)^{\epsilon'}]\}\,. \tag{A5}$$

The coefficients α, φ, λ, δ, ϵ and the similar coefficients of Eq. (A5) are obtained by a least-square fitting of $P^+(n)$ or $P^-(n)$ against the p points curve obtained by integration of Eq. (A1). This extrapolation process must be carefully checked because slight fitting errors in the wings of $P(n,\tau)$ can induce severe errors in the computation of $g_2(\tau)$.

Then we compute a set of values of $R(n,\tau)$, $\partial R(n,\tau)/\partial n$, $\partial^2 R(n,\tau)/\partial n^2$ for a new set of p values of n. The values of the parameters used in functions $P^+(n)$ and $P^-(n)$ show that the shape of $P(n,\tau)$ is always nearly a Gaussian curve.

The program has been run on an IBM 370/168 computer. The computing time necessary to obtain the results displayed in Fig. 1 and 2 is about one hour. We have choosen $p=71$, and the final relative accuracy of the results was about 10^{-3}.

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