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Approximate solutions of some nonlinear diffusion equations

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Approximate analytic solutions are given for several equations obtained from a similarity analysis of the nonlinear diffusion equation $\rho_t - \nabla\cdot(D\nabla\rho) = 0$, where the diffusion coefficient D is of the form $D \sim \rho^n$. The solution technique is also demonstrated for a case where D is inhomogeneous as well as nonlinear ($D \sim r^\gamma \rho^n$, where r is the radial coordinate). The approach is general, algebraically simple, and flexible enough to allow applications to many related problems. The analytic predictions are compared to previous numerical results for some cases and shown to yield good agreement.

I. INTRODUCTION

The importance of the fundamental diffusion equation (where D is the diffusion constant)

$$\rho_t - \vec{\nabla}\cdot(D\vec{\nabla}\rho) = 0 \tag{1}$$

is obvious, its applications covering almost every field of physics. The linear diffusion equation, where the diffusion coefficient D is constant, has been studied for a very long time and the analysis has become very powerful and sophisticated. In many important situations the diffusion equation becomes nonlinear and/or inhomogeneous. One very topical example of this is found in fusion plasma physics where the diffusion equations governing particle- and heat-flow are highly nonlinear and for some transport models the diffusion coefficients are inhomogeneous as well.[1]

However, the interest in nonlinear diffusion equations is comparatively recent and much analysis remains to be done in order to obtain a good understanding of the corresponding solutions. An important tool in investigating the linear diffusion equation has been provided by similarity methods.[2] Recently several authors have studied various nonlinear diffusion equations using this approach (for references see Ref. 3). However, one of the problems connected with the use of similarity analysis is that, although a partial differential equation is converted into an ordinary differential equation, the resulting equation may still not be solvable, except by numerical methods. In a recent paper,[4] Tuck obtains approximate analytic solutions for a special class of such similarity equations. However, this technique, although very accurate, is rather complicated and furthermore is not easily extendable to other cases of similarity equations.

The purpose of the present work is to present a unified approach to the problem of finding approximate similarity solutions to the nonlinear diffusion equation under different forms of boundary conditions or restrictions on the solutions, e.g., in the form of conservation laws. The analysis employs an integral method, originating from boundary layer theory, and also recently applied to a similarity problem involving electric field penetration in a plasma.[5] The integral approach in its simplest form, as used in Ref. 5, gives a rough but correct picture of the solution. The simplicity of the approach makes it possible to apply it to a variety of similarity problems and we will, in the present paper, demonstrate its possibilities in connection with several different similarity problems associated with the nonlinear diffusion equation. However, we will also show that it is possible to recursively extend the approximation in a very simple manner. The improved approximation is shown for several explicit cases to yield very good agreement with numerical results.

II. SELF-SIMILAR EQUATIONS

We will consider in the first paragraphs the one-dimensional nonlinear, but homogeneous, diffusion equation

$$\rho_t - (\rho^n \rho_x)_x = 0\,, \tag{2}$$

where subscripts denotes differentiation with respect to time t and space x. Using similarity methods, Eq. (2) will be reduced to ordinary differential equations, the explicit form depending on boundary conditions or on restrictions in the form of conservation laws. For easy reference and to introduce the proper nomenclature, we review the main features of the similarity method. More details and other applications can be found, e.g., in the inspiring review by Lonngren.[3]

The similarity approach makes use of the invariance properties of a partial differential equation under various transformation groups, the simplest one being the linear group G:

$$\rho \to \bar{\rho} = a^{\alpha}\rho\,,$$
$$x \to \bar{x} = a^{\beta}x\,, \tag{3}$$
$$t \to \bar{t} = a^{\gamma}t\,,$$

where a is an arbitrary positive constant, and α, β, and γ are constants to be determined by the invariance properties of the equation under study, including boundary conditions. Performing the transformation defined by Eq. (3), Eq. (2) becomes

$$a^{\gamma-\alpha}\bar{\rho}_{\bar{t}} - a^{2\beta-(n+1)\alpha}(\bar{\rho}^n\bar{\rho}_{\bar{x}}) = 0\,, \tag{4}$$

implying that Eq. (2) is constant conformally invariant under the transformation group G; i.e., Eq. (2) is transformed into

$$\bar{\rho}_{\bar{t}} - (\bar{\rho}^n\bar{\rho}_{\bar{x}})_{\bar{x}} = 0\,, \tag{5}$$

provided

$$\gamma - \alpha = 2\beta - (n+1)\alpha\,. \tag{6}$$

Furthermore, the invariants of the transformation group can be shown to constitute the similarity variables. These are given by

$$\phi(\xi) = \frac{\rho(x,t)}{t^{\alpha/\gamma}}\,, \quad \xi = \frac{x}{t^{\beta/\gamma}}\,. \tag{7}$$

From Eq. (6) we obtain

$$\frac{\alpha}{\gamma} = \frac{1}{n}\left(2\frac{\beta}{\gamma} - 1\right). \tag{8}$$

In order to completely specify the similarity variables an additional relation between α/γ and β/γ is needed. This will be provided by boundary conditions or conservation laws. In terms of the similarity variables, Eq. (2) becomes

$$(\phi^n\phi')' + \frac{\beta}{\gamma}\xi\phi' - \frac{1}{n}\left(2\frac{\beta}{\gamma} - 1\right)\phi = 0, \tag{9}$$

where prime denotes differentiation with respect to ξ.

III. SOLUTIONS OF THE SELF-SIMILAR EQUATIONS

We will now present exact or approximate solutions of Eq. (9) under different conditions.

(i) As our first application we give the solution of Eqs. (2) and (9) corresponding to the conservation law

$$\int_{-\infty}^{+\infty} \rho(x,t)dx = \text{const} = Q\,. \tag{10}$$

Equation (10) implies that

$$\frac{\alpha}{\gamma} + \frac{\beta}{\gamma} = 0\,, \tag{11}$$

i.e.,

$$\frac{\beta}{\gamma} = \frac{1}{n+2}\,, \tag{12}$$

and Eq. (9) can be written

$$(\phi^n\phi')' + \frac{1}{n+2}(\xi\phi' + \phi) = 0\,, \tag{13}$$

which is exactly integrable to

$$\phi = \phi_0(1 - \xi^2/\xi_0^2)^{1/n}\,, \quad n > 0 \tag{14}$$

where

$$\xi_0^2 = 2\frac{n+2}{n}\phi_0^n\,,$$
$$\phi_0 = \left[Q\left(\frac{n}{2(n+2)}\right)^{1/2} B^{-1}\left(\frac{1}{2}, \frac{1}{n} + 1\right)\right]^{2/(2+n)}, \tag{15}$$

and $B(x,y)$ denotes the beta function.[4] This solution was first given by Ames; see Ref. 2.

(ii) Another important situation is that the diffusing quantity is kept constant at the left boundary, $x = 0$, i.e.,

$$\rho(0,t) = \text{const} = \rho_0\,, \tag{16}$$

which requires $\alpha/\gamma = 0$ or $\beta/\gamma = \frac{1}{2}$. Equation (9) then becomes

$$(\phi^n\phi')' + \tfrac{1}{2}\xi\phi' = 0\,. \tag{17}$$

An exact analytic solution as in case (i) is not possible. Approximate solutions have been given by Tuck,[5] using nonlinear diffusion coefficients which possess explicit solutions and approximate the diffusion coefficient $D \sim \rho^n$. The solution of Eq. (17) is then obtained by connecting approximations in different regions. The agreement with numerical results is good, but the analysis is rather lengthy and is not easily extendable to other cases. In the present work we will give a more direct approximation, using a flexible integral method originating in boundary layer theory and previously used in a similarity context by Ahmadi *et al.*[6] For that purpose we write Eq. (17) as

$$(\phi^{n+1})'' = -\tfrac{1}{2}(n+1)\xi\phi'. \tag{18}$$

Asymptotically as $\xi \to 0$ we have

$$(\phi^{n+1})'' \sim -\tfrac{1}{2}(n+1)\xi\phi_0'\,,$$

implying

$$\phi(\xi) \sim \phi_0\left(1 + (n+1)\phi_0^{-1}\phi_0'\xi - \frac{n+1}{12}\phi_0^{-n-1}\phi_0'\xi^3\right)^{1/(n+1)}. \tag{19}$$

The simplest form of approximation, as used in Ref. 6, is to assume the following for $\phi(\xi)$:

$$\phi(\xi) = \begin{cases} \phi_0(1 - \xi/\xi_0)^{1/(n+1)}, & \xi < \xi_0 \\ 0, & \xi \geq \xi_0 \end{cases} \tag{20}$$

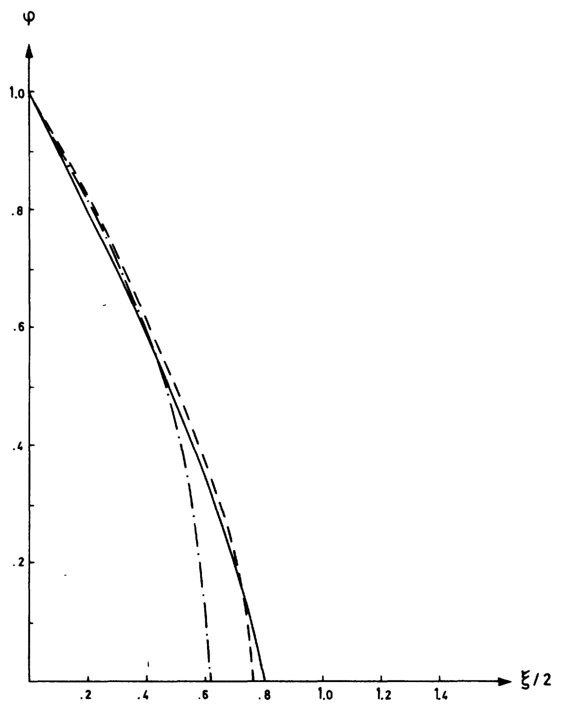

FIG. 1. Solutions of Eq. (17) for $n=1$. Exact solution (——), first approximation [Eq. (20)] (-·-·-), and extended approximation [Eq. (24)] (---).

where the unknown ξ_0 is determined by inserting the trial function Eq. (20),into Eq. (17) and integrating. Thus we obtain [assuming $\phi'(\xi_0)=0$]

$$\int_0^\infty (\phi^{n+1})' = -\frac{n+1}{2}\int_0^\infty \xi\phi' d\xi = \frac{n+1}{2}\int_0^\infty \phi d\xi\,, \tag{21}$$

which directly gives us the characteristic decay length in ξ, viz.,

$$\xi_0^2 = 2\,\frac{n+2}{(n+1)^2}\,\phi_0^n\,. \tag{22}$$

Equation (20) with ξ_0 given by Eq. (22) provides a rough but qualitatively correct picture of the solution of Eq. (17); see Fig. 1.

However, we can also in a very simple manner recursively improve the approximation by using the full asymptotic expansion, Eq. (19), together with the identification

$$(n+1)\phi_0'\phi_0^{-1} = -1/\xi_0\,, \tag{23}$$

i.e.,

$$\phi(\xi) \simeq \phi_0\left(1 - \frac{\xi}{\xi_0} + \frac{\phi_0^{-n}}{12}\frac{\xi^3}{\xi_0}\right)^{1/(n+1)}\,. \tag{24}$$

For $n=1$, the approximate solution given by Eq. (24) is compared with numerical results from Ref. 7. The agreement is very good. On the other hand, Eq. (24) is inapplicable when n becomes too small. The requirement that ϕ, as given by Eq. (24), has a minimum less than zero can be shown to be $(n+2)/(n+1)^2 < \frac{8}{9}$. Thus for $n > n_{cr} \lesssim 1$, the approximation given by Eq. (24) should be a good one. For smaller n the recursive method could be carried further to provide a power series representation of the solution in terms of the unknown $f'(0)$ or ξ_0 which is determined approximately from Eq. (22). For example for $n=0$ we obtain

$$\phi/\phi_0 \simeq 1 - \tfrac{1}{2}\xi + \tfrac{1}{24}\xi^3 - \tfrac{1}{320}\xi^5 + \cdots\,.$$

The exact solution in this case is the complementary error function

$$\phi/\phi_0 = \mathrm{erfc}(\tfrac{1}{2}\xi)$$
$$= 1 - \frac{1}{\sqrt{\pi}}\xi + \frac{1}{12\sqrt{\pi}}\xi^3 - \frac{1}{160\sqrt{\pi}}\xi^5 + \cdots\,.$$

That is, the recursive procedure yields a good approximation of the exact solution. However, the convergence properties of the power series are not favorable for small n, and we have found it more convenient in these cases to use a slightly different approach based on the exact solution for $n=0$. By substituting $y=\phi^{n+1}$, Eq. (18) can be written in the form

$$y'' = -\tfrac{1}{2}\xi y^{-n/(n+1)} y'\,. \tag{25}$$

Asymptotically as $\xi \to 0$ we have

$$y''/y' \simeq -\tfrac{1}{2}\xi y_0^{-n/(n+1)}\,, \quad y_0 = \phi_0^{n+1}\,, \tag{26}$$

with the solution

$$y = \frac{1}{\sqrt{\pi}}\, y_0^{n+2/2(n+1)} \int_\xi^\infty \exp(-\tfrac{1}{4} y_0^{-n/(n+1)} z^2)\,dz\,. \tag{27}$$

Equation (27) indicates that a trial function can be chosen in the form

$$\phi = \phi_0[\mathrm{erfc}(\lambda\xi)]^{1/(n+1)}\,, \tag{28}$$

where the parameter λ is to be determined as before by inserting Eq. (28) into Eq. (18) and integrating. This yields

$$\lambda^2 = \frac{\sqrt{\pi}}{4}(n+1)\phi_0^{-n}\int_0^\infty [\mathrm{erfc}(z)]^{1/(n+1)} dz\,. \tag{29}$$

We emphasize that for $n=0$, Eq. (29) yields $\lambda = \frac{1}{2}$ and we obtain the exact solution $\phi = \phi_0 \mathrm{erfc}(\frac{1}{2}\xi)$. For small values of n, the integral in Eq. (29) can be approximated by a series expansion around $n=0$. Taking into account only the two first terms in this expansion we find

$$\lambda^2 \simeq \frac{3n+2}{8(n+1)}\phi_0^{-n}\,, \tag{30}$$

which together with Eq. (28) represent the approximate solution.

(iii) We next consider the case of constant flow at $x=0$, i.e.,

$$\Gamma(0,t)=-\rho^n(0,t)\rho_x'(0,t)=\text{const}=\Gamma_0 .$$

This requires $(n+1)\alpha/\gamma-\beta/\gamma=0$, implying that

$$\beta=\frac{n+1}{n+2},$$

and Eq. (9) becomes

$$(\phi^n\phi')'+\frac{n+1}{n+2}\xi\phi'-\frac{1}{n+2}\phi=0 . \tag{31}$$

As before, a trial solution is sought in the form

$$\phi=\begin{cases}\phi_0(1-\xi/\xi_0)^{1/(n+1)}, & \xi<\xi_0\\ 0, & \xi\geq\xi_0\end{cases} \tag{32a}$$

and ξ_0 is determined by integrating Eq. (31). This yields

$$\xi_0^2=\frac{n+2}{(n+1)^2}\phi_0^n , \tag{32b}$$

which together with the flow condition

$$\Gamma_0=\frac{\phi_0^{n+1}}{(n+1)\xi_0} , \tag{33}$$

yields expressions for ξ_0 and ϕ_0 in terms of Γ_0, viz.,

$$\phi_0=[(n+2)\Gamma_0^2]^{1/(n+2)} ,$$
$$\xi_0^2=\frac{n+2}{(n+1)^2}[(n+2)\Gamma_0^2]^{n/(n+2)} . \tag{34}$$

An improved solution is obtained from the asymptotic equation

$$(\phi^{n+1})''\sim\frac{n+1}{n+2}(\phi_0+\phi_0'\xi)-\frac{(n+1)^2}{n+2}\xi\phi_0' , \tag{35}$$

which implies

$$\phi=\phi_0\left(1-\frac{\xi}{\xi_0}+\frac{n+1}{2(n+2)}\phi_0^{-n}\xi^2+\frac{n}{6(n+2)}\phi_0^{-n}\frac{\xi^3}{\xi_0}\right)^{1/(n+1)} . \tag{36}$$

Using the same procedure as in the previous case we can improve our approach for small values of n. Solving Eq. (35) in the limit $\xi\to 0$ indicates that the trial solution can be sought in the form

$$\Phi=\Phi_0\left(\xi\int_\xi^\infty\frac{e^{-\lambda^2z^2}}{z^2}dz\right)^{1/(n+1)}$$
$$=\Phi_0[\exp(-\lambda^2\xi^2)-\lambda\sqrt{\pi}\,\xi\,\text{erfc}(\lambda\xi)]^{1/(n+1)} . \tag{37}$$

In order to determine the unknown parameter λ we integrate Eq. (35) using Eq. (37). This yields

$$\lambda^2=\frac{(n+1)I_n}{\sqrt{\pi}}\phi_0^{-n} , \tag{38}$$

where

$$I_n=\int_0^\infty[\exp(-z^2)-\sqrt{\pi}\,z\,\text{erfc}(z)]^{1/(n+1)}dz , \tag{39}$$

which together with the flow condition

$$\Gamma_0=\frac{\sqrt{\pi}\lambda}{n+1}\phi_0^{n+1} \tag{40}$$

give the following expressions for λ and ϕ_0:

$$\lambda=\left(\frac{(n+1)I_n^{n+1}}{\sqrt{\pi}}\Gamma_0^{-n}\right)^{1/(n+2)} , \tag{41}$$

$$\phi_0=\left(\frac{(n+1)\Gamma_0^2}{\sqrt{\pi}I_n}\right)^{1/(n+2)} . \tag{42}$$

Taking into account only the two first terms in a series expansion of Eq. (39), we obtain

$$\lambda\simeq\left(\frac{(n+1)^{3/2}(1-n)}{2(n+2)}\Gamma_0^{-n}\right)^{1/(n+2)} . \tag{43}$$

Note that, for $n=0$, Eqs. (37) and (41)–(43) reduce to the exact solution

$$\phi=\phi_0\left(\xi\int_\xi^\infty\frac{e^{-z^2/4\Gamma_0}}{z^2}dz\right) ,$$
$$\phi_0=\pi^{-1/2}\Gamma_0^{3/2} .$$

(iv) Finally we investigate the case of a second fixed boundary on which $\rho(x,t)$ vanishes, i.e., $\rho(L,t)=0$ for all t. This implies $\beta/\gamma=0$ and the similarity approach reduces to a separation of variables; i.e., we can write $\rho(x,t)=\phi(x)\nu(t)$ and separate the diffusion equation to read

$$\frac{d\nu}{dt}\Big/\nu^{n+1}=\frac{1}{\phi}\frac{d}{dx}\left(\phi^n\frac{d\phi}{dx}\right)=-\frac{1}{\tau} , \tag{44}$$

where τ is the separation constant. The time evolution is easily determined $[\nu(0)=1]$:

$$\nu(t)=\begin{cases}(1+nt/\tau)^{-1/n}, & n\neq 0\\ \exp(-t/\tau), & n=0\end{cases} \tag{45}$$

and the space-dependent part satisfies the equation

$$(\phi^n\phi')'+\lambda\phi=0 , \tag{46}$$

where ϕ and λ are normalized with respect to $\phi_0=\phi(0)$ and L, and consequently $\lambda=(\tau L^{-2}\phi_0^n)^{-1}$. We consider only solutions even in x. Thus the boundary conditions are $\phi(0)=1$, $\phi'(0)=0$, $\phi(1)=0$. Equation (46) can be integrated once to yield

$$\phi'=-\left(\frac{2\lambda}{(n+2)}\right)^{1/2}\phi^{-n}(1-\phi^{2+n})^{1/2} , \tag{47}$$

but the remaining quadrature can not in general be carried out analytically. However, for special values of n, explicit solutions exist in terms of elliptic functions, as follows.

(a) $n=2$

$$x=\sqrt{\tau}\left[2E\left(\arcsin(1-\phi^2)^{1/2},\frac{1}{\sqrt{2}}\right)-F\left(\arcsin(1-\phi^2)^{1/2},\frac{1}{\sqrt{2}}\right)\right],$$

$$\tau L^{-2}\phi_0^n=\left[2E\left(\frac{\pi}{2},\frac{1}{\sqrt{2}}\right)-F\left(\frac{\pi}{2},\frac{1}{\sqrt{2}}\right)\right]^{-2}\simeq 1.39\,.$$

(b) $n=1$

$$x=\left(\frac{3\tau}{2}\right)^{1/2}(3^{-1/4}-3^{1/4})F\left(\arccos\frac{\sqrt{3}-1+\phi}{\sqrt{3}+1-\phi},\frac{(2+\sqrt{3})^{1/2}}{2}\right)$$

$$+2\times 3^{1/4}E\left(\arccos\frac{\sqrt{3}-1+\phi}{\sqrt{3}+1-\phi},\frac{(2+\sqrt{3})^{1/2}}{2}-2\frac{(1-\phi^3)^{1/2}}{\sqrt{3}+1-\phi}\right),$$

$$\tau L^2\phi_0^n=\frac{2\pi^2 16^{1/3}}{9}\left[\Gamma\left(\tfrac{3}{2}\right)\right]^{-6}\simeq 0.88\,.$$

(c) $n=-\frac{1}{2}$

$$x=(3\tau^2)^{1/4}F\left(\arccos\frac{\sqrt{3}-1+\sqrt{\phi}}{\sqrt{3}+1-\sqrt{\phi}},\frac{(2+\sqrt{3})^{1/2}}{2}\right),$$

$$\tau L^{-2}\phi_0^n=\frac{1}{\sqrt{3}}\left[F\left(\arccos\frac{\sqrt{3}-1}{\sqrt{3}+1},\frac{(2+\sqrt{3})^{1/2}}{2}\right)\right]^{-2}$$

$$\simeq 0.17\,,$$

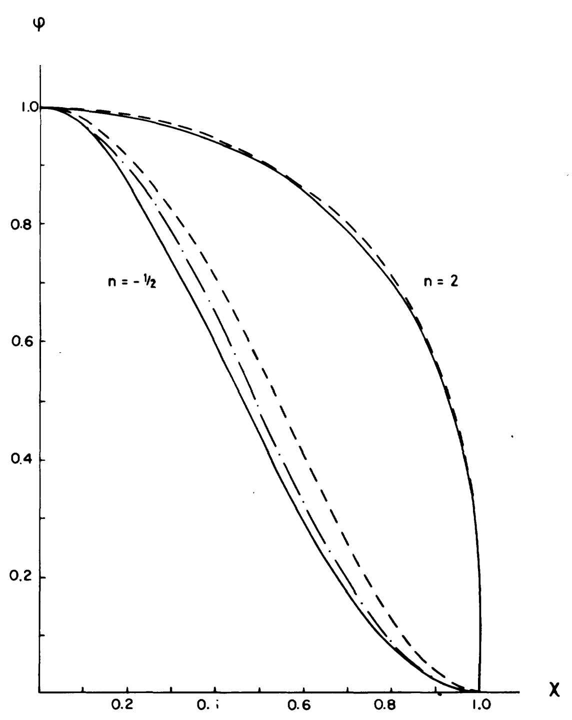

FIG. 2. Solutions of Eq. (46) for $n=-\frac{1}{2}$ and $n=2$. Exact solutions (——), first approximations [Eq. (48)] (---), and extended approximation [Eq. (50)] (-·-·-).

where F and E denote incomplete elliptic functions of the first and second kind.[8] We emphasize that even when Eq. (47) does admit analytic solutions, the result is rather complicated. Therefore simple analytic approximations will be important for all values of n. Following the approach taken in the previous paragraphs we choose the approximating functions from the asymptotic behavior for small x and consistent with the boundary conditions. The simplest possible choice is

$$\phi(x)=(1-x^2)^{1/(n+1)}\,, \tag{48}$$

and by integration we determine τ as

$$\tau L^{-2}\phi_0^n=\frac{n+1}{4}B\left(\frac{1}{2},\frac{n+2}{n+1}\right)\,. \tag{49}$$

The exact solutions and the approximations given by Eq. (48) are compared in Fig. 2. The agreement is surprisingly good, especially when considering the simplicity of the approximating functions and the range of n.

However, as in the previous sections, we can very simply extend the approximation to yield very good agreement. An improved approximating function is

$$\phi(x)=(1-\alpha_1x^2-\alpha_2x^4)^{1/(n+1)}\,. \tag{50}$$

Since $\phi(1)=0$ we must have $1-\alpha_1-\alpha_2=0$. By inserting Eq. (50) into the equation for ϕ and integrating we obtain

$$\alpha_1+2\alpha_2=\frac{\lambda}{4}(n+1)B\left(\frac{1}{2},\frac{n+2}{n+1}\right)\,, \tag{51}$$

where we have used the recursive approximation

$$\int_0^1 \phi(x)dx \simeq \int_0^1 (1-x^2)^{1/(n+1)}dx = \tfrac{1}{2}B\left(\frac{1}{2}, \frac{n+2}{n+1}\right) .$$

The third relation between α_1, α_2, and λ is obtained also using the first moment of Eq. (46); i.e., we multiply Eq. (46) with x before integrating, cf. Ref. 2. Again we approximate recursively

$$\int_0^1 x\,\phi(x)dx \simeq \int_0^1 x(1-x^2)^{1/(n+1)}dx = \frac{1}{2}\frac{n+1}{n+2} .$$

This yields

$$\alpha_1 + 3\alpha_2 = \frac{\lambda}{2}\frac{(n+1)^2}{n+2} \tag{52}$$

and we can solve for α_1, α_2, and λ:

$$\begin{aligned} \lambda^{-1} &= \frac{n+1}{2}\left[B\left(\frac{1}{2}, \frac{n+2}{n+1}\right) - \frac{n+1}{n+2}\right], \\ \alpha_1 &= 2\left[\tfrac{3}{4}B\left(\frac{1}{2}, \frac{n+2}{n+1}\right) - \frac{n+1}{n+2}\right] \\ &\quad\times\left[B\left(\frac{1}{2}, \frac{n+2}{n+1}\right) - \frac{n+1}{n+2}\right]^{-1}, \\ \alpha_2 &= \left[\frac{n+1}{n+2} - \frac{1}{2}B\left(\frac{1}{2}, \frac{n+2}{n+1}\right)\right] \\ &\quad\times\left[B\left(\frac{1}{2}, \frac{n+2}{n+1}\right) - \frac{n+1}{n+2}\right]^{-1} . \end{aligned} \tag{53}$$

The improved approximations are also compared with the exact solutions for some values of n in Fig. 2. The agreement is very good. As further illustration we compare the corresponding eigenvalues.

n	$\lambda^{-1} \sim \tau$		
	exact	first approx.	second approx.
$-\frac{1}{2}$	0.17	0.13	0.18
0	0.41	0.33	0.42
1	0.89	0.79	0.90
2	1.39	1.26	1.39

IV. A NONLINEAR AND INHOMOGENEOUS DIFFUSION EQUATION

In order to further demonstrate the flexibility of the approach we will apply it to a particular set of nonlinear and inhomogeneous diffusion equations, which is of special interest for fusion plasma physics in connection with torodial devices. For example, assuming thermal conduction to be the dominating loss mechanism, the energy equation for the plasma ions can be written as[1]

$$\frac{3}{2}\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rK\frac{\partial T}{\partial r}\right) , \tag{54}$$

where T is the ion temperature, r is the radial coordinate, the plasma density has been assumed constant, and the thermal conductivity K is of the form

$$K = K_0 r^\gamma T^n , \tag{55}$$

where the powers γ and n depend on the operating regime.[1] The proper boundary conditions for Eq. (54) are

$$T(a) = 0, \quad T(0) = T_0, \quad T'(0) = 0 , \tag{56}$$

where a is the radius of the plasma cylinder. The similarity solution of Eq. (54) is equivalent to that obtained by separation of variables. If the latter approach is used we write T as

$$T(r,t) = \phi(r)\mu(t) . \tag{57}$$

Equation (54) then separates into

$$\frac{d\mu}{dt}\Big/\mu^{n+1} = \frac{1}{r\phi}\frac{d}{dr}\left(\tfrac{2}{3}K_0 r^{\gamma+1}\phi^n\frac{d\phi}{dr}\right) = -\frac{1}{\tau} , \tag{58}$$

where the separation constant is $-1/\tau$. The equation for μ is easily solved $[\mu(0)=1]$:

$$\mu(t) = \begin{cases} (1+nt/\tau)^{-n}, & n \neq 0 \\ \exp(-t/\tau), & n = 0 . \end{cases} \tag{59}$$

We will restrict the analysis to values of $n > -1$. Note that for $-1 < n < 0$, $\mu(t)$ goes to zero for finite $t = \tau/|n|$. The space-dependent equation can be written

$$\frac{1}{r}\frac{d}{dr}\left(\tfrac{2}{3}K_0 r^{\gamma+1}\phi^n\frac{d\phi}{dr}\right) + \frac{1}{\tau}\phi = 0 . \tag{60}$$

We introduce the normalization $\phi/T(0) \to \phi$, $r/a \to r$. Then Eq. (60) can be written

$$\frac{1}{r}\frac{d}{dr}\left(r^{\gamma+1}\phi^n\frac{d\phi}{dr}\right) + \lambda\phi = 0 , \tag{61}$$

where

$$\lambda^{-1} = \tfrac{2}{3}\tau K_0 a^{\gamma-2} T_0^n \tag{62}$$

and the corresponding boundary conditions are $\phi(0)=1$, $\phi'(0)=0$, and $\phi(1)=0$. We emphasize the scaling for the ion energy confinement time τ inherent in Eq. (62)

$$\tau \sim K_0^{-1} a^{2-\gamma} T_0^{-n} . \tag{63}$$

Proceeding as before we have asymptotically for small r

$$r^{\gamma+1}\frac{d}{dr}(\phi^{n+1}) \sim -\frac{\lambda}{2}(n+1)r^2 , \tag{64}$$

implying the following trial function

$$\phi = (1 - r^{2-\gamma})^{1/(n+1)} . \tag{65}$$

We point out that the power $1/(n+1)$ in Eq. (65) is exactly what is required to make the outgoing heat flux at $r=1$ nonzero and finite. The same remark is valid for case (iv) in Sec. III.

The separation constant is now determined approximately by integrating Eq. (61). This yields

$$\lambda^{-1}=\frac{n+1}{2(2-\gamma)}\int_0^1(1-x^{2-\gamma})^{1/(n+1)}dx$$
$$=\frac{n+1}{2(2-\gamma)^2}B\left(\frac{1}{2-\gamma},\frac{n+2}{n+1}\right). \tag{66}$$

A comparison with the exact solution for $n=\gamma=0$ is made in Fig. 3. The profiles show rough but good agreement. Again the approximation can be extended to yield even better agreement. For example, take the case of $\gamma=0$. The extended approximation is then

$$\phi=(1-\alpha_1r^2-\alpha_2r^4)^{1/(n+1)}.$$

From the zero- and first-order moments and using the recursive approximation for the integrals of ϕ and $r\phi$, we obtain

$$\lambda^{-1}=\frac{n+1}{4}\left[\tfrac{3}{4}B\left(\frac{1}{2},\frac{n+2}{n+1}\right)-\frac{1}{2}\,\frac{n+1}{n+2}\right],$$
$$\alpha_1=\frac{\lambda(n+1)}{4}\left[\tfrac{3}{2}B\left(\frac{1}{2},\frac{n+2}{n+1}\right)-2\,\frac{n+1}{n+2}\right],$$
$$\alpha_2=-\frac{3\lambda}{8}(n+1)\left[\tfrac{1}{2}B\left(\frac{1}{2},\frac{n+2}{n+1}\right)-\frac{n+1}{n+2}\right].$$

For $n=0$, we obtain $\phi=1-\frac{4}{3}r^2+\frac{1}{3}r^4$ as an approximation of the zero-order Bessel function. As seen from **Fig.** 3 the agreement is very good.

As an application of the obtained results we consider the problem of the ion temperature diffusion in Tokamak plasmas. Assuming thermal conduction to be the dominant loss mechanism, the proper equation governing the evolution of the ion temperature T_i is Eq. (54). According to neoclassical transport theory the thermal conductivity can be approximated as[1]

$$K=\frac{\rho_i^2q^2}{\tau_i}\,\frac{0.7(1+0.43\nu)}{\epsilon^{3/2}+\epsilon^{3/4}\nu^{1/2}+0.2\nu}, \tag{67}$$

where $\rho_i=(2m_iT_ic^2/e^2B_\phi^2)^{1/2}$ is the ion Larmor radius in the toroidal magnetic field (B_ϕ), $q=rB_\phi/RB_\theta$ is the safety factor (B_θ is the poloidal magnetic field), $\tau_i=3m_i^{1/2}T_i^{3/2}/4\pi^{1/2}n_ez_ie^4\ln\Lambda$ is the ion collision time (n_e is the electron density and $\ln\Lambda$ the Coloumb logarithm), $\epsilon=r/R$ the inverse aspect ratio (R the major radius), and $\nu=Rq/v_i\tau_i$ [where $v_i=(2T_i/m_i)^{1/2}$ is the ion thermal velocity].

Equation (67) is reduced to simpler forms in different regions, as shown in the following examples.

(i) $\nu\gg1$ (*collision dominated region*)

$$K=K_{\rm CD}\sim m_i^{1/2}Z_i\frac{a^2}{R^2}\,\frac{n_e}{T_i^{1/2}I_0^2}. \tag{68}$$

(*ii*) $\nu\ll\epsilon^{3/2}$ (*collisionless-regime*)

$$K=K_{\rm CL}\sim m_i^{1/2}Z_i\frac{a^{7/2}}{R^{1/2}}\,\frac{n_e}{T_i^{1/2}I_0^2}\,\frac{1}{r^{3/2}}, \tag{69}$$

where we have assumed that the current density has the constant value I_0 over the cross section of the plasma.

From the previous results we obtain the following ion-temperature profiles and the characteristic decay times of the ion temperature.

(*i*)

$$T_{\rm CD}(r,t)=T_{i0}\left[1-\left(\frac{r}{a}\right)^2\right]^2\left(1-\frac{t}{2\tau_{\rm CD}}\right)^{1/2},$$
$$\tau_{\rm CD}\simeq0.6\,\frac{a^2R^2T_{i0}^{1/2}I_0^2}{Z_in_em_i^{1/2}e^2\ln\Lambda c^4}. \tag{70}$$

(*ii*)

$$T_{\rm CL}(r,t)=T_{i0}\left[1-\left(\frac{r}{a}\right)^{7/2}\right]^2\left(1-\frac{t}{2\tau_{\rm CL}}\right)^{1/2},$$
$$\tau_{\rm CL}\simeq0.9\,\frac{a^{7/2}R^{1/2}T_{i0}^{1/2}I_0^2}{Z_in_em_i^{1/2}e^2\ln\Lambda c^4}. \tag{71}$$

In a tokamak experiment the energy containment time τ_E is usually defined as $\tau_E=\frac{3}{2}\langle(n_iT_i+n_eT_e)\rangle/VI$, where $n_{i,e}$ and $T_{i,e}$ denote densities and tem-

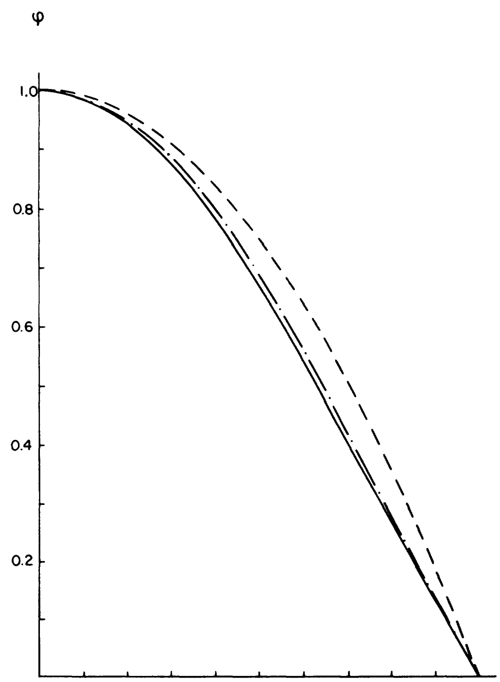

FIG. 3. Solutions of Eq. (61) for $n=0=\gamma$. Exact solution (——), first approximation [Eq. (65)] (---), and extended approximation (-·-·-).

peratures of ions and electrons respectively, V is the loop voltage, and I is the total plasma current. Brackets $\langle\ \rangle$ denotes integration over the plasma volume. For cases where $T_e \simeq T_i$ and the total energy loss is dominated by ion thermal conduction, one has $\tau_E = 2\tau_E$ where (τ_{Ei} is the ion energy containment time).

We consider two specific applications. For Jet the following data could be relevant[9]: $a = 125$ cm, $R = 296$ cm, $B_\phi = 30$ kG, plasma current $= 2.6$ MA, $\ln\Lambda = 15$, and (i) $T_{i0} = 200$ eV, $n_e = 10^{14}$ cm^{-3}, $Z_i = 6$; (ii) $T_{i0} = 1$ keV, $n_e = 2 \times 10^{13}$ cm^{-3}, $Z_i = 1$. The corresponding characteristic decay times are $\tau_{\rm CD} \simeq 0.2$ s and $\tau_{\rm CL} \simeq 2$ s, respectively. For Alcator discharges in the collision dominated regime,[10] $a \simeq 9$ cm, $R = 57.5$ cm, $B_\phi = 71.1$ kG, $I_0 = 1.5$ kA cm^{-2}, $n_e \simeq 7 \times 10^{14}$ cm^{-3}, $T_{i0} = 554$ eV, $Z_i = 1$, $\ln\Lambda = 16$, and we obtain $\tau_{\rm CD} \simeq 50$ ms, which is to be compared with half the value of the total energy confinement time $\tau_E = 12$ ms reported in Ref. 10. Thus, ion thermal conductivity is an important but not dominating energy loss mechanism for these types of Alcator discharges. This conclusion is also supported by the fact that for Alcator, $\tau_E \sim n_e$, which is not consistent with the n_e dependence of Eq. (70). Anomalous transport processes have been involved to explain the enhanced transport in this case.[10,11]

V. CONCLUSION

The method of self-similar solutions of partial differential equations has been applied to nonlinear diffusion equations under different boundary conditions or restrictions. By employing a particular class of transformation the problem is reduced to an ordinary differential equation, which is solved by using an integral method originating from boundary layer theory. It has been shown that the flexibility of the procedure makes it possible to recursively improve the approximation in a simple manner. The solution technique is also demonstrated for a case of an inhomogeneous diffusion equation. The approximate analytic solutions are compared with numerical results and show very good agreement. We conclude that the present method offers a convenient way for finding approximate solutions of nonlinear diffusion equations, and we believe that it can be used as a powerful tool for studies of various physical model equations.

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