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Brownian motion of harmonic systems with fluctuating parameters. II. Relation between moment instabilities and parametric resonance

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In this paper we establish the connection between the energetic instability encountered in stochastic oscillators with fluctuating frequencies and the parametric instability of deterministic oscillators with time-dependent frequencies.

I. INTRODUCTION

In this paper we provide a detailed physical interpretation of the instability encountered in a mechanical oscillator with a fluctuating frequency. This instability has been noted by a number of investigators.[1-7] Its physical origin, however, has not in general been well understood. Oscillators with fluctuating frequencies can be used to model a variety of dynamical systems.[2,3,8-14] A few examples are (1) scalar wave propagation in a random medium, where the index of refraction in the Helmholtz equation is a stochastic variable due to fluctuations in the medium,[11-14] (2) a spin precessing about a noisy external magnetic field,[8] (3) electrical circuits with fluctuating circuit parameters,[2] (4) certain models of turbulent flow fields,[10] and (5) semiclassical models of the dynamics of level population densities in lasers.[3] An understanding of the origin of the instability is therefore essential.

The mechanical oscillator considered here, as in Refs. 5–7, is described by the equation of motion

$$\ddot{x}(t)+2\lambda\dot{x}(t)+\omega^2(t)x(t)=f(t), \tag{1.1}$$

where $x(t)$ is the oscillator displacement, λ is a damping parameter, $\omega(t)$ is a fluctuating frequency, and $f(t)$ is a random force. We assume that the random force has zero mean, i.e.,

$$\langle f(t)\rangle_f=0 \tag{1.2}$$

and is statistically stationary in time with the correlation function given by

$$\langle f(t)f(t')\rangle_f=2\tilde{D}\delta(t-t')\,. \tag{1.3}$$

The $\langle\ \rangle_f$ in (1.2) and (1.3) denote averages over an ensemble of realizations of $f(t)$. For convenience we have chosen a delta-correlated driving force since the correlation properties of $f(t)$ do not affect the stability properties of the oscillator.[5]

We partition the fluctuating frequency as

$$\omega^2(t)=\Omega_0^2+\gamma(t), \tag{1.4}$$

where Ω_0^2 is the average of $\omega^2(t)$. The average of the fluctuating quantity $\gamma(t)$ then vanishes by definition, i.e.,

$$\langle\gamma(t)\rangle=0\,. \tag{1.5}$$

The unsubscripted brackets $\langle\ \rangle$ denote an average over an ensemble of realizations of the frequency fluctuations. The cumulants of the fluctuations $\gamma(t)$ are all assumed to be delta correlated, with

$$\langle\langle\gamma(t_1)\gamma(t_2)\cdots\gamma(t_n)\rangle\rangle=2^nD_n\delta(t_1-t_2)\cdots\delta(t_{n-1}-t_n), \tag{1.6}$$

where the double brackets denote a cumulant average. The fluctuating quantities $f(t)$ and $\gamma(t)$ are assumed to be mutually statistically independent. This independence occurs physically when the sources of the fluctuations are different. However, we have shown that the results to be discussed here are unchanged if $f(t)$ and $\gamma(t)$ are delta correlated.[5,6]

The most interesting feature of the oscillator (1.1) is that the second-order moments $\langle\langle x^2(t)\rangle_f\rangle$, $\langle\langle p^2(t)\rangle_f\rangle$, and $\langle\langle x(t)p(t)\rangle_f\rangle$ become unstable when the second cumulant coefficient D_2 in (1.6) exceeds the value $2\lambda\Omega_0^2$. Thus, when the frequency fluctuations are sufficiently large, the oscillator (1.1) is energetically unstable.[1,4-7] In the unstable parameter regime the second moments grow exponentially.[1,6] We interpret this instability in terms of the parametric resonance instability encountered in deterministic oscillators with time-dependent frequencies.[15,16] This connection has been observed qualitatively by Stratonovich[2] but seems not to have been explored otherwise in the literature. Here we establish the quantitative connection between the stochastic oscillator instability and the deterministic oscillator parametric instability.

In Sec. II we review the properties of the stochastic oscillator and summarize previous results relevant to this study. Section III presents a discussion of parametric resonance for an oscillator which adiabatically switches between two frequencies. In Sec. IV we establish the detailed connection between the resonances in the stochastic and deterministic systems. Section V contains a discussion of extensions of these ideas to other systems.

II. A REVIEW OF FIRST- AND SECOND-MOMENT RESULTS FOR STOCHASTIC OSCILLATORS

The oscillator equation (1.1) can be rewritten as the set of two first-order equations:

$$\dot{x}(t)=p(t), \tag{2.1a}$$

$$\dot{p}(t)=-\Omega_0^2x(t)-\gamma(t)x(t)-2\lambda p(t)+f(t). \tag{2.1b}$$

Moment transport equations for the oscillator can be obtained by a variety of methods. One method is to construct the equation of evolution for the probability density $P(x,p,t\,|\,x_0,p_0)$ for the oscillator displacement and momentum to lie between (x,p) and $(x+dx,\ p+dp)$ at time t conditional on the initial values (x_0,p_0).[7] The equation of evolution is[7,17-19]

$$\begin{aligned}\frac{\partial}{\partial t}P(x,p,t\,|\,x_0,p_0) &= \left(-p\,\frac{\partial}{\partial x}+\frac{\partial}{\partial p}(2\lambda p+\Omega_0^2x)+\tilde{D}\,\frac{\partial^2}{\partial p^2}\right)\\ &\quad\times P(x,p,t\,|\,x_0,p_0)\\ &\quad+\sum_{n=2}^{\infty}D_n x^n\,\frac{\partial^n}{\partial p^n}P(x,p,t\,|\,x_0,p_0).\end{aligned} \tag{2.2}$$

In (2.2) we have assumed $f(t)$ to be Gaussian. This assumption does not limit the generality of our results since only the second cumulant of $f(t)$ enters the oscillator moments of concern here.[5-7] The terms in the large parentheses on the right side of (2.2) then correspond to those of the usual Ornstein-Uhlenbeck-Wang equation for a damped oscillator of constant frequency Ω_0.[20] The remaining terms result from the frequency fluctuations. Transport equations for the mean oscillator displacement and momentum are obtained by multiplying (2.2) by x and p and integrating over x and p. This procedure leads to the moment equations

$$\dot{\underline{Y}}_1(t)=\underline{M}_1\underline{Y}_1(t), \tag{2.3}$$

where

$$\underline{Y}_1(t)=\begin{pmatrix}\langle\langle x(t)\rangle_f\rangle\\ \langle\langle p(t)\rangle_f\rangle\end{pmatrix} \tag{2.4}$$

and

$$\underline{M}_1=\begin{pmatrix}0 & 1\\ -\Omega_0^2 & -2\lambda\end{pmatrix}. \tag{2.5}$$

The second moments obey the transport equation

$$\dot{\underline{Y}}_2(t)=\underline{M}_2\underline{Y}_2(t)+\underline{d}_2, \tag{2.6}$$

where

$$\underline{Y}_2(t)=\begin{pmatrix}\langle\langle x^2(t)\rangle_f\rangle\\ \langle\langle x(t)p(t)\rangle_f\rangle\\ \langle\langle p^2(t)\rangle_f\rangle\end{pmatrix}, \tag{2.7}$$

$$\underline{M}_2=\begin{pmatrix}0 & 2 & 0\\ -\Omega_0^2 & -2\lambda & 1\\ 2D_2 & -2\Omega_0^2 & -4\lambda\end{pmatrix}, \tag{2.8}$$

and

$$\underline{d}_2=\begin{pmatrix}0\\ 0\\ 2\tilde{D}\end{pmatrix}. \tag{2.9}$$

Equation (2.6) is obtained by multiplying (2.2) by x^2, p^2, and xp and integrating over all x and p.

The solution of Eq. (2.3) for the average oscillator displacement and momentum is found by standard methods to be[5]

$$\langle\langle x(t)\rangle_f\rangle=e^{-\lambda t}\left[\left(\cos\omega_1 t+\frac{\lambda}{\omega_1}\sin\omega_1 t\right)x_0+\left(\frac{1}{\omega_1}\sin\omega_1 t\right)p_0\right], \tag{2.10a}$$

$$\langle\langle p(t)\rangle_f\rangle=e^{-\lambda t}\left[\left(\frac{-\Omega_0^2}{\omega_1}\sin\omega_1 t\right)x_0+\left(\cos\omega_1 t-\frac{\lambda}{\omega_1}\sin\omega_1 t\right)p_0\right], \tag{2.10b}$$

where

$$\omega_1=(\Omega_0^2-\lambda^2)^{1/2}. \tag{2.11}$$

These results are identical in form with those of Wang and Uhlenbeck[20] for a damped harmonic oscillator. The frequency fluctuations only appear in the frequency shift from ω_0^2 to Ω_0^2 wherever the former appears in the Wang-Uhlenbeck results. The fluctuations $\gamma(t)$ thus do *not* affect the average dynamics of the oscillator.

The transport equation (2.6) for the second moments can also be solved straightforwardly. If the solution exists as $t\to\infty$, then it can be found directly from (2.6) by setting $\dot{\underline{Y}}_2(t)=0$ and performing a simple matrix inversion. This procedure leads to the expression[1,5-7]

$$\lim_{t\to\infty} \underline{Y}_2(t) = \frac{\tilde{D}}{2\lambda\Omega_0^2 - D_2}\begin{pmatrix}1\\0\\\Omega_0^2\end{pmatrix}. \quad (2.12)$$

The interesting feature of (2.12) is that the second moments are stable as $t\to\infty$ only if the frequency fluctuations are sufficiently small, i.e., if

$$D_2 < 2\lambda\Omega_0^2 . \quad (2.13)$$

The approach of $\underline{Y}_2(t)$ to the result (2.12) is determined by the full solution of Eq. (2.6),

$$\underline{Y}_2(t) = e^{\underline{M}_2 t}[\underline{Y}_2(0) + \underline{M}_2^{-1}\underline{d}_2] - \underline{M}_2^{-1}\underline{d}_2 . \quad (2.14)$$

The eigenvalues of the matrix $\underline{M}_2$ determine the stability of $\underline{Y}_2(t)$ and are given by[6]

$$\epsilon_1 = A + B - 2\lambda, \quad (2.15a)$$

$$\epsilon_2 = -2\lambda - \tfrac{1}{2}(A+B) + \tfrac{1}{2}i\sqrt{3}(A-B), \quad (2.15b)$$

$$\epsilon_3 = -2\lambda - \tfrac{1}{2}(A+B) - \tfrac{1}{2}i\sqrt{3}(A-B), \quad (2.15c)$$

where

$$A = \{2D_2 + [4D_2^2 + (\tfrac{4}{3}\omega_1^2)^3]^{1/2}\}^{1/3} \quad (2.16a)$$

and

$$B = \{2D_2 - [4D_2^2 + (\tfrac{4}{3}\omega_1^2)^3]^{1/2}\}^{1/3} . \quad (2.16b)$$

We note that the second-order statistics of the oscillator and, in particular results (2.12)–(2.15), depend only on the second cumulant coefficient D_2 and not on the coefficients of the higher cumulants. The quantities A and B are real and $A>B$. Therefore, the real parts of ϵ_2 and ϵ_3 are negative for all parameter values. The stability of the second moments is thus determined by the eigenvalue ϵ_1. We have shown elsewhere[6] that ϵ_1 is negative if (2.13) is satisfied and becomes positive when $D_2 > 2\lambda\Omega_0^2$. The second moments then grow exponentially in this parameter regime [cf. (2.14)].

It is our purpose in this paper to relate the second-moment instability described above to the instabilities encountered in parametrically resonant systems. In the next section we describe the behavior of a purely deterministic oscillator with a time-dependent frequency and in Sec. IV we explore the relation between the stochastic and deterministic systems.

III. PARAMETRIC RESONANCE FOR DETERMINISTIC OSCILLATORS

A simple example of an oscillatory system with a time varying frequency has been described by Landau and Lifschitz.[15] They specifically consider an oscillator obeying the dynamical equation

$$\ddot{x} + \omega^2(t)x = 0 , \quad (3.1)$$

where the time-dependent frequency $\omega(t)$ is periodic. The form

$$\omega^2(t) = \omega_0^2(1 + h\cos\beta t) \quad (3.2)$$

with $h \ll 1$ is considered in detail and leads to exponential growth of $x(t)$, i.e., to parametric resonance, provided the parameter β lies within a specified h-dependent range. They also consider the effect of damping on the behavior of the oscillator and find that the range of values of β leading to parametric resonance is narrowed. For sufficiently large damping, the range of parametric resonance disappears and the motion is stable.

A somewhat simpler system that exhibits parametric resonance is an oscillator with an "adiabatically switched" frequency, i.e., one in which $\omega(t) = \Omega_a$ for a time t_a, then it switches to Ω_b and remains at that value for a time t_b, and continues to switch back and forth in this fashion. Thus,

$$\omega(t) = \begin{cases}\Omega_a, & m(t_a+t_b) < t < (m+1)t_a + m t_b \\ \Omega_b, & (m+1)t_a + m t_b < t < (m+1)(t_a+t_b),\end{cases} \quad (3.3)$$

where

$$m = 0, 1, 2, \ldots, .$$

This model was considered in detail by Tabor[16] in the absence of damping in the context of his studies of chaotic motion in dynamical systems. We extend his analysis to include damping, i.e., we consider the solution to the equation

$$\ddot{x}(t) + 2\lambda\dot{x}(t) + \omega^2(t)x(t) = 0 , \quad (3.4)$$

with $\omega(t)$ given by (3.3).

The solution of (3.4) at times that are multiples of the basic period (t_a+t_b) with initial conditions $x(0)=x_0$ and $\dot{x}(0)=p_0$ can be written in matrix form as

$$\begin{pmatrix}x(t)\\p(t)\end{pmatrix} = (\underline{M}_b\underline{M}_a)^m\begin{pmatrix}x_0\\p_0\end{pmatrix}, \quad (3.5)$$

for $t = m(t_a+t_b)$ with $m=0,1,2,\ldots,$. The solution at these times is sufficient to examine the stability properties of the oscillator (3.4). The matrix $\underline{M}_a$ is given by

$$\underline{M}_a = \frac{e^{-\lambda t_a}}{\omega_a}\begin{pmatrix}\lambda\sin\omega_a t_a + \omega_a\cos\omega_a t_a & \sin\omega_a t_a \\ -(\omega_a^2+\lambda^2)\sin\omega_a t_a & -\lambda\sin\omega_a t_a + \omega_a\cos\omega_a t_a\end{pmatrix}, \quad (3.6)$$

where $\omega_a \equiv (\Omega_a^2 - \lambda^2)^{1/2}$. The matrix $\underline{M}_b$ is of the same form with the replacement $(\omega_a, t_a) \to (\omega_b, t_b)$. The stability of the motion is determined by the eigenvalues of the evolution operator $(\underline{M}_b \underline{M}_a)^m$. If we denote the eigenvalues of the product $\underline{M}_b \underline{M}_a$ by ϵ_+ and ϵ_-, then the eigenvalues of the evolution operator are $(\epsilon_+)^m$ and $(\epsilon_-)^m$. The solution (3.5) is clearly unstable with increasing t if one (or both) of $|\epsilon_+|$ and $|\epsilon_-|$ is greater than unity.

The eigenvalues of $\underline{M}_b \underline{M}_a$ are given by

$$\epsilon_\pm = e^{-\lambda(t_a+t_b)}[Q \pm (Q^2-1)^{1/2}], \tag{3.7}$$

where

$$Q \equiv \cos\omega_a t_a \cos\omega_b t_b - \frac{(\omega_a^2+\omega_b^2)}{2\omega_a\omega_b} \sin\omega_a t_a \sin\omega_b t_b. \tag{3.8}$$

If $Q<1$, then from (3.7) we conclude that $|\epsilon_\pm| = e^{-\lambda(t_a+t_b)}$. The motion in this case is stable and the oscillator position and momentum are damped to zero as $t \to \infty$. Parametric resonance occurs if $Q>1$ and $\epsilon_+>1$. The condition $Q>1$ places restrictions on the values of the switching times t_a and t_b for given frequencies ω_a and ω_b, giving ranges of these times where parametric resonance is at all possible. This is the condition found by Tabor[16] in the absence of damping (with $\omega_{a,b} = \Omega_{a,b}$ in that case). The additional condition $\epsilon_+>1$, which is automatically satisfied when $Q>1$ in the absence of damping, places a restriction on the dissipative parameter λ. According to this condition, parametric resonance can occur only if the dissipative parameter satisfies the inequality

$$\lambda < \frac{\ln[Q+(Q^2-1)^{1/2}]}{t_a+t_b}. \tag{3.9}$$

The condition (3.9) gives a limiting value of λ that depends on the parameters ω_a, ω_b (which also contain λ), t_a, and t_b. We note, however, that the right-hand side of (3.9) is bounded for all values of the parameters. If λ exceeds this absolute bound, then the dissipation will quench the parametric instability and the motion is stable for all frequencies and switching times.

The reason for the instability found above can best be understood by considering the phase plane of the oscillator shown in Fig. 1. We follow Tabor and depict two sets of orbits of the oscillator in the absence of damping. The orbits drawn with solid curves are for oscillators of frequency Ω_a with different energies; those represented by dashed curves are orbits for oscillators of frequency Ω_b. When the frequency of an oscillator is instantaneously switched from Ω_a to Ω_b, the phase-space point of the oscillator moves from its old Ω_a orbit onto a new Ω_b orbit at the point of intersection of the two orbits. If the switching is done at the proper time, one can insure that the new orbit is "larger" (i.e., corresponds to a higher maximum displacement and momentum and hence to higher energy) than the old one. This is shown by the thick line with arrows on the figure. If the next switch is again performed at an appropriate time, the oscillator can move from the Ω_b orbit onto an Ω_a orbit corresponding to even higher energy. The condition $Q>1$ insures that the switching will cause the oscillator to move onto orbits of higher and higher energy, thus causing the parametric instability. The arguments in the presence of damping are entirely similar except that now the phase-space orbits are spirals towards the origin rather than ellipses. The instability can now occur only if the inward spiraling is not too rapid, as given quantitatively by Eq. (3.9).

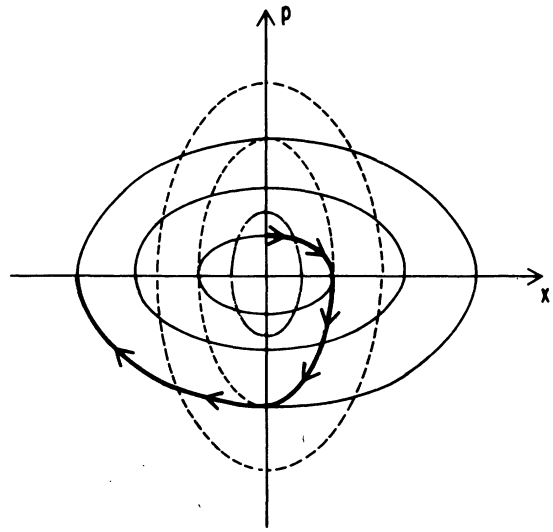

FIG. 1. Phase-space orbits for undamped mechanical oscillator of frequency Ω_a (solid lines) and Ω_b (dashed lines). The orbit of the adiabatically switched oscillator is indicated by the arrows.

IV. PARAMETRIC RESONANCE AND STOCHASTIC OSCILLATORS

In this section we interpret the instability described in Sec. II for the stochastic oscillator in terms of the parametric instability discussed in Sec. III. We view the stochastic oscillator of Sec. II as having a time-dependent frequency that can take on values according to a distribution, with switches among these frequency values occurring at random times. In Sec. III we saw that a deterministic oscillator whose frequency can take on two values exhibits parametric resonance if the switching times between these values are chosen appropriately (and the damping is not too large). Had we considered switching among several frequencies, appropriate conditions on the switching times would also have led to parametric resonance for sufficiently small damping.

In a stochastic oscillator, switchings among a prescribed finite or infinite set of frequencies occur at random times governed by a distribution function for each switch. Some of the switching times occurring in these distributions fall outside of the ranges leading to parametric resonance (in the sense of Sec. III) and others are within these ranges. Consider an ensemble of undamped oscillators for each of which a particular sequence of switching times is chosen from the distributions of switching times. Some of the oscillators of the ensemble are stable while others have exponentially growing displacement and momentum. The oscillators that exhibit the unstable behavior are those for which a finite fraction of the switching times are within the ranges that lead to parametric resonance. This description is also valid in the presence of damping provided the dissipation parameter is sufficiently small.

We associate the ensemble of oscillators discussed above with the ensemble of realizations of the stochastic oscillator of Sec. II. An average of the displacements and momenta of all the oscillators in the ensemble can then be identified with the first-order statistics of the stochastic oscillator. We found in Eq. (2.10) that the first-order moments of the stochastic oscillator are always stable even in the absence of damping. The explanation of this stability in terms of the parametric oscillator description is that in the averaging procedure, displacements (momenta) that grow exponentially in the positive direction are canceled by others that grow exponentially in the negative direction. Because of this cancellation, the effects of parametric resonance first appear in ensemble averages of the square of the oscillator positions and momenta. These averages are identified with the second-order statistical quantities that were found in Sec. II to be unstable for small damping [cf. Eqs. (2.12) and (2.13)]. The correspondence of the model described here and the stochastic oscillator of Sec. II is established quantitatively below.

In Sec. II we pointed out that for delta-correlated frequency fluctuations only the average square frequency Ω_0^2 and the second cumulant coefficient $2D_2$ enter the first- and second-order oscillator statistics. One can therefore choose any convenient stochastic process $\omega(t)$ for the oscillator frequency provided the average of $\omega^2(t)$ is Ω_0^2 and provided the fluctuations $\gamma(t)\equiv\omega^2(t)-\Omega_0^2$ are delta correlated with the second cumulant coefficient equal to $2D_2$. Here we choose a dichotomous Markov process,[1,21] which is the stochastic analog of the adiabatic switching process between two frequencies considered in Sec. III.

In the dichotomous Markov process the oscillator frequency $\omega(t)$ switches between the values Ω_a and Ω_b at random times governed by a distribution $\Psi(t)$ which we choose to be exponential, i.e., the probability density for the time between two consecutive switches to be t is given by

$$\Psi(t)=\nu e^{-\nu t}\,. \tag{4.1}$$

The average time between switches is $1/\nu$. The probability that n switches occur in the time interval $(0,t)$ with the first switch occurring at time t_1, the second at time t_2, etc., is

$$\phi(t_1,t_2,\ldots,t_n;t)=\nu\int_t^\infty dt_+\int_{-\infty}^0 dt_-\Psi(t_+-t_n)\cdots\Psi(t_2-t_1)\Psi(t_1-t_-)\,. \tag{4.2}$$

For the probability density (4.1), Eq. (4.2) yields the simple form

$$\phi(t_1,t_2,\ldots,t_n;t)=\nu^n e^{-\nu t}\,, \tag{4.3}$$

independent of the switching times. An integral of (4.3) over all the intermediate times $t_1<t_2<\cdots<t_n$ with $t_1>0$ and $t_n<t$ leads to the Poisson distribution for the probability that exactly n switches occur in the time interval $(0,t)$, i.e.,

$$\Phi(n;t)\equiv\int_0^t dt_n \times\int_0^{t_n}dt_{n-1}\cdots\int_0^{t_2}dt_1\phi(t_1,t_2,\ldots,t_n;t) =\frac{(\nu t)^n}{n!}e^{-\nu t}\,. \tag{4.4}$$

The probability that an even (odd) number of switches have occurred in the time $(0,t)$ is given by the sum of (4.4) over all even (odd) n. Performing these sums we therefore obtain

$$\mathrm{Prob}\{\omega(t+\tau)=\Omega_b\,|\,\omega(t)=\Omega_a\} =\mathrm{Prob}\{\omega(t+\tau)=\Omega_a\,|\,\omega(t)=\Omega_b\}=\tfrac{1}{2}(1-e^{-2\nu\tau})\,, \tag{4.5a}$$

$$\mathrm{Prob}\{\omega(t+\tau)=\Omega_b\,|\,\omega(t)=\Omega_b\} =\mathrm{Prob}\{\omega(t+\tau)=\Omega_a\,|\,\omega(t)=\Omega_a\}=\tfrac{1}{2}(1+e^{-2\nu\tau})\,. \tag{4.5b}$$

From these expressions it is then straightforward to obtain the correlation function for the frequency fluctuations $\gamma(t) \equiv \omega^2(t) - \Omega_0^2$ as

$$\langle \gamma(t)\gamma(t+\tau)\rangle = g^2\Omega_0^4 e^{-2\nu\tau}\,, \tag{4.6}$$

where the average square frequency is

$$\Omega_0^2 = \tfrac{1}{2}(\Omega_a^2 + \Omega_b^2)\,, \tag{4.7a}$$

and where

$$g^2 \equiv (\Omega_a^2 - \Omega_b^2)^2/4\Omega_0^4\,. \tag{4.7b}$$

In order to compare the results obtained here with those of Sec. II, we must go to the limit of delta-correlated frequency fluctuations, i.e., to the limit $\nu \to \infty$. This limit must be taken with some caution for the following reason. As switching between fixed frequencies Ω_a and Ω_b becomes more rapid, the effects of switching ultimately disappear and the oscillator in effect behaves like an ordinary oscillator of constant frequency. In order to preserve the features that we have been discussing (e.g., parametric resonance), as switching becomes more rapid we must also increase Ω_a and/or Ω_b, i.e., we must *scale* the frequencies as well as the mean switching time. A full discussion of the implications of various scaling relationships that can be chosen will be given elsewhere.[22] We will here consider the case where the dissipation parameter λ is fixed and unaffected by the scaling.[23] This is the assumption that is generally made explicitly or implicitly, and it fixes the relative scaling of the other parameters. To see this, we identify the cumulant coefficient D_2 of the delta-correlated frequency fluctuations of Sec. II with the integral of (4.6) over τ:

$$\begin{aligned} D_2 &= \int_0^\infty 2D_2\delta(\tau)d\tau \\ &\equiv \lim_{\nu\to\infty,\,\Omega_0\to\infty} \int_0^\infty g^2\Omega_0^4 e^{-2\nu\tau}d\tau \\ &\equiv \lim_{\nu\to\infty,\,\Omega_0\to\infty} g^2\frac{\Omega_0^4}{2\nu}\,. \end{aligned} \tag{4.8}$$

Since λ is unaffected by the scaling, to ultimately recover the stability criterion of Eq. (2.13) we must require that D_2 and Ω_0^2 scale the same way. Hence the scaling relation between ν and Ω_0^2 must be such that $\Omega_0^2/\nu \to$ const as each separately goes to infinity.

Consider then the oscillator (3.4) with $\omega(t)$ described by the dichotomous Markov process (4.5). We will construct the first and second moments of the oscillator by direct integration of the equation of motion followed by an average over switching times. We note that the stability properties of the oscillator can be discussed independently of the stochastic forcing function $f(t)$ of Eq. (2.1), so that with no loss of generality, we have set $f(t)$ equal to zero.

A. First-order properties

We begin observing our oscillator at time $t=0$ with initial displacement $x(0)=x_0$ and momentum $p(0)=p_0$. We take the initial frequency to be Ω_a (our conclusions are independent of the initial state of the oscillator). The evolution of the oscillator from the initial time $t=0$ until the first frequency switch at time t_1 is described by the matrix equation

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \underline{M}_a(t)\begin{pmatrix} x_0 \\ p_0 \end{pmatrix}, \quad 0 \le t < t_1 \tag{4.9}$$

where

$$\underline{M}_a(t) = \frac{e^{-\lambda t}}{\omega_a}\begin{pmatrix} \omega_a\cos\omega_a t + \lambda\sin\omega_a t & \sin\omega_a t \\ -(\omega_a^2+\lambda^2)\sin\omega_a t & \omega_a\cos\omega_a t - \lambda\sin\omega_a t \end{pmatrix} \tag{4.10}$$

and where $\omega_a^2 = \Omega_a^2 - \lambda^2$. Note that this is precisely the matrix (3.6) occurring in the parametric oscillator of Sec. III. At time t_1 the frequency switches to Ω_b and the oscillator position and momentum are given by

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \underline{M}_b(t-t_1)\underline{M}_a(t_1)\begin{pmatrix} x_0 \\ p_0 \end{pmatrix}, \quad t_1 \le t < t_2 \tag{4.11}$$

where $\underline{M}_b(t-t_1)$ is of the same form as $\underline{M}_a$ with the replacement $\omega_a \to \omega_b$, and where t_2 is the time at which the next switch occurs. For a sequence of n switches we obtain

$$\begin{aligned} \begin{pmatrix} x(t) \\ p(t) \end{pmatrix} &= \underline{M}_i(t-t_n)\cdots\underline{M}_b(t_4-t_3)\underline{M}_a(t_3-t_2) \\ &\quad\times \underline{M}_b(t_2-t_1)\underline{M}_a(t_1)\begin{pmatrix} x_0 \\ p_0 \end{pmatrix}, \quad t_n \le t < t_{n+1} \end{aligned} \tag{4.12}$$

where $i=a$ if n is even and $i=b$ if n is odd. Since the switching times $t_1, t_2, \ldots, t_n$ are random, we must average them over the distribution (4.3) and sum over all possible numbers n of switches:

$$\begin{pmatrix}\langle x(t)\rangle \\ \langle p(t)\rangle\end{pmatrix} = \sum_{n=0}^{\infty}\int_0^t dt_n \cdots \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \phi_n(t_1, t_2, \ldots, t_n; t) \underline{M}_i(t-t_n) \cdots \underline{M}_b(t_2-t_1)\underline{M}_a(t_1)\begin{pmatrix}x_0 \\ p_0\end{pmatrix}. \quad (4.13)$$

Because of the convolutions in (4.13), it is convenient to consider the Laplace transform of the oscillator displacement and momentum with respect to time,

$$\begin{pmatrix}\langle \tilde{x}(s)\rangle \\ \langle \tilde{p}(s)\rangle\end{pmatrix} = \sum_{n=0}^{\infty} \nu^n \underline{\tilde{M}}_i(s+\nu)\cdots\underline{\tilde{M}}_b(s+\nu) \times \underline{\tilde{M}}_a(s+\nu)\begin{pmatrix}x_0 \\ p_0\end{pmatrix}, \quad (4.14)$$

where

$$\begin{aligned}\underline{\tilde{M}}_i(s+\nu) &= \frac{1}{\nu}\int_0^{\infty} e^{-st}\Psi(t)\underline{M}_i(t)dt \\ &= \int_0^{\infty} e^{-(s+\nu)t}\underline{M}_i(t)dt \\ &= \frac{1}{(\lambda+\nu+s)^2+\omega_i^2}\begin{pmatrix}2\lambda+\nu+s & 1 \\ -(\omega_i^2+\lambda^2) & \nu+s\end{pmatrix},\end{aligned} \quad (4.15)$$

with $i=a, b$. The sum in (4.14) must be carried out separately over even n and odd n. The result is

$$\begin{pmatrix}\langle \tilde{x}(s)\rangle \\ \langle \tilde{p}(s)\rangle\end{pmatrix} = [(\underline{\tilde{M}}_a + I/\nu)(\underline{I} - \nu^2\underline{\tilde{M}}_b\underline{\tilde{M}}_a)^{-1} - \underline{I}/\nu]\begin{pmatrix}x_0 \\ p_0\end{pmatrix}, \quad (4.16)$$

where $\underline{I}$ is the 2×2 identity matrix and the argument $(s+\nu)$ of the matrices $\underline{\tilde{M}}_i$ has been suppressed.

The time evolution of the average moments can now be found by performing the inverse Laplace transform of (4.16). To do this one must find the poles of (4.16) in the s plane. These poles arise from the inverse matrix $(\underline{I} - \nu^2\underline{\tilde{M}}_b\underline{\tilde{M}}_a)^{-1}$ occurring in (4.16), i.e., they are the solutions of the determinantal equation

$$\det[\underline{I} - \nu^2\underline{\tilde{M}}_b(s+\nu)\underline{\tilde{M}}_a(s+\nu)] = 0. \quad (4.17)$$

In general, Eq. (4.17) gives an eighth degree polynomial equation in s. The eight roots $S_1, \ldots, S_8$ then determine the time evolution of $\langle x(t)\rangle$ and $\langle p(t)\rangle$. We note that with these eight roots one can construct the first-order statistical properties of the oscillator with a dichotomous Markov frequency having an *arbitrary* correlation time. Since our primary interest here is the recovery of the results of Sec. II, we consider (4.17) in the limit corresponding to delta-correlated frequency fluctuations, i.e., we scale the variables according to the arguments presented earlier in this section. If we introduce a scaling parameter α, then $\nu\sim\alpha^2$, $\omega_i\sim\alpha$, and $\lambda\sim\alpha^0$. Expanding s in a power series in α of the form $s = s_0 + s_1\alpha + s_2\alpha^2 + \cdots$ and considering (4.17) in the limit $\alpha\to\infty$, we find that the eight roots degenerate to only two distinct ones, given by

$$S_\pm = -\lambda \pm i\frac{1}{\sqrt{2}}(\omega_a^2 + \omega_b^2)^{1/2}. \quad (4.18)$$

Using the definition of Ω_0^2 given in (4.7), we see that $S_\pm = -\lambda \pm i\omega_1$ with ω_1 given by (2.11). Thus these are precisely the roots that yield the time evolution (2.10) for the mean motion of the stochastic oscillator.

B. Second-order properties

The evolution of the second-order quantities of the dichotomous oscillator for a sequence of n switches is given in analogy with Eq. (4.12) by

$$\begin{pmatrix}x^2(t) \\ x(t)p(t) \\ p^2(t)\end{pmatrix} = \underline{N}_i(t-t_n)\cdots\underline{N}_b(t_2-t_1)\underline{N}_a(t_1)\begin{pmatrix}x_0^2 \\ x_0p_0 \\ p_0^2\end{pmatrix}, \quad t_n \le t < t_{n+1}. \quad (4.19)$$

We have again assumed the oscillator to initially have the frequency Ω_a. The evolution matrices $\underline{N}_i$ can be constructed directly from the matrices $\underline{M}_i$ occurring in the calculation of the first-order properties and are given by

$$\underline{N}_i(t) = \frac{e^{-2\lambda t}}{\omega_i^2}\begin{pmatrix}(g_i^+)^2 & 2g_i^+\sin\omega_i t & \sin^2\omega_i t \\ -g_i^+(\omega_i^2+\lambda^2)\sin\omega_i t & g_i^+g_i^- - (\omega_i^2+\lambda^2)\sin^2\omega_i t & g_i^-\sin\omega_i t \\ (\omega_i^2+\lambda^2)^2\sin^2\omega_i t & -2g_i^-(\omega_i^2+\lambda^2)\sin\omega_i t & (g_i^-)^2\end{pmatrix}, \quad (4.20)$$

where

$$g_i^{\pm}(t) \equiv \omega_i \cos\omega_i t \pm \lambda \sin\omega_i t. \tag{4.21}$$

Averaging (4.20) over all switching times $t_1, \ldots, t_n$ and summing over n gives the second-order moments, i.e.,

$$\begin{bmatrix} \langle x^2(t)\rangle \\ \langle x(t)p(t)\rangle \\ \langle p^2(t)\rangle \end{bmatrix} = \sum_{n=0}^{\infty} \int_0^t dt_n \cdots \int_0^{t_2} dt\, \phi_n(t_1, t_2, \ldots, t_n; t) \underline{N}_i(t-t_n) \cdots \underline{N}_b(t_2-t_1)\underline{N}_a(t_1) \begin{bmatrix} x_0^2 \\ x_0 p_0 \\ p_0^2 \end{bmatrix}. \tag{4.22}$$

Following our earlier procedure, we Laplace transform (4.22) and arrive at the analog of (4.16):

$$\begin{bmatrix} \langle \tilde{x}^2(s)\rangle \\ \langle \tilde{x}p(s)\rangle \\ \langle \tilde{p}^2(s)\rangle \end{bmatrix} = [(\underline{\tilde{N}}_a + \underline{I}/\nu)(\underline{I} - \nu^2 \underline{\tilde{N}}_b \underline{\tilde{N}}_a)^{-1} - \underline{I}/\nu] \begin{bmatrix} x_0^2 \\ x_0 p_0 \\ p_0^2 \end{bmatrix}. \tag{4.23}$$

In (4.23) the column vector on the left side denotes the Laplace transform of the column vector on the left side of (4.22), $\underline{I}$ is the 3×3 unit matrix, and

$$\begin{aligned} \underline{\tilde{N}}_i &\equiv \underline{\tilde{N}}_i(s+\nu) \\ &= \frac{1}{\nu}\int_0^{\infty} e^{-st}\Psi(t)\underline{N}_i(t)\,dt \\ &= \int_0^{\infty} e^{-(s+\nu)t}\underline{N}_i(t)\,dt. \end{aligned} \tag{4.24}$$

To find the inverse Laplace transform of (4.23) one again needs the roots of a determinantal equation, i.e.,

$$\det[\underline{I} - \nu^2 \underline{\tilde{N}}_b(s+\nu)\underline{\tilde{N}}_a(s+\nu)] = 0. \tag{4.25}$$

This equation is an 18th degree polynomial in s. If we restrict our attention to delta-correlated fluctuations by again introducing a scaling parameter α, then in the limit $\alpha \to \infty$, Eq. (4.25) has only three district roots corresponding to the eigenvalues $\epsilon_1, \epsilon_2, \epsilon_3$ of Eq. (2.15). The eigenvalue ϵ_1 is the one that leads to the instability of the second moments. It is this instability that we want to explain in the context of the above model. It is therefore necessary to identify the root of (4.25) that corresponds to this eigenvalue. To establish a criterion that will aid us in this identification we analyze the behavior of the eigenvalue ϵ_1 in terms of the scaling parameter α.

With the scaling relations $\nu \sim \alpha^2$, $\omega_i \sim \alpha$, $\lambda \sim \alpha^0$, $g^2 \sim \alpha^0$, and $D_2 \sim \alpha^2$ one can easily establish from Eqs. (2.15) and (2.16) that $\epsilon_1 \sim \alpha^0$, Reϵ_2 and Re$\epsilon_3 \sim \alpha^0$, and Imϵ_2 and Im$\epsilon_3 \sim \alpha$. An expansion of ϵ_1 in inverse powers of α leads to

$$\epsilon_1 = -2\lambda + D_2/\Omega_0^2 + O(1/\alpha). \tag{4.26}$$

This expansion in orders of the scaling parameter is necessary because the model of Sec. II is a phenomenological construction not based on an underlying process with a finite correlation time.[23] Since ϵ_1 is the only eigenvalue whose leading contribution is independent of α, our procedure is to search for a root S of (4.25) that is also of $O(\alpha^0)$. If one then assumes in (4.25) that s is of $O(\alpha^0)$ and collects terms in powers of α, one obtains a linear equation for s whose solution is exactly (4.26):

$$S = -2\lambda + D_2/\Omega_0^2 \tag{4.27}$$

Our procedure therefore leads to the *same* condition for stability and to the *same* exponential growth rate as in Sec. II. We finally note that a direct comparison between the stability bound $\lambda = D_2/2\Omega_0^2$ found here and in Sec. II, and that given by (3.9), is not possible, because the latter is appropriate for specific switching times, whereas in the former an average over switching times has been performed.

V. DISCUSSION

In this paper we have explained the physical origin of the energetic instability of stochastic oscillators with fluctuating frequencies in terms of the parametric resonance of deterministic oscillators with time-dependent frequencies. While the details have here been presented only for a mechanical harmonic oscillator, the same physical picture can be used to understand more complicated mechanical systems. For example, a harmonic oscillator with a fluctuating dissipation parameter can also exhibit an energetic instability under appropriate conditions.[24] This can be understood in terms of the orbit pictures used in Sec. III. These arguments can also be extended to systems with more than two degrees of freedom.

We noted in Ref. 5 that not all physical systems that are described as a harmonic oscillator with a fluctuating frequency correspond to the mechanical oscillator.[3] For example, the Kubo oscillator[8] governed by the mode amplitude equation

$$\dot{A}(t) = -i\omega(t)A(t) \tag{5.1}$$

and its complex conjugate cannot be cast in the form (1.1). Oscillators of this type do not exhibit the instability discussed in this paper. The reason can again be given by picturing the orbits describing the motion of the oscillator. The effect of changing the frequency (5.1) with time is simply to change the phase point of the system from one location to another on a given orbit. In other words, the frequency fluctuations in this case affect the oscillator phase but not its energy.

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