Perturbation theory for a Dirac particle in a central field

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We present a perturbation theory for an arbitrary bound state of a Dirac particle in a central field by reducing the Dirac equation to a Ricatti equation, following a method first introduced by Mikhailov and Polikanov. All corrections to the energies and wave functions, including corrections to the positions of the nodes in excited states, are expressed in quadrature in a hierarchical scheme, without the use of either the Green's function or the sum over intermediate states.

I. INTRODUCTION

The relativistic bound-state wave function of a Dirac particle in a central field can be written as[1,2]

$$\psi_{jlm}=\frac{1}{r}\begin{pmatrix} i\mathcal{G}(r)\Omega_{jlm}(\theta,\phi) \\ -\mathcal{F}(r)\Omega_{jl'm}(\theta,\phi)\end{pmatrix}, \tag{1}$$

where $l'=2j-l$, and $\Omega_{jlm}(\theta,\phi)$ is a spherical spinor in the direction (θ,ϕ). In natural units, $\hbar=c=1$, the radial wave functions $\mathcal{G}$ and $\mathcal{F}$ are determined by the two coupled first-order differential equations

$$\mathcal{G}'+\frac{k}{r}\mathcal{G}-(E-V+m)\mathcal{F}=0$$

and (2)

$$\mathcal{F}'-\frac{k}{r}\mathcal{F}+(E-V-m)\mathcal{G}=0,$$

where the prime indicates a differentiation with respect to its argument r and $k\equiv\mp(j+\frac{1}{2})$ for $j=l\pm\frac{1}{2}$, E is the energy eigenvalue, m is the rest mass of the Dirac particle, and $V\equiv V(r)\equiv V_0(r)+\lambda V_1(r)$ is the spherically symmetric potential with λV_1 introduced as a perturbation to the potential V_0. In order that the bound-state wave functions be normalizable and regular at the origin, we impose the boundary conditions that $\mathcal{G}$ and $\mathcal{F}$ be 0 as $r\to 0$ and as $r\to\infty$. We also assume that in the absence of the perturbation V_1, the solution to the coupled differential equations (2) are known

$$\mathcal{G}_0'+\frac{k}{r}\mathcal{G}_0-(E_0-V_0+m)\mathcal{F}_0=0$$

and (3)

$$\mathcal{F}_0'-\frac{k}{r}\mathcal{F}_0+(E_0-V_0+m)\mathcal{G}_0=0;$$

that is, the quantities $\mathcal{G}_0$, $\mathcal{F}_0$, and E_0 are known exactly. Here we shall give the perturbative solution to $\mathcal{G}$, $\mathcal{F}$, and E as the perturbation V_1 is introduced for an arbitrary bound state. This will be achieved through the reduction of the coupled differential equations (2) to a Ricatti equation.

In the case of the ground state, where the wave function does not possess any node, the perturbative solution has been given by Mikhailov and Polikanov.[3] In Sec. II, we would like to discuss two variants of the method of Mikhailov and Polikanov. The first one, though less rapidly convergent, seems more straightforward to use. The second one is the first-order perturbation iteration method (FOPIM), first introduced by Hirschfelder[4] in the discussion on the Rayleigh-Schrödinger perturbation theory, and recently applied to the Ricatti equation by Au.[5] This second method offers accelerated convergence, if convergence exists.

In the case of excited bound states, where the wave functions possess nodes, the method of Mikhailov and Polikanov[3] must be modified. As emphasized previously[6,7] in the case of the Schrödinger equation, the zeros need to be located and factored out first. We shall then show that, similar to nonrelativistic problems reducible to one dimension,[7] the energy shifts, the wave function corrections, as well as the corrections to the positions of the nodes, to any order, are expressible in quadrature in a hierarchical scheme. This will be done in Sec. III for the first excited state. In Sec. IV we extend the technique to an arbitrary bound excited state. Finally, in Sec. V, we make some concluding remarks.

II. THE GROUND STATE OF A DIRAC PARTICLE IN A CENTRAL FIELD

In this case, as noted earlier, the functions $\mathcal{G}$ and $\mathcal{F}$ have no node. As pointed out by Mikhailov and Polikanov,[3] by introducing a function Φ through

$$\mathcal{F}=\mathcal{G}\Phi, \tag{4}$$

the set of differential equations (2) decouple into

$$\Phi'-\frac{2k}{r}\Phi+B\Phi^2+B-2m=0 \tag{5}$$

and

$$\mathcal{G}' + \left(\frac{k}{r} - B\Phi\right)\mathcal{G} = 0 , \tag{6}$$

where we have also set $B \equiv E - V + m$, following Mikhailov and Polikanov. Equation (5) is an eigenvalue problem in the Ricatti form. Once solved, it will enable Eq. (6) to be readily integrated, provided $\mathcal{G}$ has no zeros, as is the case for the ground state. A perturbation theory on Eq. (5) was given by Mikhailov and Polikanov.[3] However, in their treatment, the higher order corrections are not explicitly given because of increasing complexity. Here we discuss two variants of their method, where we manage to retain relative simplicity in treating the higher order corrections.

A. The naive method

The conventional perturbation theory on an eigenvalue problem is to expand both the eigenfunction and the eigenvalue in power series of the coupling constant λ. We therefore write

$$\Phi = \sum_{i=0}^{\infty} \lambda^i \Phi_i \tag{7}$$

and

$$B = \sum_{i=0}^{\infty} \lambda^i B_i , \tag{8}$$

where

$$\begin{aligned} B_0 &= E_0 - V_0 + m , \\ B_1 &= E_1 - V_1 , \\ B_i &= E_i , \quad i \geq 2 . \end{aligned} \tag{9}$$

On substituting the expansions (7) and (8) into the Ricatti equation (5), we obtain the following hierarchy of equations to each order in λ:

$$\Phi_0' - \frac{2k}{r}\Phi_0 + B_0\Phi_0^2 + B_0 - 2m = 0 , \tag{10}$$

$$\Phi_1' - \frac{2k}{r}\Phi_1 + B_1\Phi_0^2 + 2B_0\Phi_0\Phi_1 + B_1 = 0 , \tag{11}$$

and for $i \geq 2$,

$$\Phi_i' - \frac{2k}{r}\Phi_i + B_i\Phi_0^2 + 2\Phi_0\sum_{j=1}^{i} B_{i-j}\Phi_j + \sum_{j=2}^{i} B_{i-j}\sum_{k=1}^{j-1}\Phi_k\Phi_{j-k} + B_i = 0 , \tag{12}$$

where the unperturbed solutions, Φ_0 and B_0, and hence $\mathcal{G}_0$ and $\mathcal{F}_0$ are known. One then observes, through Eq. (6) to zeroth order, that the square of the wave function $\mathcal{G}_0$, serves as an integrating factor in this hierarchy:

$$\begin{aligned} (\mathcal{G}_0^2\Phi_i)' &= -(E_i - V_i)(1 + \Phi_0^2)\mathcal{G}_0^2 \\ &= -(E_i - V_i)(\mathcal{F}_0^2 + \mathcal{G}_0^2) , \end{aligned} \tag{13}$$

where for $i \geq 2$,

$$-V_i(1 + \Phi_0^2) = 2\Phi_0\sum_{j=1}^{i-1} B_{i-j}\Phi_j + \sum_{j=2}^{i} B_{i-j}\sum_{k=1}^{j-1}\Phi_k\Phi_{j-k} . \tag{14}$$

Equation (13) can be readily integrated. On using the normalization and boundary conditions imposed on $\mathcal{F}_0$ and $\mathcal{G}_0$, we obtain

$$E_i = \int_0^{\infty} V_i(\mathcal{F}_0^2 + \mathcal{G}_0^2)\,dr \tag{15}$$

and

$$\Phi_i = -\frac{1}{\mathcal{G}_0^2}\int_0^r (E_i - V_i)(1 + \Phi_0^2)\mathcal{G}_0^2\,dr . \tag{16}$$

For $i = 1$, our present result reduces to that of Mikhailov and Polikanov.[3] The reason for the choice of the lower limit in the integration in Eq. (16) has been given by these authors.[3] We have thus established that the corrections to the energy E and the eigenfunction Φ are obtainable in a hierarchical scheme to any arbitrary order. It can then be seen from Eq. (6) that once the corrections to E and Φ are obtained to a given order, the corrections to the logarithm of $\mathcal{G}$, and hence the correction to $\mathcal{G}$, are obtained to the same order;

$$\ln\mathcal{G} = \int^r \left(B\Phi - \frac{k}{r}\right)dr . \tag{17}$$

The lower limit of integration is an additive constant to $\ln\mathcal{G}$, and is hence a mulitplicative constant to $\mathcal{G}$ which can be fixed by the normalization condition

$$\int_0^{\infty} (\mathcal{G}^2 + \mathcal{F}^2)\,dr = \int_0^{\infty} (1 + \Phi^2)\mathcal{G}^2\,dr = 1 . \tag{18}$$

B. First-order perturbation iteration method (FOPIM)

This method was first introduced by Hirschfelder[4] in the discussion of Rayleigh-Schrödinger perturbation theory and was recently applied to the Ricatti equation by Au.[5] Here we first follow the naive method to find the first-order corrections Φ_1 and E_1. Next we seek a potential $V'(r)$ $\equiv V(r) - \Delta V'(r) = V_0 + \lambda V_1 - \Delta V'$ such that $\Phi_0 + \lambda\Phi_1$ is an exact solution to the corresponding Ricatti equation with an exact energy eigenvalue $E_0 + E_1$, that is,

$$\begin{aligned} &(\Phi_0 + \lambda\Phi_1)' - \frac{2k}{r}(\Phi_0 + \lambda\Phi_1) \\ &\quad + (B_0 + \lambda B_1 + \Delta V')(\Phi_0 + \lambda\Phi_1)^2 \\ &\quad + (B_0 + \lambda B_1 + \Delta V') - 2m = 0 . \end{aligned} \tag{19}$$

On using Eqs. (10) and (11), we readily find

$$\Delta V^I = -\lambda^2 \frac{[B_0\Phi_1^2 + B_1(2\Phi_0\Phi_1 + \lambda\Phi_1^2)]}{1+(\Phi_0+\lambda\Phi_1)^2} \equiv \lambda^2 V_1^I . \tag{20}$$

We next define

$$\Phi_0^I \equiv \Phi_0 + \lambda\Phi_1 , \tag{21}$$

$$E_0^I \equiv E_0 + \lambda E_1 , \tag{22}$$

$$V_0^I \equiv V_0 + \lambda V_1 - \Delta V^I , \tag{23}$$

and

$$B_0^I \equiv E_0^I - V_0^I + m . \tag{24}$$

Then

$$\mathcal{G}_0^I = C^I r^{-k} \exp \int_0^r B_0^I \Phi_0^I dr \tag{25}$$

and

$$\mathcal{F}_0^I = \Phi_0^I \mathcal{G}_0^I , \tag{26}$$

where the constant C^I is determined by the normalization condition

$$\int_0^\infty [(\mathcal{G}_0^I)^2 + (\mathcal{F}_0^I)^2] dr = 1 . \tag{27}$$

$\mathcal{G}_0^I$ and $\mathcal{F}_0^I$ are the normalized wave functions for the Dirac particle in the central potential $V - \Delta V^I$. Thus, from the knowledge of the exact solution to the unperturbed Hamiltonian H_0, where the perturbation is given by λV_1, we are able to construct the exact solution to the "new unperturbed Hamiltonian" $H_0^I \equiv H - \Delta V^I = H_0 + \lambda V_1 - \lambda^2 V_1^I$. Hence as far as the original Hamiltonian H is concerned, the perturbation is given by ΔV^I which is of order λ^2, since $H = H_0^I + \lambda^2 V_1^I$ and for this particular state the exact solutions to H_0^I are known. Therefore we have succeeded in transforming a problem with a perturbation of order λ to one with a perturbation of order λ^2. Definitely, this process can be continued. On executing this procedure for a second time, we can construct the exact solution to the next in line new unperturbed Hamiltonian $H_0^{II} \equiv H - \Delta V^{II}$ such that the leading term in ΔV^{II} is of the order λ^4. All one needs to do is to replace the set ($\mathcal{G}_0$, $\mathcal{F}_0$, Φ_0, E_0, V_0, B_0, V_1, λ) by the set ($\mathcal{G}_0^I$, $\mathcal{F}_0^I$, Φ_0^I, E_0^I, V_0^I, B_0^I, V_1^I, λ^2). Specifically,

$$\Delta V^{II} = -\frac{\lambda^4[B_0^I(\Phi_1^I)^2 + B_1^I(2\Phi_0^I\Phi_1^I) + \lambda^2(\Phi_1^I)^2]}{1+(\Phi_0^I+\lambda^2\Phi_1^I)^2} \tag{28}$$

$$\equiv \lambda^4 V_1^{II} , \tag{29}$$

where

$$B_1^I = E_1^I - V_1^I , \tag{30}$$

$$E_1^I = \int_0^\infty V_1^I[(\mathcal{F}_0^I)^2 + (\mathcal{G}_0^I)^2] dr , \tag{31}$$

and

$$\Phi_1^I = -\frac{1}{(\mathcal{G}_0^I)^2} \int_0^r (E_1^I - V_1^I)[1 + (\Phi_0^I)^2](\mathcal{G}_0^I)^2 dr . \tag{32}$$

One next defines

$$\Phi_0^{II} \equiv \Phi_0^I + \lambda^2 \Phi_1^I , \tag{33}$$

$$E_0^{II} \equiv E_0^I + \lambda^2 E_1^I , \tag{34}$$

$$V_0^{II} \equiv V_0^I + \lambda^2 V_1^I - \Delta V^{II} , \tag{35}$$

and

$$B_0^{II} \equiv E_0^{II} - V_0^{II} + m . \tag{36}$$

Then

$$\mathcal{G}_0^{II} = C^{II} r^{-k} \exp \int_0^r B_0^{II} \Phi_0^{II} dr \tag{37}$$

and

$$\mathcal{F}_0^{II} = \Phi_0^{II} \mathcal{G}_0^{II} , \tag{38}$$

such that

$$\int_0^\infty [(\mathcal{G}_0^{II})^2 + (\mathcal{F}_0^{II})^2] dr = 1 . \tag{39}$$

$\mathcal{G}_0^{II}$ and $\mathcal{F}_0^{II}$ are the normalized wave functions for the next in line new unperturbed Hamiltonian H_0^{II}. Obviously, at the end of the nth step, we have the exact solution to the "nth order unperturbed Hamiltonian $H_0^N \equiv H - \Delta V^N$ which differs from the total Hamiltonian H by ΔV^N, a term of order λ^{2^n}. Hence, we have the solution to order $\lambda^{(2^n-1)}$ after n steps in this iteration scheme. As Dalgarno and Stewart have shown,[8] using the usual perturbation theory, we can estimate the energy to order $2(2^n-1)+1$ since we know the wave function to order 2^n-1. This method thus offers accelerated convergence, if convergence exists, and is particularly attractive when the integrations in the naive method cannot be done analytically.

III. THE FIRST EXCITED BOUND STATE OF A DIRAC PARTICLE IN A CENTRAL FIELD

We now extend our technique to the excited bound states where the wave functions have nodes. For simplicity, we focus our attention on the first excited state where the wave function has only one node, which is sufficient to illustrate the principles involved. We shall see that in the present approach, the corrections to the nodal positions of the wave function of the excited states are expressible as quadratures. Owing to the accelerated convergence of the FOPIM, we will also limit our discussion to this method for the excited states.

The wave functions $\mathcal{G}$ and $\mathcal{F}$ for the first excited state have one node each. We write

$$\mathcal{G}=(r-\alpha)G \tag{40}$$

and

$$\mathcal{F}=(r-\beta)F, \tag{41}$$

where $\mathcal{G}$ and $\mathcal{F}$ do not possess any zeros. The unperturbed solutions are indicated by

$$\mathcal{G}_0=(r-\alpha_0)G_0 \tag{42}$$

and

$$\mathcal{F}_0=(r-\beta_0)F_0. \tag{43}$$

On substituting Eqs. (40) and (41) into the coupled differential Eq. (2), we obtain

$$G+(r-\alpha)G'+\frac{k}{r}(r-\alpha)G-B(r-\beta)F=0 \tag{44}$$

and

$$F+(r-\beta)F'-\frac{k}{r}(r-\beta)F+(B-2m)(r-\alpha)G=0, \tag{45}$$

where, as before, $B=E-V+m$. In analogy to Eq. (4), we introduce a function Φ which does not contain any zeros except possibly at the origin and infinity, through

$$F=G\Phi. \tag{46}$$

On substituting Eq. (46) into Eqs. (44) and (45), we obtain the logarithmic derivative of G,

$$u\equiv-(\ln G)'=\frac{k}{r}+\frac{1}{r-\alpha}[1-B\Phi(r-\beta)], \tag{47}$$

which is the one-node analog of Eq. (6), and the one-node analog of Eq. (5):

$$\Phi+(r-\beta)(\Phi'-u\Phi)-\frac{k}{r}(r-\beta)\Phi+(B-2m)(r-\alpha)=0. \tag{48}$$

Equation (48) can be rewritten in either one of the more symmetric forms

$$(r-\alpha)(r-\beta)\left(\Phi'-\frac{2k}{r}\Phi\right)-(\alpha-\beta)\Phi+B\Phi^2(r-\beta)^2+(B-2m)(r-\alpha)^2=0 \tag{49}$$

or

$$\Phi'+\Phi\left(\frac{1}{r-\beta}-\frac{1}{r-\alpha}-\frac{2k}{r}\right)+B\Phi^2\left(\frac{r-\beta}{r-\alpha}\right)+(B-2m)\left(\frac{r-\alpha}{r-\beta}\right)=0. \tag{50}$$

Next we write

$$\alpha=\sum_{i=0}^{\infty}\lambda^i\alpha_i,\qquad \beta=\sum_{i=0}^{\infty}\lambda^i\beta_i,\qquad \Phi=\sum_{i=0}^{\infty}\lambda^i\Phi_i, \tag{51}$$

and

$$B=\sum_{i=0}^{\infty}\lambda^iB_i.$$

Since we confine ourselves to FOPIM, we only need the zeroth- and first-order coefficients, with respect to the coupling constant λ, of Eq. (49). We obtain

$$(r-\alpha_0)(r-\beta_0)\left(\Phi_0'-\frac{2k}{r}\Phi_0\right)-(\alpha_0-\beta_0)\Phi_0+B_0\Phi_0^2(r-\beta_0)^2+(B_0-2m)(r-\alpha_0)^2=0 \tag{52}$$

and

$$(r-\alpha_0)(r-\beta_0)\left(\Phi_1'-\frac{2k}{r}\Phi_1\right)-[\alpha_1(r-\beta_0)+\beta_1(r-\alpha_0)]\left(\Phi_0'-\frac{2k}{r}\Phi_0\right)-(\alpha_1-\beta_1)\Phi_0-(\alpha_0-\beta_0)\Phi_1+2B_0\Phi_0\Phi_1(r-\beta_0)^2-2B_0\Phi_0^2\beta_1(r-\beta_0)-2(B_0-2m)(r-\alpha_0)\alpha_1+B_1[\Phi_0^2(r-\beta_0)^2+(r-\alpha_0)^2]=0. \tag{53}$$

We next observe

$$-(\alpha_0-\beta_0)\Phi_1=[(r-\alpha_0)+(r-\beta_0)-2(r-\beta_0)]\Phi_1, \tag{54}$$

$$u_0=\frac{k}{r}+\frac{1}{r-\alpha_0}[1-B_0\Phi_0(r-\beta_0)], \tag{55}$$

$$-(B_0-2m)(r-\alpha_0)G_0^2=\Phi_0G_0^2\left[1-(r-\beta_0)\left(u_0+\frac{k}{r}\right)\right]+(r-\beta_0)G_0^2\Phi_0', \tag{56}$$

and

$$B_0\Phi_0^2G_0^2(r-\beta_0)=\Phi_0G_0^2\left[1-(r-\alpha_0)\left(u_0-\frac{k}{r}\right)\right]. \tag{57}$$

Then G_0^2 again serves as an integrating factor to Eq. (53). We obtain the important result

$$\{[\alpha_1(r-\beta_0)-\beta_1(r-\alpha_0)]\Phi_0\mathcal{G}_0^2\}'+[(r-\alpha_0)(r-\beta_0)\Phi_1\mathcal{G}_0^2]'=-B_1G_0^2[\Phi_0^2(r-\beta_0)^2+(r-\alpha_0)^2]$$
$$=-B_1(\mathcal{G}_0^2+\mathcal{F}_0^2). \tag{58}$$

Equation (58) enables us to express every interesting physical quantity in quadrature. On integrating both sides from 0 to ∞, the left-hand side vanishes because of the boundary conditions, and we have, upon using the normalization condition

$$E_1=\int_0^\infty V_1(\mathcal{G}_0^2+\mathcal{F}_0^2)dr, \tag{59}$$

which agrees with the conventional perturbation theory. On integrating both sides from 0 to the unperturbed nodal positions, we obtain the first-order shifts in the nodes

$$\alpha_1(\alpha_0-\beta_0)\Phi_0(\alpha_0)G_0^2(\alpha_0)=\int_0^{\alpha_0}(V_1-E_1)(\mathcal{G}_0^2+\mathcal{F}_0^2)dr \tag{60}$$

and

$$\beta_1(\alpha_0-\beta_0)\Phi_0(\beta_0)G_0^2(\beta_0)=\int_0^{\beta_0}(V_1-E_1)(\mathcal{G}_0^2+\mathcal{F}_0^2)dr. \tag{61}$$

On integrating both sides from 0 to r, we obtain the solution to Φ_1,

$$(r-\alpha_0)(r-\beta_0)\Phi_1(r)G_0^2(r)=[\beta_1(r-\alpha_0)-\alpha_1(r-\beta_0)]\Phi_0(r)G_0^2(r)+\int_0^r(V_1-E_1)(\mathcal{G}_0^2+\mathcal{F}_0^2)dr. \tag{62}$$

Equations (59)–(62) express the main results of this section.

In principle, one can follow the naive method to find E_i, α_i, β_i, and Φ_i to all orders. This will be very similar to the nonrelativistic case discussed by Aharonov and Au.[7] Because of the accelerated convergence of the FOPIM, we shall limit our discussion to this method. As in the case of the ground state, we therefore seek a potential $V^I(r)\equiv V(r)-\Delta V^I(r)=V_0+\lambda V_1-\Delta V^I$ such that $\alpha_0+\lambda\alpha_1$, $\beta_0+\lambda\beta_1$, and $\Phi_0+\lambda\Phi_1$ constitute the exact solution to the Ricatti equation (49) with an exact energy eigenvalue $E_1+\lambda E_1$, that is,

$$(r-\alpha_0-\lambda\alpha_1)(r-\beta_0-\lambda\beta_1)\left((\Phi_0+\lambda\Phi_1)'-\frac{2k}{r}(\Phi_0+\lambda\Phi_1)\right)-[(\alpha_0-\beta_0)+\lambda(\alpha_1-\beta_1)](\Phi_0+\lambda\Phi_1)$$
$$+(B_0+\lambda B_1+\Delta V^I)(\Phi_0+\lambda\Phi_1)^2(r-\beta_0-\lambda\beta_1)^2+(B_0+\lambda B_1+\Delta V^I-2m)(r-\alpha_0-\lambda\alpha_1)^2=0. \tag{63}$$

We then define

$$\begin{aligned}&\Phi_0^I=\Phi_0+\lambda\Phi_1,\\ &E_0^I=E_0+\lambda E_1,\\ &V_0^I=V_0+\lambda V_1-\Delta V^I,\\ &B_0^I=E_0^I-V_0^I+m,\\ &\alpha_0^I=\alpha_0+\lambda\alpha_1,\\ &\beta_0^I=\beta_0+\lambda\beta_1.\end{aligned} \tag{64}$$

Then $\Delta V^I\equiv\lambda^2V_1^I$ is given by

$$-V_1^I[(\Phi_0^I)^2(r-\beta_0^I)^2+(r-\alpha_0^I)^2]=[\alpha_1(r-\beta_0)+\beta_1(r-\alpha_0)]\left(\frac{2k}{r}\Phi_1-\Phi_1'\right)+\alpha_1\beta_1\left(\Phi_0^{I\prime}-\frac{2k}{r}\Phi_0^I\right)$$
$$-(\alpha_1-\beta_1)\Phi_1+\beta_1^2B_0\Phi_0^2+(B_0+\lambda B_1-2m)\alpha_1^2-2B_1\alpha_1(r-\alpha_0)$$
$$+(B_0\Phi_1^2+2B_1\Phi_0\Phi_1+\lambda\Phi_1^2B_1)(r-\beta_0^I)^2-(B_1\Phi_0+2B_0\Phi_0\Phi_1)[2(r-\beta_0)\beta_1-\lambda\beta_1^2]. \tag{65}$$

We next observe

$$u_0^I\equiv-(\ln G_0^I)'$$
$$=\frac{k}{r}+\frac{1}{r-\alpha_0^I}[1-B_0^I\Phi_0^I(r-\beta_0^I)], \tag{66}$$

from which G_0^I can be obtained,

$$G_0^I=C^Ir^{-k}\exp\int_0^r[B_0^I\Phi_0^I(r-\beta_0^I)-1]/(r-\alpha_0^I)dr. \tag{67}$$

Because of Eq. (44), the integral in Eq. (67) is singularity free at $r=\alpha_0^I$. The integration constant C^I in Eq. (67) can be determined by the normalization condition

$$\int_0^\infty [(\mathcal{G}_0^I)^2+(\mathcal{F}_0^I)^2]dr=1\,, \tag{68}$$

where

$$\mathcal{G}_0^I=(r-\alpha_0^I)G_0^I \tag{69}$$

and

$$\mathcal{F}_0^I=\Phi_0^I G_0^I(r-\beta_0^I)\,. \tag{70}$$

Thus, once again, we are able to construct the exact solution to the new unperturbed Hamiltonian $H_0^I\equiv H_0+\lambda V_1-\lambda^2 V_1^I$, and we have succeeded in transforming a problem with a perturbation of order λ to one with a perturbation of order λ^2. This process can then be continued, as discussed earlier for the case of the ground state, to obtain the corrections to any desired order.

IV. THE Nth EXCITED BOUND STATE

The only difference between this case and that of the first excited state is in the number of nodes in the wave function. Here, we have N nodes instead of one. At first sight, the complexity seems awesome, but the outcome is surprisingly simple and elegant.

The multinode analogs of Eqs. (40) and (41) are

$$\mathcal{G}=\prod_{i=1}^{N}(r-\alpha^i)G\,, \tag{71}$$

$$\mathcal{F}=\prod_{i=1}^{N}(r-\beta^i)F\equiv\prod_{i=1}^{N}(r-\beta^i)G\Phi\,. \tag{72}$$

On substituting Eqs. (71) and (72) into Eqs. (2), we obtain

$$\sum_j\prod_{i\neq j}(r-\alpha^i)G+\prod_i(r-\alpha^i)G'+\frac{k}{r}\prod_i(r-\alpha^i)G - B\Phi G\prod_i(r-\beta^i)=0\,, \tag{73}$$

from which we get

$$u\equiv-(\ln G)'=\frac{k}{r}+\sum_i\frac{1}{r-\alpha^i}-B\Phi\prod_i\left(\frac{r-\beta^i}{r-\alpha^i}\right), \tag{74}$$

and from the second equation of (2):

$$\Phi'+\Phi\left(\sum_i\frac{1}{r-\beta^i}-\sum_i\frac{1}{r-\alpha^i}-\frac{2k}{r}\right)+B\Phi^2\prod_i\left(\frac{r-\beta^i}{r-\alpha^i}\right) +(B-2m)\prod_i\left(\frac{r-\alpha^i}{r-\beta^i}\right)=0\,. \tag{75}$$

Equations (73)–(75) are the multinode analogs of Eqs. (44), (47), and (50). We then repeat the procedure in Eqs. (51)–(58) and arrive at the following elegant result:

$$\left[\left(\sum_j\alpha_1^j\sum_{i\neq j}(r-\alpha_0^i)\prod_n(r-\beta_0^n)-\sum_j\beta_1^j\prod_{i\neq j}(r-\beta_0^i)\prod_n(r-\alpha_0^n)\right)\Phi_0G_0^2\right]'+\left(\prod_i(r-\alpha_0^i)(r-\beta_0^i)\Phi_1G_0^2\right)' =-B_1G_0^2\left(\Phi_0\prod_i(r-\beta_0^i)^2+\prod_i(r-\alpha_0^i)^2\right)=-B_1(\mathcal{G}_0^2+\mathcal{F}_0^2)\,. \tag{76}$$

By integrating Eq. (76) on both sides from 0 to ∞, we get

$$E_1=\int_0^\infty V_1(\mathcal{G}_0^2+\mathcal{F}_0^2)dr \tag{77}$$

in agreement with the conventional result. The first-order shifts in nodal positions are given by

$$\alpha_1^j\prod_{i\neq j}(\alpha_0^j-\alpha_0^i)\prod_n(\alpha_0^j-\beta_0^n)\Phi_0(\alpha_0^j)G_0^2(\alpha_0^j)=\int_0^{\alpha_0^j}(V_1-E_1)(\mathcal{G}_0^2+\mathcal{F}_0^2)dr \tag{78}$$

and

$$\beta_1^j\prod_{i\neq j}(\beta_0^j-\beta_0^i)\prod_n(\beta_0^j-\alpha_0^n)\Phi_0(\beta_0^j)G_0^2(\beta_0^j)=\int_0^{\beta_0^j}(E_1-V_1)(\mathcal{G}_0^2+\mathcal{F}_0^2)dr\,. \tag{79}$$

The solution $\Phi_1(r)$ is given by

$$\Phi_1(r)G_0^2(r)\prod_i(r-\alpha_0^i)(r-\beta_0^i)=\left(\sum_j\beta_1^j\prod_{i\neq j}(r-\beta_0^i)\prod_n(r-\alpha_0^n)-\sum_j\alpha_1^j\prod_{i\neq j}(r-\alpha_0^i)\prod_n(r-\beta_0^n)\right)\Phi_0(r)G_0^2(r) +\int_0^r(V_1-E_1)(\mathcal{G}_0^2+\mathcal{F}_0^2)dr\,. \tag{80}$$

Thus we have expressed all physically interesting quantities in quadrature. Now one can use FOPIM as outlined in Eqs. (63)–(70) to obtain the higher corrections to any desired order.

V. CONCLUDING REMARKS

In this paper, we have succeeded in obtaining higher corrections to the energies and wave functions, including the positions of the nodes for excited states, of a Dirac particle in a central field, to any arbitrary order in perturbation theory, without the use of either the Green's function or a sum over intermediate states. This was achieved by the reduction of the coupled first-order differential equations on the large and small component wave functions to the Ricatti equation, a technique first suggested by Mikhailov and Polikanov.[3] We have extended the work of these authors to the excited states and have also given variants of their method, which we believe to be simpler to use.

The results achieved in this paper are similar to what has been done in the nonrelativistic case of the Schrödinger equation, where a reduction to the Ricatti equation is possible by the transformation to the logarithm of the wave function.[7,9,10,11] It seems appropriate to examine the connection between the present treatment on the Dirac equation and that on the Schrödinger equation.

In the case of the Dirac equation, we deal with two linear, first-order but coupled differential equations. In the case of the Schrödinger equation, we deal with one linear second-order differential equation. It is well known that by labeling the first derivative with a different function, the second-order differential equation becomes two coupled linear first-order differential equations. In the present work on the Dirac equation, we see that the introduction of Φ through

$$\mathcal{F} = \mathcal{G}\Phi \tag{4}$$

decouples the equation. In the nonrelativistic case of the Schrödinger equation, if we have taken $\mathcal{F}$ to be $\mathcal{G}'$, then Φ will be the logarithmic derivative of $\mathcal{G}$. A perturbation expansion on Φ then is a logarithmic perturbation expansion. Thus the use of the logarithmic derivative of the wave function in the Schrödinger equation decouples the set of linear first order differential equation derivable from it.

The technique of logarithmic perturbation expansion also works in an eigenvalue problem of the Sturm-Liouville type, an example of which is the problem of the Stark effect in hydrogen in parabolic coordinates. Some discussion on the use of logarithmic perturbation expansion in the Sturm-Liouville equation is planned to be reported by one of the authors in a master thesis.[12]

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