PHYSICAL REVIEW A VOLUME 22, NUMBER 4 OCTOBER 1980

Optimal configuration of an irreversible heat engine with fixed compression ratio

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(Received 21 January 1980)

The cycle that maximizes the average power output of a class of irreversible heat engines has been obtained using optimal control theory. This class of heat engines is distinguished by being endoreversible, having a fixed compression ratio and being irreversible because of linear heat conduction. The optimal cycle is found to have eight branches including two fixed-volume branches, two isothermal branches, and two maximum-power branches. Maximum-power branches are defined and discussed in detail. It is shown that the maximum-power cycle contains no adiabatic branches. Two special limits are analyzed in detail, the Curzon-Ahlborn limit and the large-compression-ratio limit. The fixed-compression-ratio constraint and the periodicity of the engine require special attention which lends this problem some purely mathematical interest.

I. INTRODUCTION

A standard problem in thermodynamics is to determine the maximum useful work that can be obtained from a given process which is specified by placing constraints on some of the state variables of the system. In order to obtain the maximum useful work it is necessary for the process to be reversible, which means that the process must be carried out in an infinite amount of time. Recently several papers have appeared which consider the problem of optimal performance of heat engines for processes of finite durations.[1-6] In this paper we extend the results of Ref. 4 to a class of heat engines with fixed compression ratios. This is an interesting extension of the previous work because of the constraint imposed by the fixed compression ratio.

The problem of obtaining bounds on physically interesting quantities is usually attacked by either obtaining inequalities through direct mathematical or physical reasoning or by employing the machinery of variational calculus. In the past twenty-five years mathematicians and engineers have developed and applied an extension of variational calculus to the study of extremum problems under the name of optimal control theory.[7] In this paper I mainly will rely on the standard techniques of the calculus of variation; however, I shall not hesitate to use results from optimal control theory where they are advantageous. From a mathematical point of view the problem formulated here has two interesting and challenging parts. One of these arises from the constraint implied by a fixed compression ratio. This type of problem has been discussed extensively[8] for certain cases, but does not seem to have been thoroughly analyzed for periodic systems. The other interesting point is that the quantity being maximized is a time averaged quantity.[9] As with Refs. 1–6 the purpose of this paper is to gain some understanding of the effect of the requirement that processes occur in a finite time has on thermodynamic bounds.

The plan of this paper is as follows. In Sec. II the model heat engine to be studied is presented. Section III begins with a formulation of the optimization problem. The optimization procedure is carried out in several steps during the remainder of this section. Particular attention is given to the corner conditions which determine how the various branches of the maximum-power cycle are linked together. Control theory in the form of the Pontryagin maximum principle is used in this section to determine the possible values of the control variables along the cycle. The section concludes with the derivation of the maximum-power cycle. Anyone not interested in the mathematical details of the derivation may skip to Sec. IV where the maximum-power cycle is analyzed and the final results are presented. The paper concludes with a summary and three technical appendices.

II. MODEL HEAT ENGINE

The heat engine we discuss is a subclass of the class of endoreversible engines.[3] An endoreversible engine is an engine whose working fluid undergoes reversible transformations so that any irreversible processes must occur through the coupling of the engine to the environment. Here we shall be interested in the case for which the irreversible process is linear heat conduction between the working fluid and a thermal reservoir at temperature T_R.

Our model heat engine is defined by the following conditions.

(1) The engine is endoreversible.

(2) During each branch of the process the walls have a constant thermal conductivity ρ subject to

$$0 \leqslant \rho \leqslant \rho_0 . \tag{2.1}$$

(3) When the engine is in thermal contact with a heat reservoir of absolute temperature T_R, the heat flux into the working fluid is given by a linear law

$$\dot{q} = \rho(T_R - T) , \tag{2.2}$$

where T is the absolute temperature of the working fluid.

(4) Each heat reservoir has a constant temperature T_R where

$$T_L \leqslant T_R \leqslant T_H . \tag{2.3}$$

(5) The work done by the engine in one cycle is given by

$$W = \int_0^\tau P\dot{V}\,dt , \tag{2.4}$$

where P and V are the pressure and volume of the working fluid, the time derivative of V is denoted by $\dot{V}$, and τ is the cycling period of the engine.

(6) The working fluid is an ideal gas with constant heat capacities.

To state our next assumption it will be convenient to define

$$\beta = (\gamma - 1)\ln V/V_0 , \tag{2.5}$$

where $\gamma = C_P/C_V$ is the ratio of the constant pressure and constant volume heat capacities of the gas, and V_0 is a constant reference volume. We then require that the rate of change of β, $\dot{\beta} = c$, be bounded.

(7) Let c_m and c_M be arbitrary constant positive numbers. Then we require that $c(t)$ be restricted such that

$$-c_m \leqslant c \leqslant c_M . \tag{2.6}$$

(8) The engine has a fixed compression ratio. If V_0 in Eq. (2.5) is the minimum volume of the cylinder this constraint may be written as

$$0 \leqslant \beta \leqslant \beta_M . \tag{2.7}$$

It is now a simple matter to show[4] that the equations of motion of the system are

$$\dot{T} = -cT + \hat{\rho}(T_R - T) , \tag{2.8}$$

$$\dot{\beta} = c , \tag{2.9}$$

where $\hat{\rho} = \rho/C_V$ and the average power output is

$$P = C_V \int_0^\tau \frac{1}{\tau} cT\,dT . \tag{2.10}$$

The problem now is to maximize P subject to all the conditions listed above. Because of the constraint on β, the maximization problem is not as straightforward as the problem discussed in Ref. 4.

We conclude this section with a few words about the order of magnitude of $\hat{\rho}$, c_M, and c_m. To estimate the order of magnitude of these quantities we assume that the engine is a simple piston and cylinder. If the thermal conductivity of the walls is taken to be 5 W/m K, the thickness of the walls to be 0.5 cm, and the effective area of the cylinder to be 0.1 cm^2, then $\rho_0 \simeq 100$ W/K. If we take C_V to be approximately 5 J/K, then $\hat{\rho}_0 \simeq 20\ \text{s}^{-1}$. This means that the relaxation time for thermal conductivity is about $\frac{1}{20}$ s, which is probably an upper limit.

To estimate c_M and c_m, we recall that the assumption of endoreversibility means that the working fluid must remain in internal equilibrium. This puts constraints on all the rate processes. In particular, the volume must not change by a large fraction in the time it takes for a sound wave to cross the cylinder. Taking the speed of sound, a, to be 350 m/s and a typical dimension of the cyclinder, L, to be 0.1 m, we must require that c_M and c_m are much less than $3.5\times10^3\ \text{s}^{-1}$.

These estimates are quite rough; however, it seems plausible that if we let ϵ lie between $\frac{1}{2}$ and 10^{-3} where

$$\epsilon = \hat{\rho}_0/\min(c_M, c_m) , \tag{2.11}$$

then we should cover most cases of interest.

III. OPTIMIZATION PROCEDURE

Our problem is to maximize P defined by Eq. (2.10) subject to the constraints imposed by Eqs. (2.1), (2.3), (2.6), (2.7) and the equations of motion (2.8) and (2.9). In the language of optimal control theory the variables T_R, c, and $\hat{\rho} = \rho/C_V$ are called control variables and T and β are called state variables. The solution of our problem in the space of state variables is called the optimal trajectory and the corresponding controls are the optimal controls.[7]

Anyone not interested in the mathematical details may wish to skip the remainder of this section and proceed to Sec. IV.

The technique for solving the problem stated above is to introduce time-dependent Lagrange multipliers called adjoint or co-state variables and to treat the equations of motion as constraints. Thus, define

$$L = H - \psi_1\dot{T} - \psi_2\dot{\beta} , \tag{3.1}$$

where

$$H=(1/\tau)cT+\psi_1[-cT+\hat{\rho}(T_R-T)]$$
$$+\psi_2 c-\mu G\,, \tag{3.2}$$
$$G=\beta(\beta-\beta_M)\,, \tag{3.3}$$

and μ is a time-dependent Lagrange multiplier.[10] Maximizing Eq. (2.10) is now transformed into maximizing

$$J=\int_0^\tau dt\,L\,. \tag{3.4}$$

H is the Hamiltonian of our problem, the ψ's are the adjoint variables, and the constraint (2.7) is equivalent to the requirement that

$$G\leq 0\,. \tag{3.5}$$

The state and control variables are required to be periodic with period τ.

In the usual fashion, we now assume that the optimal trajectory and optimal controls are known and consider small deviations from the optimal functions. It will be convenient to divide the problem in several parts starting with the variations of the state variables. The variations in the control variables, the corner conditions which indicate how the cycle is pieced together, and the variation in the period τ will be studied in that order. Of these procedures the first is standard and should be familiar to anyone who has studied the calculus of variation. The calculation of the control variables is most easily done using control theory, although the problem may be formulated as a calculus of variations problem. The final two parts of the problem employ standard mathematical techniques of the calculus of variations but may be unfamiliar.

Before beginning the calculation, it should be noted that trajectories obtained by solving Eqs. (2.8) and (2.9) subject to the constraints are continuous functions of time by virtue of the fact that they satisfy differential equations. The derivatives of the state variables are piecewise continuous because the control variables are allowed to be discontinuous.[12] Thus the trajectories are in general made up of arcs which begin and end at points, called switching points or corners, where the controls change discontinuously. Because of this it is convenient to write Eq. (3.4) as

$$J=\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} dt\,L\,, \tag{3.6}$$

where $t_{n+1}=t_0+\tau$ and $t_0\leq t_1\leq\cdots\leq t_{n+1}$ are the times at which some control changes discontinuously.

A. Variation of the state variables

We first consider variations which leave the end points of the arcs unchanged. After the usual integration by parts, we find

$$dJ=\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} dt\left[\left(\frac{\partial H}{\partial T}+\dot{\psi}_1\right)\delta T+\left(\frac{\partial H}{\partial \beta}+\dot{\psi}_2\right)\delta\beta\right] \tag{3.7}$$

For J to be a maximum, $dJ\leq 0$; therefore, since δT may be positive or negative we must require that the coefficient of δT vanish. The variation of β is complicated by the constraint (3.5).

There are two cases that must be considered. First, if $G<0$ then $\delta\beta$ is arbitrary. In this case μ is set equal to zero and the coefficient of $\delta\beta$ in Eq. (3.7) must vanish. Secondly, if $G=0$ somewhere on the optimal trajectory, then $\delta\beta$ is not arbitrary because the allowed variations of β must not violate the constraint. This implied that $\delta G=(\partial G/\partial\beta)\delta\beta\leq 0$ at such a point. This constraint is enforced by choosing $\mu(t)$ so the coefficient of $\delta\beta$ in Eq. (3.7) vanishes. It is not difficult to show that in order for J to be a maximum $\mu(t)\geq 0$.[8] Thus by appropriately choosing μ, along the optimal trajectory $\mu G=0$ and the first-order variation of the state variables yield the equations of motion of the adjoint variables:

$$\dot{\psi}_1=-\frac{\partial H}{\partial T}=-c\left(\frac{1}{\tau}-\psi_1\right)+\hat{\rho}\psi_1\,, \tag{3.8}$$

$$\dot{\psi}_2=-\frac{\partial H}{\partial \beta}=\mu(2\beta-\beta_M)\,. \tag{3.9}$$

B. Determination of the control variables

It is possible to treat the control variables in a manner analogous to the treatment[11] of β; however, it is simpler to use the central result of optimal control theory, the Pontryagin maximum principle. We refer the reader to Ref. 4 and the list of references contained in that paper for a discussion of this principle. We shall simply state the maximum principle for the case we are studying here.

Suppose we have solved our problem and determined the optimal trajectory, the optimal controls, and the corresponding solution to the adjoint equations. Let H^* be the value of H evaluated at any time by using the optimal solution. Let H_t be the value of the Hamiltonian evaluated in the same way as H^* with the exception that the optimal controls may be replaced by any value of the controls consistent with the constraints on the control variables. Then the maximum principle states that $H^*\geq H_t$.

To see how this works in practice consider

$$H^* - H_t = [(1/\tau - \psi_1)T + \psi_2](c - c_t) + \psi_1[\hat{\rho}(T_R - T) - \hat{\rho}_t(T_{R_t} - T)] \geq 0, \qquad (3.10)$$

where the subscript t denotes a trial value and all the other functions are evaluated at any point on the optimal trajectory.

First set $\hat{\rho}_t = \hat{\rho}$ and $T_{R_t} = T_R$; then, we immediately obtain

$$c = \begin{cases} c_M, & >0 \\ -c_m \text{ if } (1/\tau - \psi_1)T + \psi_2, & <0 \\ \text{undetermined}, & =0. \end{cases} \qquad (3.11)$$

The last possibility corresponds to what is called the singular control problem and is discussed in Ref. 4. Next, set $c_t = c$ and $\hat{\rho}_t = \hat{\rho}$, then Eq. (3.10) implies that for $\hat{\rho} \neq 0$

$$T_R = \begin{cases} T_H, & >0 \\ T_L \text{ if } \psi_1, & <0 \\ \text{undetermined}, & =0. \end{cases} \qquad (3.12)$$

It is shown in Ref. 4 that ψ_1 cannot vanish over a finite time interval.[13] Finally, set $c_t = c$ and $T_{R_t} = T_R$; then Eq. (3.10) yields

$$\hat{\rho} = \begin{cases} \hat{\rho}_0, & >0 \\ 0 \text{ if } \psi_1(T_R - T), & <0 \\ \text{undetermined}, & =0. \end{cases} \qquad (3.13)$$

It is a simple matter to show that the last case in Eq. (3.13) is not possible on a finite time interval since it would lead to a time-independent state of the system.

Thus there are four classes of optimal arcs which may be used to construct the optimal trajectory.

(1) Adiabatic branches:

$\hat{\rho} = 0$, $c = c_M$ or $-c_m$.

(2) Maximum power branches:

$\hat{\rho} = \hat{\rho}_0$, $T_R = T_H$ or T_L, and $c = c_M$ or $-c_m$.

(3) Isothermal branches:

$\hat{\rho} = \hat{\rho}_0$, $T_R = T_H$ or T_L.

(4) Boundary branches:

$\beta = 0$ or β_M, $T_R = T_H$ or T_L, and $c = 0$.

The equations of motion and the adjoint equations for each of these cases are easy to solve.[4,13] We find the following results

(1) Adiabatic branches:

$$\hat{\rho} = 0, \; c = c_M \text{ or } -c_m, \quad T(t) = T(t_1)e^{-c(t-t_1)}, \; \beta(t) = \beta(t_1) + c(t - t_1). \qquad (3.14)$$

(2) Maximum-power branches:

$$\hat{\rho} = \hat{\rho}_0, \; c = c_M \text{ or } -c_m, \; T_R = T_H \text{ or } T_L,$$
$$T(t) = \frac{\hat{\rho}_0}{c + \hat{\rho}_0} T_R + \left(T(t_1) - \frac{\hat{\rho}_0}{c + \hat{\rho}_0} T_R\right)e^{-(c+\hat{\rho}_0)(t-t_1)}, \qquad (3.15)$$
$$\beta(t) = \beta(t_1) + c(t - t_1).$$

(3) Isothermal branches:

$$\hat{\rho} = \hat{\rho}_0, \; c_r = \hat{\rho}_0(T_R/T_r - 1), \; T_R = T_H \text{ or } T_L, \quad T(t) = T_r, \; \beta(t) = \beta(t_1) + c_r(t - t_1), \qquad (3.16)$$

where the subscript $r = h$ or l when $R = H$ or L, respectively. T_r and c_r are constants.

(4) Boundary or constant volume branches:

$$\hat{\rho} = \hat{\rho}_0, \; c = 0, \; T_R = T_H \text{ or } T_L, \quad T(t) = T_R - [T_R - T(t_1)]e^{-\hat{\rho}_0(t-t_1)}, \; \beta = 0 \text{ or } \beta_M. \qquad (3.17)$$

Note that the control variables $\hat{\rho}$ and T_R only take on their extreme values along optimal branches. Thus we do not need a continuum of heat reservoirs but only the hottest and coldest ones available. The third control variable c only takes its extreme values in cases (1) and (2) above; however, in the third case there is a relation between c and the temperature of the working fluid which is an immediate consequence of Eq. (2.8) and the constancy of T. It is obvious that there are other solutions to Eqs. (2.8) and (2.9) which satisfy the constraints, but these solutions do not contribute to the maximum-power cycle. The isothermal and boundary branches correspond to singular cases of Eq. (3.11). In Appendix A we argue that the optimal trajectory cannot obtain adiabatic branches.[14] Thus the maximum-power cycle may be constructed from isothermal branches, maximum-power branches, and boundary branches of the types shown above.

The maximum-power branch is unfamiliar to most people, so in Appendix B we have analyzed the thermodynamics of the process in the spirit of Ref. 15. We note that these processes differ from adiabatic processes because the cylinder remains in thermal contact with a reservoir during a rapid expansion or compression. When $\hat{\rho}_0/|c| \ll 1$, the difference between a maximum-power process and an adiabatic process is negligible; however, if $\hat{\rho}_0/|c|$ is not too small then the difference is significant. In order to determine how many times each branch occurs, in what sequence the branches occur, and the duration of each branch, it is necessary to return to the variation of Eq. (3.6) and consider the corner conditions that must be satisfied at points where the branches join.

C. Corner conditions

Consider now the variation of Eq. (3.6) with respect to the state variables, this time letting the end points of the arcs vary and also letting the t_j vary.

$$dJ = [L(t_{n+1}) - P/\tau]\, d\tau + \sum_{j=0}^{n} [(\Delta\psi_1 \delta T)_j + (\Delta\psi_2 \delta\beta)_j - (\Delta L)_j dt_j], \quad (3.18)$$

where $(\Delta f)_j = f(t_j +) - f(t_j -)$ for $j = 1, \ldots, n$, $(\Delta f)_0 = f(t_0+) - f(t_{n+1} -)$, and $t_j\pm$ means that t_j is approached through positive or negative values, respectively. To obtain Eq. (3.18), Eqs. (3.8) and (3.9) have been used to eliminate the variations along the arcs and dt_{n+1} has been replaced by $dt_0 + d\tau$.

The differentials of T and β at the corners are given by[11,16]

$$dT(t_j) = \delta T(t_j \pm) + \dot{T}(t_j \pm) dt_j , \qquad d\beta(t_j) = \delta\beta(t_j \pm) + \dot{\beta}(t_j \pm) dt_j . \quad (3.19)$$

Recall that the variation in a function f, $\delta f(t)$, corresponds to displacing f at fixed time, while the differential $df(t_j)$ corresponds to the shift in the location of the corner which originally occurred at time t_j. The conditions on the differentials ensure the continuity of the varied trajectory.

Substituting Eq. (3.19) into Eq. (3.18) yields

$$dJ = [H(t_{n+1}) - P/\tau]\, d\tau + \sum_{j=0}^{n} [(\Delta\psi_1)_j dT(t_j) + (\Delta\psi_2)_j d\beta(t_j) - (\Delta H)_j dt_j], \quad (3.20)$$

where the periodicity of the system has been used to set $dT(t_0) = dT(t_{n+1})$ and $d\beta(t_0) = d\beta(t_{n+1})$. At all corners where $G < 0$, the differentials of T, β, t_j are arbitrary which implies that ψ_1, ψ_2, and H are continuous at these points. In particular, ψ_1 and ψ_2 are periodic with period τ. The variation of the period τ is also arbitrary which then implies that[9]

$$H(t_{n+1}) = P/\tau. \quad (3.21)$$

An immediate consequence of Eq. (3.21) is that H is a constant along the optimal trajectory and, therefore, is certainly continuous across the corners. This follows from the periodic nature of the trajectory and the fact that $H(t_{n+1})$ can be evaluated at any point along the trajectory by a simple shift of t_0, the initial time.[17]

It is now necessary to examine the corners which are entry and exit points of the boundaries determined by $G = 0$. These points must be corners because c changes at such points and c only changes discontinuously. First, observe that if dT is arbitrary ψ_1 must be continuous at the entry or exit point. Because of the constraint (3.5), $d\beta$ is not arbitrary but must satisfy $dG = (\partial G/\partial\beta) d\beta \leq 0$. Thus $dJ \leq 0$ if

$$\Delta\psi_2 = \nu \partial G/\partial\beta , \quad (3.22)$$

where ν, which may be different at each entry or exit point, is a non-negative constant. Here and in the following $\partial G/\partial\beta$ is evaluated at the boundary. At an exit or entry point Eq. (3.11) implies that

$$\Delta\left[\left(\frac{1}{\tau} - \psi_1\right) T + \psi_2\right] = -\nu' \frac{\partial G}{\partial\beta}, \quad (3.23)$$

where ν' is a non-negative constant. To see why this is so, suppose $\beta = \beta_M$, then at the entry point $c(t-) > 0$, and $c(t+) \leq 0$ where the equality holds if the optimal trajectory includes a boundary arc and the inequality holds if the trajectory just touches the boundary at one point. In this case $\Delta[(1/\tau - \psi_1)T + \psi_2] \leq 0$, which is conveniently expressed as $-\nu' \partial G/\partial\beta$. A similar argument holds at the exit and entry points of both boundaries. Finally, since T and ψ_1 are continuous, Eq. (3.23) implies that $\Delta\psi_2 = -\nu' \partial G/\partial\beta$ and with Eq. (3.22) in turn implies that $\nu' = \nu = 0$ or that ψ_2 is continuous at the boundary corners.

The argument just given depends on the continuity of ψ_1 which followed from Eq. (3.20) and assumed the arbitrariness of dT. It is not always true that dT is arbitrary at the entry point of the boundary. To see how this complication may arise, refer to Fig. 1 where the arc a-b is a segment of the optimal trajectory which strikes the boundary $\beta = \beta_M$. Suppose—as is in fact the case in problem under study—that a-b is a minimum time arc, that is, for the allowed controls all other arcs take longer to go from a to b than the optimal arc. Furthermore, suppose that all arcs hit the boundary at higher temperatures than T_b; then for all variations $dT_b \geq$ and $dt_b \geq 0$. Thus, in general, the requirement that $dJ \leq 0$ implies that $(\Delta\psi_1)_b \leq 0$ and $(\Delta H)_b \geq 0$. At the entry to the other boundary $(\Delta\psi_1) \geq 0$ and $(\Delta H) \geq 0$ because there $dT \leq 0$ and $dt \geq 0$. For the problem treated in this paper, it is again possible to show that ψ_1 and ψ_2 are continuous. The continuity of H follows from Eq. (3.21); however, for a periodic system the continuity can be proved directly.[18] The proof that ψ_1 and ψ_2 are continuous follows from Eq. (3.23). First, to take into account both boundaries, it is convenient to write $\Delta\psi_1 = -\nu'' \partial G/\partial\beta$ where $\nu'' \geq 0$. Then the left-hand side of Eq. (3.23) becomes $(T\nu'' + \nu')\partial G/\partial\beta$. Since $T\nu'' + \nu' \geq 0$ and the right-hand side of the equation is not positive, $\nu = \nu' = \nu'' = 0$ and ψ_1 and ψ_2 are continuous.

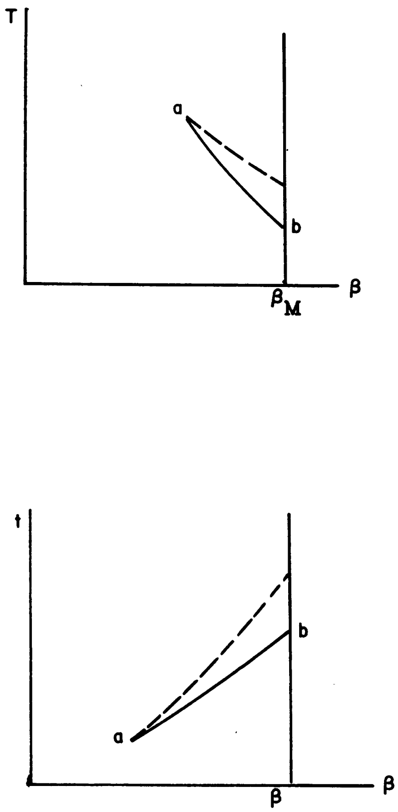

FIG. 1. T-β and time-β sketches of the minimum time arc ab. The dashed lines represent an allowed neighboring arc. Such an arc meets the boundary at a higher temperature than T_b and is of a longer duration than $t_b - t_a$.

In summary, in this rather long section it has been shown that the adjoint variables and the Hamiltonian are continuous along the optimal trajectory. Eq. (3.21) shows that H is in fact constant.

D. Switchings

To construct the optimal trajectory it is necessary to find which branches can be connected together at the switching times t_j. In doing this we also find the duration of each branch and the period of the cycle. Arguments like those given in Ref. 4 show that the possible switches between pairs of branches are limited. To make the discussion of the switchings more compact it is convenient to label the allowed arcs by denoting the basic type by their number given in Sec. III B, using subscripts H and L to denote whether the system is in contact with the high- or low-temperature reservoir and using superscripts $\pm$ to denote the sign of c along the maximum-power branches. For the boundary branch the superscript labels the boundary $\beta = 0$ or $\beta = \beta_M$. For example, 2_H^+ is a maximum-power-expansion process with $T_R = T_H$.

TABLE I. Switchings.

	2	3	4
2	a or b	b	a
3	a	c	b
4	a	b	c

[a] $\Delta c = 0$ and $\psi_1 = 0$.
[b] $\Delta T_R = 0$ and $(1/\tau - \psi_1)T + \psi_2 = 0$.
[c] Forbidden switchings.

The switchings are determined by Eqs. (3.11)–(3.13) and are summarized in Table I. Certain switchings would require both c and T_R to change simultaneously which would force H to vanish, but Eq. (3.21) and the constancy of H along the optimal trajectory forbid this. For example, the end points of isotherms are determined by the fact that Eq. (3.11) implies that $(1/\tau - \psi_1)T + \psi_2$ vanishes along an isotherm. Since H is not zero, Eq. (3.2) implies that ψ_1 cannot vanish along an isotherm; consequently, T_R must remain constant at the corner which leads to the possible switchings involving isotherms listed in Table I. Analogous arguments generate the remainder of the table.

The maximum-power cycle is shown in Fig. 2 and is presented in detail in Appendix C. It may be constructed by starting at the minimum volume and considering the possible paths which are allowed. As will be discussed in Sec. IV, it is sometimes necessary to resort to numerical calculations to eliminate certain possible cycles.

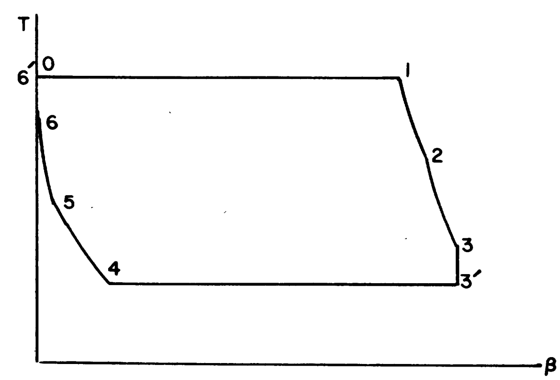

FIG. 2. T-β diagram of the maximum-power cycle. The arcs 6–6′, 0–2 correspond to the cylinder being in contact with the hottest reservoir and arcs 2–6 correspond to contact with the low-temperature reservoir. 1–3 and 4–6 are, respectively, expansion and compression maximum-power lines.

IV. MAXIMUM-POWER CYCLE

A. Maximum-power cycle description

The maximum-power cycle is composed of eight branches. In Fig. 2 we sketch the T-β diagram for the maximum-power cycle and in Appendix C we give the mathematical details of the cycle. Recall that Eq. (2.10) implies that the area under an arc in the T-β plane is proportional to the work done by the corresponding process. We will now describe the eight branches starting at 0, where the volume of the cylinder is at its minimum and the working fluid is at its maximum temperature T_h. The arcs 6-0-1-2 represent processes for which the cylinder is in contact with the high-temperature reservoir, while along the remaining arcs, 2-6, the cylinder is in contact with the low-temperature reservoir.

The 0-1 branch of the cycle represents an isothermal expansion. This is followed by two maximum-power branches. During the next interval heat is withdrawn from the working fluid until the minimum temperature T_l is reached.

The compressional part of the cycle is composed of an isothermal branch followed by two maximum power branches which terminate at point 6 when the cylinder has reached is minimum volume. Finally the branch ends with heat being added at constant volume to raise the working fluid temperature to T_h.

As explained in Appendix C the constancy of H along the trajectory gives a relation between T_h and T_l. This relation leads to a one parameter set of cycles parametrized by Δ [see Eq. (C1)]. For $\Delta = 0$, the boundary arcs reduce to points and, in general, $H \geq P/\tau$, violating Eq. (3.21). As Δ increases, H decreases, τ increases, and P first increases and then decreases. The equality is satisfied at the maximum value of P.

Once Δ is fixed, as shown in Appendix C, the durations of the maximum-power arcs are determined, which in turn fixed the switching points 2 and 5 of Fig. 2. Finally, the conditions $\beta(t_3) = \beta_M$ and $\beta(t_6) = 0$ fix the length of the isothermal branches and, consequently, yield τ.

It is actually possible to construct a cycle without isotherms. For $\beta_M \geq 0.40$ such cycles do not produce the maximum average power and in general do not expand enough for β to reach β_M. However, as β_M decreases the isotherms for the optimal cycle decrease and for sufficiently small β_M (between 0.40 and 0.39) the low-temperature isotherm vanishes. Since this corresponds to a very small compression ratio, I have not bothered to work out the details of such a cycle.

One unexpected feature of the maximum-power cycle is the existence of two constant volume arcs. This may be understood by noting that the amount of work per cycle that the engine does increases with decreasing T_l/T_h. In contrast lengthening the cycling time of the engine tends to decrease the power generated. Increasing T_h and decreasing T_l increases the period of the engine for two reasons. First, the heating or cooling along the constant volume arcs increases the period, and secondly, the isothermal processes slow down as $T_H - T_h$ and $T_l - T_L$ decrease due to the changes in T_h and T_l. The balance between the increasing work and the lengthening cycling time determines the duration of the constant volume arcs in such a way as to maximize the average-power output.

Finally we note that the replacement of adiabatic branches by maximum-power branches is simply a matter of increasing the work done at no cost in time.

B. Power generated by the maximum-power cycle

If we first introduce a dimensionless quantity $\bar{P} = P/C_V \hat{\rho}_0 T_H$, then

$$\bar{P} = (1 - T_h/T_H)^2 , \tag{4.1}$$

where T_h is the temperature along the high-temperature isotherm in the cycle. It is convenient to write

$$T_h/T_H = \tfrac{1}{2}(1+x) + x\Delta , \tag{4.2}$$

where $x = (T_L/T_H)^{1/2}$. When Δ vanishes we obtain the Curzon-Ahlborn result[1]

$$\bar{P}_{CA} = \tfrac{1}{4}(1-x)^2 . \tag{4.3}$$

As discussed in Sec. IV A the equation determining the value of Δ can only be solved numerically; however, the limiting form of Δ may be obtained in the case of large compression ratio and in the case of small ϵ [recall Eq. (2.11)]. We shall refer to the small ϵ limit as the Curzon-Ahlborn limit. We treat the latter, more physical case, first.

1. The Curzon-Ahlborn limit

In their paper Curzon and Ahlborn calculated the maximum average power output for a class of heat engines similar to those discussed here for which the cycling time was fixed. In their engines the working fluid was assumed to undergo a Carnot cycle with the duration of the adiabatic transformations assumed to be proportional to the duration of the isothermal transformations. They obtained the result given in Eq. (4.3) when the adiabatic transformations occurred in a negligible time. Their work was extended in Refs. 2–4 and it was shown more generally that in the limit that

TABLE II. The large-compression-ratio behavior of the maximum average power cycle for $\epsilon = 0.10$ and $T_L/T_H = 0.36$.

β_M	r ($\gamma = 1.4$)	$\tilde{P}$ (10^2)	η	$\hat{\rho}_0 \tau$
0.80	7.39	3.95	0.390	3.05
1.00	12.2	3.97	0.394	4.54
1.50	42.5	3.98	0.397	8.39
2.00	148.4	3.99	0.398	12.33
2.50	518.0	3.99	0.399	16.29

ϵ vanishes, Eq. (4.3) yields the maximum average power output.

In the case of fixed compression ratio, we find the same result. In particular, if $\epsilon_M = \hat{\rho}_0/c_M$ and $\epsilon_m = \hat{\rho}_0/c_m$ are both much less than one,

$$\tilde{P} = \tilde{P}_{CA}[1 - 4x/(1-x)\,\Delta_1 + O(\epsilon^2)], \tag{4.4}$$

where

$$\Delta_1 = (\epsilon_M + \epsilon_m) \times [1 - x + (1-x)\ln\tfrac{1}{2}(1+x) + x\ln x]/2x(\beta_M + \ln x).$$

For reasonable compression ratios and reservoir temperature ratios the correction term is quite small. For example, if the compression ratio is 10, $\gamma = 1.4$, $T_L/T_H = 0.36$, and $\epsilon_M = \epsilon_m = \epsilon$, the second term in the brackets in Eq. (4.4) is 0.10ϵ.

The fact that $\tilde{P}_{CA} \geq \tilde{P}$ is a mathematical consequence of relaxing the constraint of Eq. (2.6). The new maximum average power must be greater than or equal to the old $\tilde{P}$ because the space of allowed cycles has been increased in such a way that all the previously allowed cycles are included.

2. *Infinite compression ratio limit*

As claimed in Ref. 6, $\tilde{P}$ goes to $\tilde{P}_{CA}$ in the infinite-compression-ratio limit. In that paper the authors show that if one assumes a cycle composed of two isotherms and two adiabats which occur in fixed finite time then one obtains $\tilde{P}_{CA}$ in the infinite compression limit. Of course, in this limit the process is not a finite-time process. The authors of Ref. 6 further show that their cycle is optimal for large but finite compression ratio with respect to small variations around the isotherms. Since these authors restrict their considerations to a specific type of cycle the maximum power they find is smaller than that given by Eq. (C2).[19] In Table II the result of a numerical calculation of the large compression limit is presented. It is seen that the average power approaches its limiting value very slowly because of the logarithmic dependence of β_M on the compression ratio r.

V. SUMMARY

The simplest way to summarize the results of this paper is to refer to Fig. 2 and Table III. Table III contains numerical results which convey some idea of how the power generated depends on the parameters that enter the problem. The results in the Table are for $\epsilon_M = \epsilon_m = \epsilon$ and $T_L/T_H = 0.36$. If $\gamma = 1.4$ the values of β_M correspond to compression ratios of roughly 4.5, 7.4, and 12.2. Δ is defined in Appendix C, Eq. (C1) and $\tilde{P}$ is given by Eq. (C2).

In the limit that ϵ vanishes, Δ vanishes and $\tilde{P}$ becomes $\tilde{P}_{CA}$. For $T_L/T_H = 0.36$, this leads to $\tilde{P}_{CA} = 4.00 \times 10^{-2}$ and the efficiency $\eta_{CA} = 0.40$. It is evident from Table II and Sec. IV that $\tilde{P}$ approaches $\tilde{P}_{CA}$ rapidly as ϵ decreases, the working fluid temperatures along the isothermal branches of the cycle approach the values found in a CA cycle, and the efficiency increases to η_{CA}.

If ϵ is held constant at a moderate value and β_M is allowed to increase, then, as discussed in Sec. IV, $\tilde{P}$ approaches $\tilde{P}_{CA}$ slowly. This is illustrated in Table II. Because β_M depends logarithmically on the compression ratio, it is necessary to reach enormous values of the compression ratio before β_M becomes very large.

It should be pointed out that in the usual way we have found our cycle by using necessary conditions

TABLE III. Results for the maximum average power cycle.

ϵ	β_M	$\tilde{P}$ (10^2)	η	$\hat{\rho}_0 \tau$	Δ (10^3)
0.50	0.6	3.70	0.343	2.75	13.0
	0.8	3.81	0.366	4.03	8.01
	1.0	3.87	0.376	5.43	5.46
0.25	0.6	3.80	0.361	2.21	8.44
	0.8	3.89	0.378	3.51	4.61
	1.0	3.92	0.387	4.95	3.35
0.10	0.6	3.90	0.378	1.71	4.19
	0.8	3.95	0.390	3.05	2.09
	1.0	3.97	0.394	4.54	1.42
0.01	0.6	3.99	0.396	1.07	0.60
	0.8	3.99	0.399	2.54	0.23
	1.0	4.00	0.399	4.06	0.14

TABLE IV. The comparison of the rate of work done along maximum-power lines and adiabats. Positive values of ϵ correspond to expansions and negative values to compressions. The two cases correspond to a cylinder in contact with reservoir at higher and lower temperatures than the working fluid.

	ϵ	0.10	0.50	−0.10	−0.50
$T/T_R=0.8$	R	2.6×10^{-2}	0.14	-2.5×10^{-2}	-5.9×10^{-2}
$T/T_R=1.33$	R	-2.5×10^{-2}	−0.11	2.42×10^{-2}	6.61×10^{-2}

for a maximum; we have not proven that this cycle is unique.

APPENDIX A

In this Appendix it is shown that no adiabatic branch occurs in the maximum-power cycle. To begin, consider Eq. (3.10) with $c_t=c$, $\hat\rho=0$, and $\hat\rho_t>0$:

$$H^*-H_t=-\psi_1\hat\rho_t(T_{R_t}-T)\geq 0\,. \tag{A1}$$

This imples that if $\psi_1>0$, $T\geq T_H$ and if $\psi_1<0$, $T\leq T_L$, since T_{R_t} may take any value allowed by Eq. (2.3). Thus an adiabat may occur only if the working fluid temperature T lies outside the range of reservoir tmperatures $[T_L,\ T_H]$.

In Sec. III it is shown that H^* is a positive constant, so if $\hat\rho=0$

$$H^*=c[(1/\tau-\psi_1)T+\psi_2]>0\,, \tag{A2}$$

which in turn implies that $c=c_M$ or $-c_m$. Furthermore, it is impossible to switch from one case to the other since according to Eq. (3.11) this would require the vanishing of H^*. This argument and the equation of motion for T, Eq. (2.8), show that along an adiabat T is monotonic, $\dot T<0$ if $c=c_M$, and $\dot T>0$ if $c=c_m$.

It is now simple to see why there are no adiabatic branches in the optimal cycle. The only working fluid temperature at which such a branch can begin is either T_H or T_L. If the branch starts at T_L (T_H) then $c=c_M$ ($c=-c_m$), T decreases (increases) monotonically and there is no exit from the branch.

APPENDIX B

It has been shown in Ref. 15 that it is possible to derive potentials for finite-time processes. In this Appendix we shall derive the potentials for a maximum-power process.

We begin by deriving the equation for maximum-power lines in a TV diagram. We obtain a differential equation for these lines by dividing Eq. (2.8) by (2.9):

$$\frac{dT}{d\beta}=-T+\frac{\hat\rho}{c}(T_R-T)\,. \tag{B1}$$

Since $\hat\rho$, c, and T_R are all constant, we may integrate this equation:

$$\left(T-\frac{\epsilon}{1+\epsilon}T_R\right)V^{(\gamma-1)(1+\epsilon)}=\text{constant}\,, \tag{B2}$$

where we have used Eq. (2.5) to replace β by V and we have introduced $\epsilon=\hat\rho/c$. If $\hat\rho=0$ this reduces to the equation for an adiabat.

Since for an isentropic transformation $(dT/d\beta)_s=-T$, Eq. (B1) states that for an expansion, $c>0$, the slope of the maximum-power line is greater (less) than an adiabatic line at the same point if $T_R>T$ ($T_R<T$). For a contraction the opposite is true.

The work done along a maximum-power branch may be written as the change in a finite-time thermodynamic potential.[15] This may be seen by integrating

$$W_{A\to B}=C_V\int_A^B cT\,dt=C_V\int_A^B T\,d\beta\,.$$

Using Eq. (B1) we find $W_{A\to B}=\mathcal{P}_B-\mathcal{P}_A$ where

$$\mathcal{P}=-C_V\left(T+\frac{\epsilon}{1+\epsilon}T_R\ln\left|\frac{T}{T_R}-\frac{\epsilon}{1+\epsilon}\right|\right)\Big/(1+\epsilon) \tag{B3}$$

We may of course add an arbitrary constant to $\mathcal{P}$. If $\hat\rho=0$, $\mathcal{P}=\mathcal{P}_S=-C_VT=-U$ which is correct for an isentropic transformation. In order to get an idea of how the work for a maximum-power transformation compares with the work along an adiabat we consider

$$R=\frac{d\mathcal{P}}{d\mathcal{P}_S}-1=\frac{\epsilon}{1+\epsilon}\left(1-\frac{T}{T_R}\right)\Big/\left(\frac{T}{T_R}-\frac{\epsilon}{1+\epsilon}\right).$$

In Table IV we give some sample values of R. We see that R is very small unless the ratio of relaxation times, ϵ, is large in the sense of being of order one.

It is now a simple matter to write the heat flow into a fluid as a change in the potential $\mathcal{P}^*=\mathcal{P}+U$ and from this to obtain the change in the entropy of the fluid by integrating $dS=d\mathcal{P}^*/T$. We find $\Delta S_{A\to B}=\sigma_B-\sigma_A$ where

$$\sigma=-C_V\left(\frac{\epsilon}{1+\epsilon}\ln\frac{T}{T_R}-\frac{1}{1+\epsilon}\ln\left|1-\frac{\epsilon}{1+\epsilon}\frac{T_R}{T}\right|\right).$$

As expected σ vanishes if $\hat\rho=0$. Finally, for completeness we note that the total increase in entropy for a maximum-power transformation may be written as $\Delta S_{\text{tot}}=\Delta(\sigma-\mathcal{P}^*/T_R)$.

APPENDIX C

In this Appendix we present the detailed form of the maximum-power cycle starting from the minimum volume and the maximum internal temperature, point 0 of Fig. 2. It is convenient to introduce the variables:

$$x=(T_L/T_H)^{1/2}\,,\ \ y_h=T_h/T_H=\tfrac{1}{2}(1+x)+x\Delta\,,\ \ y_l=T_l/T_L=\tfrac{1}{2}(1+1/x)-\Delta\,,$$
$$\alpha_+=c_M+\hat{\rho}_0\,,\ \ \alpha_-=c_m-\hat{\rho}_0\,, \tag{C1}$$
$$\epsilon_M=\hat{\rho}_0/c_M\,,\ \ \epsilon_m=\hat{\rho}_0/c_m\,,$$
$$Z_M=\epsilon_M/1+\epsilon_M\,,\ \ z_m=\epsilon_m/1+\epsilon_m\,.$$

In the following, all functions will be expressed in terms of the quantities given in Eq. (C1). All these quantities are given except Δ. The fact that T_h and T_l depend on Δ in the way given above follows from the constancy of the Hamiltonian along the optimal trajectory. This result applied to the two isotherms implies that $T_H(1-y_h)^2=T_L(y_l-1)^2$. It is then convenient to parametrize y_l in the manner shown in Eq. (C1) and the form of y_h follows.

All the other quantities given below follow from the solutions to the equations of motion and the continuity of the state and adjoint variables at corners. In particular, the continuity of ψ_2 at points 3′ and 6′ of Fig. 2 are used to solve for the rather complicated expressions for $\psi_1(t_3)$ and $\psi_1(t_6)$ given below. The time subscripts correspond to the labels of Fig. 2, where for convenience I have set $t_0=0$.

The control variable T_R is given by

$$T_R=\begin{cases}T_H & 0\le t\le t_2\,,\ \ t_6\le t\le t_6'\,,\\ T_L & t_2\le t\le t_6\,.\end{cases}$$

$\hat{\rho}$ equals $\hat{\rho}_0$ for the entire cycle, and the control variable c is given by

$$c=\begin{cases}c_h=\hat{\rho}_0(1/y_h-1)\,, & 0\le t\le t_1\\ c_M\,, & t_1\le t\le t_3\\ 0\,, & t_3\le t\le t_3'\,,\ \ t_6\le t\le t_6'\\ c_l=-\hat{\rho}_0(1-1/y_l)\,, & t_3'\le t\le t_4\\ -c_m\,, & t_4\le t\le t_6'\,.\end{cases}$$

The state and adjoint variables are given by

$$0\le t\le t_1\!:\ \ T=T_h\,,\ \ \beta=c_h t\,,\ \ \psi_1=(1-y_h)/\tau\,,\ \ \psi_2=-y_h^2T_H/\tau\,.$$
$$t_1\le t\le t_2\!:\ \ T/T_H=z_M+(y_h-z_M)e^{-\alpha_+(t-t_1)}\,,\ \ \beta=c_h t_1+c_M(t-t_1)\,,$$
$$\psi_1=[z_M/\epsilon_M+(z_M-y_h)e^{\alpha_+(t-t_1)}]/\tau\,,\ \ \psi_2=-y_h^2T_H/\tau\,.$$
$$t_2\le t\le t_3\!:\ \ T/T_L=z_M+(y_2-z_M)e^{-\alpha_+(t-t_2)}\,,\ \ \beta=c_h t_1+c_M(t-t_1)\,,$$
$$\psi_1=-(e^{\alpha_+(t-t_2)}-1)/\tau(1+\epsilon_M)\,,\ \ \psi_2=-y_h^2T_H/\tau\,,$$

where

$$y_2=T(t_2)/T_H=z_M+(y_h-z_M)^2(1+\epsilon_M)\,.$$
$$t_3\le t\le t_3'\!:\ \ T/T_L=1+(y_3-1)e^{-\hat{\rho}_0(t-t_3)}\,,\ \ \beta=\beta_M\,,\ \ \psi_1=\psi_1(t_3)e^{\hat{\rho}_0(t-t_3)}\,,$$
$$\psi_2=-(1/\tau-\psi_1)T\,,$$

where

$$y_3=T(t_3)/T_L=1-(y_l-1)^2/\tau\psi_1(t_3)\,,$$
$$\psi_1(t_3)=-\{1/x-1+2\Delta/x-[4\Delta(1+\Delta)(1/x^2-1)]^{1/2}\}/2\tau\,.$$
$$t_3'\le t\le t_4\!:\ \ T=T_l\,,\ \ \beta=\beta_M+c_l(t-t_3')\,,\ \ \psi_1=-(y_l-1)/\tau\,,\ \ \psi_2=-y_l^2T_L/\tau\,.$$
$$t_4\le t\le t_5\!:\ \ T/T_L=-z_m+(y_l+z_m)e^{\alpha_-(t-t_4)}\,,\ \ \beta=\beta_M+c_l(t_4-t_3')-c_m(t-t_4)\,,$$
$$\psi_1=[z_m/\epsilon_m-(y_l+z_m)e^{-\alpha_-(t-t_4)}]/\tau\,,\ \ \psi_2=-y_l^2T_L/\tau\,.$$
$$t_5\le t\le t_6\!:\ \ T/T_H=-z_m+(y_5+z_m)e^{\alpha_-(t-t_5)}\,,\ \ \beta=\beta_M+c_l(t_4-t_3')-c_m(t-t_4)\,,$$
$$\psi_1=(1-e^{-\alpha_-(t-t_5)})/\tau(1-\epsilon_m)\,,\ \ \psi_2=-y_l^2T_L/\tau\,,$$

where

$$y_5 = T(t_5)/T_L = -z_m + (y_l + z_m)^2(1-\epsilon_m) .$$

$$t_6 \le t \le t_6' = \tau: \quad T/T_H = 1-(1-y_6)e^{-\hat{\rho}_0(t-t_6)}, \quad \beta = 0,$$

$$\psi_1 = \psi_1(t_6)e^{\hat{\rho}_0(t-t_6)}, \quad \psi_2 = -(1/\tau - \psi_1)T ,$$

where

$$y_6 = T(t_6)/T_H = 1-(y_h-1)^2/\tau\psi_1(t_6)$$

and

$$\psi_1(t_6) = \{1 - x + 2\Delta x^2 - [4\Delta x(1-x^2)(1-\Delta x)]^{1/2}\}/2\tau .$$

The duration of each arc may also be expressed in terms of Δ. $t_2 - t_1$ and $t_5 - t_4$ are determined by the continuity of ψ_1 at t_1 and t_4, respectively:

$$t_2 - t_1 = -(1/\alpha_+)\ln[y_h - \epsilon_M(1-y_h)],$$

$$t_5 - t_4 = (1/\alpha_-)\ln[y_l - \epsilon_m(y_l - 1)] .$$

$t_3 - t_2$ and $t_6 - t_5$ are determined by the continuity of ψ_1 at t_3 and t_6, respectively:

$$t_3 - t_2 = (1/\alpha_+)\ln[1-\psi_1(t_3)\tau(1+\epsilon_M)],$$

$$t_6 - t_5 = -(1/\alpha_-)\ln[1-\psi_1(t_6)\tau(1-\epsilon_m)] .$$

$t_3' - t_3$ and $t_6' - t_6$ are determined by the continuity of ψ_1 at t_3' and t_6', respectively:

$$t_3' - t_3 = \frac{1}{\hat{\rho}_0}\frac{\ln(y_l - 1)}{[-\tau\psi_1(t_3)]},$$

$$t_6' - t_6 = \frac{1}{\hat{\rho}_0}\frac{\ln(1-y_h)}{\tau\psi_1(t_6)} .$$

Finally, t_1 and $t_4 - t_3'$ are determined by $\beta(t_3) = \beta_M$ and $\beta(t_6) = 0$, respectively.

The last step in the complete determination of the solution is to determine the value of Δ that maximizes P. This is done by using Eq. (3.21). To do this P must be expressed in terms of Δ since H and τ are readily evaluated in terms of Δ from the foregoing. By direct evaluation Eq. (2.10), it is straightforward to show that $\tilde{P} \equiv P/C_V\hat{\rho}_0 T_H$ is given by

$$\tilde{P} = \{(1-y_h)[t_1 - x(t_4 - t_3')] + [(t_2 - t_1) + x^2(t_3 - t_2)]/(1+\epsilon_M) + [(t_6 - t_5) + x^2(t_5 - t_4)]/(1-\epsilon_m) + (y_h - x^2 y_3)/(1+\epsilon_M) - (y_6 - x^2 y_l)/(1-\epsilon_m)\}/\tau, \quad \text{(C2)}$$

where y_3, y_6, and the time intervals are given above. For completeness, we have

$$H = C_V\rho_0 T_H(1-y_h)^2/\tau .$$

Equation (3.21) was solved numerically to generate Table III.

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[12]It is possible to avoid discontinuous changes in control variables; however, we have not done so and follow the classical thermodynamic procedure of assuming controls can be changed, instantaneously.

[13]There are slight differences between the formula in

Ref. 4 and those which must be used here which arise from the $1/\tau$ in Eq. (2.10).

[14]In Ref. 4 we found that no adiabatic branches occur for the maximum-power cycle for fixed cycling time but that they do occur in the cycle that maximizes the efficiency.

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[17]Since the Hamiltonian does not depend explicitly on time, the equations of motion for the state and the adjoint variables imply that the Hamiltonian is constant along any arc.

[18]Since $\Delta H \geq 0$ at each corner, the periodicity implies that the quality sign must hold.

[19]The upper bound obtained in Eq. (4.10) of Ref. 6 exceeds the Curzon-Ahlborn upper bound for the process. I do not believe that using inequality methods such as those used by the authors are sufficiently powerful to obtain useful upper bounds to finite-time processes unless the cycles are first prescribed in terms of a few parameters such as is done in Refs. 2, 3, and 5.