Hyperfine structure in muonic helium

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A calculation of the ground-state hyperfine frequency in muonic helium is given. Perturbation theory is applied to the Schrödinger equation to obtain a series in m_e/m_μ. The value obtained is 4462.6 ± 3 MHz, which is consistent with experiments at the Swiss Institute for Nuclear Research and the Los Alamos Meson Physics Facility. The theoretical value includes a -46-MHz contribution from excited states of the effective $(\alpha\mu^-)^+$ nucleus.

I. INTRODUCTION

Muonic helium, an atom consisting of an α particle, a negative muon, and an electron, was first formed and detected by Souder *et al.* in 1975.[1] Recent experiments at the Swiss Institute for Nuclear Research SIN[2] and the Los Alamos Meson Physics Facility LAMPF[3] have measured the ground-state hyperfine splitting in this system. Various theoretical studies of muonic helium have been made.[4-8] In this paper, we apply perturbation theory in a simple nonrelativistic calculation of the hyperfine frequency. This approach can readily be generalized to a relativistic calculation, and may eventually give a precise value for the frequency. Our calculation is similar to perturbation theory studies of hyperfine structure made by Low for deuterium,[9] and by Drell and Sullivan for hydrogen.[10]

Because the α particle and the muon are much more massive than the electron, muonic helium may be regarded as a one-electron atom with an effective nucleus consisting of an $(\alpha\mu^-)^+$ atom. The hyperfine splitting is due to the spin-spin interaction of the muon and electron. In this picture, corrections to hyperfine structure associated with the size of the effective nucleus are of relative order (nuclear size)/(Bohr radius), analogous to the well-known nuclear size corrections in deuterium,[11,12] in hydrogen,[13-15] and in heavy atoms.[16-18] However, in muonic helium there is a larger correction from excited states of the effective nucleus, which is not surprising since the $(\alpha\mu^-)^+$ system is weakly bound.

II. CALCULATION

The structure of muonic helium is described, to a good approximation, by the nonrelativistic Schrödinger equation (units in which $c=\hbar=1$ are employed here)

$$\left(-\frac{1}{2M_\mu}\nabla_\mu^2-\frac{1}{2M_e}\nabla_e^2-\frac{2\alpha}{x_\mu}-\frac{2\alpha}{x_e}+\frac{\alpha}{x_{\mu e}}\right)\psi(\vec{x}_\mu,\vec{x}_e)=E\psi(\vec{x}_\mu,\vec{x}_e)\,, \quad (1)$$

where $\vec{x}_\mu$ and $\vec{x}_e$ are the position vectors of the muon and electron relative to the α particle, and $M_\mu=m_\mu m_\alpha/(m_\mu+m_\alpha)$ and $M_e=m_e m_\alpha/(m_e+m_\alpha)$ are the reduced masses of the muon and electron with respect to the α particle. The mass polarization term $(-1/m_\alpha)\vec{\nabla}_\mu\cdot\vec{\nabla}_e$ is negligible to the accuracy considered here. The hyperfine shift of the ground state is given by the expectation value of the contact term of the spin-spin interaction

$$\delta H=-\tfrac{8}{3}\pi\vec{\mu}_\mu\cdot\vec{\mu}_e\delta(\vec{x}_\mu-\vec{x}_e) \quad (2)$$

obtained in the nonrelativistic reduction of the Breit equation.[19] In (2), $\vec{\mu}_\mu=-g_\mu(m_e/m_\mu)\mu_0\vec{s}_\mu$ and $\vec{\mu}_e=-g_e\mu_0\vec{s}_e$ are the magnetic moment vectors of the muon and electron. In view of the factorization of the nonrelativistic wave function into coordinate space and spin parts, the difference between the hyperfine shifts of the ground-state levels with total angular momentum 1 and 0 is proportional to the difference $\langle\vec{s}_\mu\cdot\vec{s}_e\rangle_1-\langle\vec{s}_\mu\cdot\vec{s}_e\rangle_0=1$. Hence, we have

$$\Delta\nu=\tfrac{8}{3}\pi g_\mu g_e(m_e/m_\mu)\mu_0^2\langle\delta(\vec{x}_\mu-\vec{x}_e)\rangle \approx\tfrac{8}{3}\pi(\alpha/m_\mu m_e)\langle\delta(\vec{x}_\mu-\vec{x}_e)\rangle \quad (3)$$

for the magnitude of the hyperfine splitting, where the expectation value is evaluated with just the coordinate space portion of the wave function which is the lowest energy eigenfunction for Eq. (1).

To evaluate the expectation value in (3), we apply perturbation theory to the ground-state wave function. The "pseudonucleus" picture suggests a natural division of the Hamiltonian into a zero-order part and a perturbation, $H=H_0+\delta V$, in which

$$H_0=-\frac{1}{2M_\mu}\nabla_\mu^2-\frac{1}{2M_e}\nabla_e^2-\frac{2\alpha}{x_\mu}-\frac{\alpha}{x_e} \quad (4)$$

and

$$\delta V(\vec{x}_\mu,\vec{x}_e)=\frac{\alpha}{x_{\mu e}}-\frac{\alpha}{x_e}. \quad (5)$$

The zero-order wave function for the ground state is the product of normalized 1s hydrogenic wave functions

$$\psi_0(\vec{x}_\mu, \vec{x}_e) = \psi_{\mu 0}(\vec{x}_\mu)\psi_{e0}(\vec{x}_e) = (1/\pi)(2\alpha^2 M_\mu M_e)^{3/2} e^{-2\alpha M_\mu x_\mu} e^{-\alpha M_e x_e}, \tag{6}$$

which has the sum of the hydrogenic 1s state energies as its energy $E_0 = E_{\mu 0} + E_{e0}$. This wave function gives the zero-order hyperfine splitting

$$\Delta\nu_0 = \tfrac{8}{3}\pi \frac{\alpha}{m_\mu m_e} \int d\vec{x}_\mu \int d\vec{x}_e \psi_0^\dagger(\vec{x}_\mu, \vec{x}_e)\delta(\vec{x}_\mu - \vec{x}_e)\psi_0(\vec{x}_\mu, \vec{x}_e)$$
$$= \tfrac{8}{3} \frac{\alpha}{m_\mu m_e} (\alpha M_e)^3 (1 + M_e/2M_\mu)^{-3}$$
$$= \Delta\nu_F (1 + M_e/2M_\mu)^{-3}, \tag{7}$$

where $\Delta\nu_F$ is the Fermi value.

The first-order correction $\Delta\nu_1$ to the hyperfine splitting, due to the first-order correction to the wave function, is

$$\Delta\nu_1 = 2\tfrac{8}{3}\pi \frac{\alpha}{m_\mu m_e} \int d\vec{x}_\mu \int d\vec{x}_e \psi_0^\dagger(\vec{x}_\mu, \vec{x}_e)\delta(\vec{x}_\mu - \vec{x}_e)\psi_1(\vec{x}_\mu, \vec{x}_e)$$
$$= \tfrac{16}{3}\pi \frac{\alpha}{m_\mu m_e} \int d\vec{x}_3 \int d\vec{x}_2 \int d\vec{x}_1 \psi_{\mu 0}^\dagger(\vec{x}_3)\psi_{e0}^\dagger(\vec{x}_3)$$
$$\times \sum_{n, n' \neq 0, 0} \frac{\psi_{\mu n}(\vec{x}_3)\psi_{en'}(\vec{x}_3)\psi_{\mu n}^\dagger(\vec{x}_2)\psi_{en'}^\dagger(\vec{x}_1)}{E_{\mu 0} + E_{e0} - E_{\mu n} - E_{en'}} \delta V(\vec{x}_2, \vec{x}_1)\psi_{\mu 0}(\vec{x}_2)\psi_{e0}(\vec{x}_1). \tag{8}$$

It is convenient to divide the sum over muon states in (8) into two parts $\Delta\nu_1 = \Delta\nu_1^g + \Delta\nu_1^e$, where $\Delta\nu_1^g$ is the contribution to $\Delta\nu_1$ from the term with $n = 0$, i.e., where the intermediate muon state is the 1s state. For this part, we have

$$\Delta\nu_1^g = \tfrac{16}{3}\pi \frac{\alpha}{m_\mu m_e} \int d\vec{x}_2 \int d\vec{x}_1 \psi_{e0}^\dagger(\vec{x}_2) |\psi_{\mu 0}(\vec{x}_2)|^2 \sum_{n \neq 0} \frac{\psi_{en}(\vec{x}_2)\psi_{en}^\dagger(\vec{x}_1)}{E_{e0} - E_{en}} V_\mu(x_1)\psi_{e0}(\vec{x}_1) \tag{9}$$

with

$$V_\mu(x) = \int d\vec{x}_\mu \psi_{\mu 0}^\dagger(\vec{x}_\mu)\delta V(\vec{x}_\mu, \vec{x})\psi_{\mu 0}(\vec{x}_\mu) = -(\alpha/x)(1 + 2\alpha M_\mu x)e^{-4\alpha M_\mu x}. \tag{10}$$

Only s states contribute to the sum over n in (9), so we may replace the sum by the s state reduced Green's function for the electron[20]

$$\sum_{n \neq 0} \frac{\psi_{ens}(\vec{x}_2)\psi_{ens}^\dagger(\vec{x}_1)}{E_{e0} - E_{ens}} = -\frac{\alpha M_e^2}{\pi} e^{-\alpha M_e(x_1 + x_2)} \left(\frac{1}{2\alpha M_e x_>} - \ln(2\alpha M_e x_>) + \tfrac{5}{2} - \gamma - \alpha M_e(x_1 + x_2) + \sum_{i=1}^{\infty} \frac{(2\alpha M_e x_<)^i}{i(i+1)!} \right), \tag{11}$$

where $x_> = \max(x_1, x_2)$, $x_< = \min(x_1, x_2)$, and $\gamma = 0.5772\ldots$ is Euler's constant. Evaluation of Eq. (9), with the aid of (10) and (11), yields

$$\Delta\nu_1^g = \Delta\nu_F [\tfrac{11}{16} M_e/M_\mu + (M_e/M_\mu)^2 \ln(M_\mu/M_e) - \tfrac{7}{64}(M_e/M_\mu)^2 + O(M_e/M_\mu)^3 \ln(M_\mu/M_e)]. \tag{12}$$

The part $\Delta\nu_1^e$ corresponding to excited muon intermediate states may be written as

$$\Delta\nu_1^e = -\tfrac{16}{3}\pi \frac{\alpha}{m_\mu m_e} \int d\vec{x}_3 \int d\vec{x}_2 \int d\vec{x}_1 \psi_{\mu 0}^\dagger(\vec{x}_3)\psi_{e0}^\dagger(\vec{x}_3)$$
$$\times \sum_{n \neq 0} \psi_{\mu n}(\vec{x}_3)\psi_{\mu n}^\dagger(\vec{x}_2) G_e(\vec{x}_3, \vec{x}_1, E_{\mu 0} + E_{e0} - E_{\mu n}) \frac{\alpha}{|\vec{x}_2 - \vec{x}_1|} \psi_{\mu 0}(\vec{x}_2)\psi_{e0}(\vec{x}_1), \tag{13}$$

where

$$G_e(\vec{x}_3, \vec{x}_1, z) = \sum_n \frac{\psi_{en}(\vec{x}_3)\psi_{en}^\dagger(\vec{x}_1)}{E_{en} - z} \tag{14}$$

is the electron Coulomb Green's function. In (13), there is no contribution from the term $-\alpha/x_1$ in $\delta V(\vec{x}_2, \vec{x}_1)$ due to the orthogonality of $\psi_{\mu n}(\vec{x}_2)$ and $\psi_{\mu 0}(\vec{x}_2)$ for $n \neq 0$. To evaluate (13), we make two approximations. First, we replace the electron Coulomb Green's function G_e by the free electron Green's function

$$G_e^0(\vec{x}_3, \vec{x}_1, E_{\mu 0} + E_{e0} - E_{\mu n}) = \frac{M_e}{2\pi} \frac{e^{-b|\vec{x}_3 - \vec{x}_1|}}{|\vec{x}_3 - \vec{x}_1|} \tag{15}$$

in which $b=[2M_e(E_{\mu n}-E_{\mu 0}-E_{e0})]^{1/2}$, $b>0$. Second, we replace the electron wave functions by their value at the origin $\psi_{e0}(0)$. Both replacements correctly reproduce the behavior of the exact electron functions near the origin. An examination of the corrections to these approximations indicates that they are of order $\Delta\nu_F(M_e/M_\mu)^2\ln(M_\mu/M_e)$ or smaller. Preliminary results of a numerical evaluation of $\Delta\nu_1^e$ indicate that the corrections are numerically small. We thus have

$$\Delta\nu_1^e \approx -\Delta\nu_F\frac{\alpha M_e}{\pi}\int d\vec{x}_3\int d\vec{x}_2\int d\vec{x}_1\psi^\dagger_{\mu 0}(\vec{x}_3)\sum_{n\neq 0}\psi_{\mu n}(\vec{x}_3)\psi^\dagger_{\mu n}(\vec{x}_2)\frac{e^{-b|\vec{x}_3-\vec{x}_1|}}{|\vec{x}_3-\vec{x}_1|}\frac{1}{|\vec{x}_2-\vec{x}_1|}\psi_{\mu 0}(\vec{x}_2). \tag{16}$$

Integration over $\vec{x}_1$ yields

$$\int d\vec{x}_1\frac{e^{-b|\vec{x}_3-\vec{x}_1|}}{|\vec{x}_3-\vec{x}_1|}\frac{1}{|\vec{x}_2-\vec{x}_1|}=\frac{4\pi}{b^2}\frac{1}{|\vec{x}_3-\vec{x}_2|}(1-e^{-b|\vec{x}_3-\vec{x}_2|})$$

$$=4\pi(1/b-\tfrac{1}{2}|\vec{x}_3-\vec{x}_2|+\tfrac{1}{6}b|\vec{x}_3-\vec{x}_2|^2-\tfrac{1}{24}b^2|\vec{x}_3-\vec{x}_2|^3+\cdots). \tag{17}$$

In view of the exponential falloff of the muon wave functions, the main contribution to (16) in the integration over x_2 and x_3 comes from the region in which x_2 and x_3 are of order $1/M_\mu\alpha$. The order of magnitudes $|\vec{x}_3-\vec{x}_2|\sim 1/M_\mu\alpha$ and $b\sim(M_eM_\mu)^{1/2}\alpha$, suggest that the series in (17) gives a series in increasing powers of $(M_e/M_\mu)^{1/2}$ for $\Delta\nu_1^e$. The leading term b^{-1} gives no contribution because of the orthogonality of the muon wave functions. In view of the completeness of the muon wave functions, we have

$$\sum_{n\neq 0}\psi_{\mu n}(\vec{x}_3)\psi^\dagger_{\mu n}(\vec{x}_2)=\delta(\vec{x}_3-\vec{x}_2)-\psi_{\mu 0}(\vec{x}_3)\psi^\dagger_{\mu 0}(\vec{x}_2),$$

so the second term in (17) yields

$$-\Delta\nu_F 2\alpha M_e\int d\vec{x}_3\int d\vec{x}_2|\psi_{\mu 0}(\vec{x}_3)|^2|\vec{x}_3-\vec{x}_2||\psi_{\mu 0}(\vec{x}_2)|^2=-\Delta\nu_F\tfrac{35}{16}M_e/M_\mu \tag{18}$$

in (16). In the third term in the series in (17), we neglect E_{e0} compared to $E_{\mu n}-E_{\mu 0}$ in b, and again because of orthogonality of the wave functions in (16), we may replace $|\vec{x}_3-\vec{x}_2|^2$ by $-2\vec{x}_3\cdot\vec{x}_2$. Hence, in (16) this term contributes

$$\Delta\nu_F\tfrac{4}{3}\alpha M_e\int d\vec{x}_3\int d\vec{x}_2\psi^\dagger_{\mu 0}(\vec{x}_3)\sum_n[2M_e(E_{\mu n}-E_{\mu 0})]^{1/2}\vec{x}_3\psi_{\mu n}(\vec{x}_3)\cdot\psi^\dagger_{\mu n}(\vec{x}_2)\vec{x}_2\psi_{\mu 0}(\vec{x}_2)=\Delta\nu_F\tfrac{2}{3}(M_e/M_\mu)^{3/2}S_{1/2}. \tag{19}$$

where we define

$$S_p=\sum_n\left(\frac{E_{\mu n}-E_{\mu 0}}{R_\mu}\right)^p\left|\left\langle\mu 0\left|\frac{\vec{x}}{a_\mu}\right|\mu n\right\rangle\right|^2 \tag{20}$$

with $R_\mu=2\alpha^2M_\mu$, $a_\mu=1/2\alpha M_\mu$, the effective Rydberg and Bohr radius for the muon. For a simple estimate of $S_{1/2}$, we note that the standard sum rules[19] $S_0=S_1=3$, together with $S_{1/2}\leq\frac{1}{2}(S_0+S_1)$, give $S_{1/2}\leq 3$. A lower bound on $S_{1/2}$ is given by

$$S_{1/2}\geq\min_{n\neq 0}\left[\left(\frac{E_{\mu n}-E_{\mu 0}}{R_\mu}\right)^{1/2}\right]\sum_{m\neq 0}\left|\left\langle\mu 0\left|\frac{\vec{x}}{a_\mu}\right|\mu m\right\rangle\right|^2$$

$$=3(\tfrac{3}{4})^{1/2}. \tag{21}$$

Hence, $S_{1/2}=2.8\pm 0.2$. The leading contribution to $\Delta\nu_1^e$ from the fourth term in (17), nominally of order $\Delta\nu_F(M_e/M_\mu)^2$, vanishes.

III. CONCLUSION

The leading contributions in the nonrelativistic formulation are thus

$$\Delta\nu=\Delta\nu_F[1-3M_e/M_\mu+\tfrac{2}{3}S_{1/2}(M_e/M_\mu)^{3/2}] \tag{22}$$

from (7), (12), (18), and (19). The Fermi value is $\Delta\nu_F=4516.9$ MHz based on the constants $R_\infty=3.289\,842\times 10^9$ MHz, $\alpha^{-1}=137.0360$, $m_\mu/m_e=206.7686$, and $m_\alpha/m_e=7294$. Evaluation of Eq. (22) yields $\Delta\nu=4452.5$ MHz. The Fermi term $\Delta\nu_F$ corresponds to the hyperfine interaction of the electron with a point muon charge and magnetic-dipole moment at the origin. The corrections to $\Delta\nu_F$ from $\Delta\nu_0$ and $\Delta\nu_1^f$, which account for the distributed charge and magnetic moment of the muon in the ground state, including distortion of the electron wave function, amount to -18 MHz. The remaining correction, from excited muon states, is $\Delta\nu_1^e=-46$ MHz. The main correction to the nonrelativistic result is due to the lowest order anomalous magnetic moments of the electron and muon, which when included in g_e and g_μ shift the hyperfine frequency by $\Delta\nu_F(\alpha/\pi)=10.5$ MHz. Higher-order self-energy and vacuum-polarization corrections are roughly approximated by the hydrogenic value[21] $\Delta\nu_F(\ln 2-\frac{5}{2})\alpha^2=-0.4$ MHz. Our estimate of the uncertainty in the value for the hyperfine splitting due to uncalculated terms, including terms of order $\Delta\nu_F(M_e/M_\mu)^2\times\ln(M_\mu/M_e)$ in the nonrelativistic part, is 3 MHz. The total result is $\Delta\nu=4462.6\pm 3$ MHz.

Our result for the contribution from the static distribution of charge and magnetization of the muon is consistent with earlier work of Otten,[5] which is based on the Bohr-Weisskopf formulation. Our total result, including excited muon states, is in agreement with the result of the experiment at SIN which measures the hyperfine splitting in a weak magnetic field: $\Delta\nu = 4464.95(6)$ MHz.[2] A preliminary result of the strong field measurement at LAMPF is $\Delta\nu = 4464.99(4)$ MHz,[3] which agrees with the above values. Both experimental values assume a hydrogenic pressure shift. Huang and Hughes have applied a variational calculation to the nonrelativistic expression for the hyperfine splitting and obtain for the total splitting $\Delta\nu = 4465.1(1.0)$ MHz,[8] in agreement with experiment. Borie[7] and Huang and Hughes[8] have calculated additional relativistic, radiative, and recoil corrections to the hyperfine structure; however, the values they obtain are smaller than the current uncertainty in the nonrelativistic value for the splitting. A recent calculation by Drachman confirms the leading correction in Eq. (22) by an independent method.[22]

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[1] P. A. Souder, D. E. Casperson, T. W. Crane, V. W. Hughes, D. C. Lu, H. Orth, H. W. Reist, M. H. Yam, and G. zu Putlitz, Phys. Rev. Lett. 34, 1417 (1975).

[2] K. P. Arnold, P. O. Egan, M. Gladisch, W. Jacobs, H. Orth, G. zu Putlitz, W. Wahl, M. Wigand, V. W. Hughes, and J. Vetter, in Abstracts of the Eighth International Conference on High-Energy Physics and Nuclear Structure, Vancouver, 1979 (unpublished); and private communication.

[3] C. K. Gardner, W. Beer, P. Bolton, B. Dichter, P. O. Egan, V. W. Hughes, D. C. Lu, F. G. Mariam, P. A. Souder, H. Orth, G. zu Putlitz, and J. Vetter, Bull. Am. Phys. Soc. 25, 19 (1980).

[4] V. W. Hughes and S. Penman, Bull. Am. Phys. Soc. 4, 80 (1959).

[5] E. W. Otten, Z. Phys. 225, 393 (1969).

[6] K.-N. Huang, Ph.D. thesis, Yale University, 1974 (unpublished).

[7] E. Borie, Z. Phys. A 291, 107 (1979).

[8] K.-N. Huang and V. W. Hughes, Phys. Rev. A 20, 706 (1979); 21, 1071 (1980).

[9] F. Low, Phys. Rev. 77, 361 (1950).

[10] S. D. Drell and J. D. Sullivan, Phys. Rev. 154, 1477 (1967).

[11] J. E. Nafe, E. B. Nelson, and I. I. Rabi, Phys. Rev. 71, 914 (1947).

[12] A. Bohr, Phys. Rev. 73, 1109 (1948).

[13] G. E. Brown and G. B. Arfken, Phys. Rev. 76, 1305 (1949).

[14] A. C. Zemach, Phys. Rev. 104, 1771 (1956).

[15] H. Grotch and D. R. Yennie, Rev. Mod. Phys. 41, 350 (1969).

[16] E. Rosenthal and G. Breit, Phys. Rev. 41, 459 (1932).

[17] M. F. Crawford and A. L. Schawlow, Phys. Rev. 76, 1310 (1949).

[18] A. Bohr and V. F. Weisskopf, Phys. Rev. 77, 94 (1950).

[19] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer, Berlin, 1957).

[20] H. F. Hameka, J. Chem. Phys. 47, 2728 (1967); 48, 4810 (1968).

[21] For a review of the analogous effects in hydrogen, see B. E. Lautrup, A. Peterman, and E. de Rafael, Phys. Rep. 3, 193 (1972).

[22] R. J. Drachman, Phys. Rev. A 22, 1751 (1980).