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Instability of large-amplitude electromagnetic waves in plasmas

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Electromagnetic sawtooth waves, supposedly generated in pulsar environments, have been the subject of numerous publications. These waves are of such magnitude as to drive all plasma particles with relativistic velocities. We show analytically that such waves are subject to fast instabilities that destroy them in a time comparable to the oscillation period. They are stabilized in the presence of a strong magnetic field.

Electromagnetic waves in unmagnetized plasmas are known to be subject to parametric instabilities which cause backscattering or filamentation of these waves.[1] Here we investigate waves at sufficiently large amplitude as to drive the particles with relativistic velocities.[2,3] A very simple calculation shows that these waves have growth rates of the order of the oscillation period. Such waves are believed to exist in pulsar environments,[4] and possibly in future laser-plasma fusion experiments.

We consider the perpendicular propagation of an electromagnetic plane wave in a magnetized plasma with some phase velocity $v_{\text{ph}} > c$. It is useful to study such a wave in a Lorentz frame moving with velocity c^2/v_{ph}, where spatial dependence, as well as the wave magnetic field, has been transformed away. For linearly polarized strong waves, a solution exists which describes an electric field with a sawtooth-shaped time dependence and relativistic ions and electrons with velocities resembling a square wave.[2,3,5] The electric field and the velocities are parallel to the magnetic field. The characteristic frequency of the sawtooth wave is $\omega_w = \pi\omega_{pe}/2\gamma_{e\,\max}^{1/2}$ (ω_{pe} and γ_e are defined below). It was assumed that the plasma velocity is zero in the absence of the wave in this Lorentz frame, a condition satisfied for the large-amplitude sawtooth wave.[5]

We consider now a Weibel-type electromagnetic instability,[6] with a $\vec{k}$ vector defining the x direction perpendicular to the electron and ion velocities (respectively $\vec{V}^-$ and $\vec{V}^+$) in the direction of the ambient magnetic field $\vec{B}_0 = B_0\vec{1}_y$. This is driven by current pinching arising from the mutual attraction of parallel currents.

For each particle species we write the velocity as $\vec{u} = \vec{V} + \vec{v}$ where $\vec{v}$ is a small linear perturbation. The electric field $E_t = E_w + E$, E_w is the wave field and E is the perturbed electric field; γ is the relativistic Lorentz factor $\gamma = (1 - V^2/c^2)^{-1/2}$. Each small quantity is Fourier analyzed with $\exp[i(\omega t + \vec{k}\cdot\vec{x})]$. From the equation of motion

$$\frac{d\vec{p}}{dt} = q(\vec{E}_t + \vec{u}\wedge\vec{B}) = q\left(\vec{E}_t - \frac{1}{\omega}\vec{u}\wedge(\vec{k}\wedge\vec{E}) + \vec{u}\wedge\vec{B}_0\right). \quad (1)$$

One finds after linearization

$$i\omega\gamma\left(\vec{v} + \gamma^2\vec{V}\frac{\vec{V}\cdot\vec{v}}{c^2}\right) = \frac{q}{m}\left(\vec{E} - \frac{1}{\omega}(\vec{V}\cdot\vec{E})\vec{k} + \vec{v}\wedge\vec{B}_0\right), \quad (2)$$

where in the linearized expression for $\vec{p}$, the terms $\gamma^3[(\vec{V}\cdot\vec{v})/c^2]\partial\vec{V}/\partial t$ and $\vec{v}\partial\gamma/\partial t$ are much smaller than $\omega\gamma^3[(\vec{V}\cdot\vec{v})/c^2]\vec{V}$ (if one assumes that $\vec{V}\cdot\vec{v}\neq 0$). The dot product of this equation with $\vec{k}$ can be used in the continuity equation $\omega n + N\vec{k}\cdot\vec{v} = 0$ (N is the number density and n the perturbed particle density) for each species to obtain

$$\omega n_i + \frac{\omega_{pi}^2}{\omega\gamma_i}(n_e - n_i) + \frac{ieN}{M\omega^2\gamma_i}k^2\vec{V}^+\cdot\vec{E} + \frac{Nek\omega_{ci}}{M\gamma_i\omega^2}\left(E_z - \frac{\omega n_i}{kN}B_0\right) = 0, \quad (3)$$

$$\omega n_e - \frac{\omega_{pe}^2}{\omega\gamma_e}(n_e - n_i) - \frac{ieN}{m\omega^2\gamma_e}k^2\vec{V}^-\cdot\vec{E} - \frac{Nek\omega_{ce}}{M\gamma_e\omega^2}\left(E_z - \frac{\omega n_e}{kN}B_0\right) = 0, \quad (4)$$

where M, m are the ion and electron masses, respectively, ω_{pi} and ω_{pe} are the ionic and electronic plasma frequencies, ω_{ci} and ω_{ce} are the signed ionic and electronic relativistic cyclotron frequencies. Poisson's equation

$$i\vec{k}\cdot\vec{E} = (e/\epsilon_0)(n_i - n_e) \quad (5)$$

has been used. One calculates now n_i and n_e from Eqs. (3) and (4), $\vec{v}_i$ and $\vec{v}_e$ from Eq. (2) and combines it with the linearized wave equation

$$\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \nabla^2\vec{E} = -i\omega\mu_0 e[N(\vec{v}_i - \vec{v}_e) - n_e\vec{V}^- + n_i\vec{V}^+] \quad (6)$$

to obtain the dispersion relation

$$DAA_* + BA + CA_* + G = 0, \quad (7)$$

where

$$D=\left(\omega^2-\frac{\omega_{pi}^2}{\gamma_i}-\omega_{ci}^2\right)\left(\omega^2-\frac{\omega_{pe}^2}{\gamma_e}-\omega_{ce}^2\right)-\frac{\omega_{pi}^2\omega_{pe}^2}{\gamma_i\gamma_e},$$

$$A=-\omega^2+k^2c^2+\Lambda^2,$$

$$A_*=-\omega^2+k^2c^2+\Omega^2,$$

$$\Lambda^2=\frac{\omega_{pe}^2}{\gamma_e^3}+\frac{\omega_{pi}^2}{\gamma_i^3},$$

$$\Omega^2=\frac{\omega_{pe}^2}{\gamma_e}+\frac{\omega_{pi}^2}{\gamma_i},$$

$$B=-\frac{\omega_{pi}^2\omega_{pe}^2}{\gamma_i\gamma_e}(\omega_{ce}-\omega_{ci})^2+(\omega^2-\omega_{ce}^2)\frac{\omega_{pi}^2}{\gamma_i}\omega_{ci}^2+(\omega^2-\omega_{ci}^2)\frac{\omega_{pe}^2}{\gamma_e}\omega_{ce}^2,$$

$$C=k^2\left[-\frac{\omega_{pi}^2\omega_{pe}^2}{\gamma_i\gamma_e}(V^+-V^-)^2+(\omega^2-\omega_{ce}^2)\frac{\omega_{pi}^2}{\gamma_i}V^{+2}+(\omega^2-\omega_{ci}^2)\frac{\omega_{pe}^2}{\gamma_e}V^{-2}\right],$$

$$G=k^2\omega_{pi}^2\omega_{pe}^2(V^+\omega_{ce}-V^-\omega_{ci})^2/\gamma_i\gamma_e\,.$$

(1) Zero magnetic field. The dispersion relation reduces to

$$\omega^6-\omega^4(\Omega^2+k^2c^2+\Lambda^2)+\omega^2\Lambda^2(\Omega^2+k^2c^2)+k^2(\omega_{pe}^2\omega_{pi}^2/\gamma_e\gamma_i)(\vec{V}^+-\vec{V}^-)^2=0\,. \quad (8)$$

For large values of k this becomes a biquadratic, with purely imaginary frequencies signifying a purely growing, absolutely unstable wave,

$$\omega^2=\tfrac{1}{2}\left\{\Lambda^2-\left[\Lambda^4+4\,\frac{\omega_{pe}^2\omega_{pi}^2}{\gamma_e\gamma_i}\left(\frac{V^+}{c}-\frac{V^-}{c}\right)^2\right]^{1/2}\right\}. \quad (9)$$

For an ionic electronic plasma, the growth rate Γ of the instability is of the order

$$\Gamma^2\simeq\omega_{pe}^2\left[2\left(\frac{m_e}{m_i}\frac{1}{\gamma_e\gamma_i}\right)^{1/2}-\frac{1}{2}\left(\frac{1}{\gamma_e^3}+\frac{m_e}{m_i}\frac{1}{\gamma_i^3}\right)\right] \quad (10)$$

This is comparable to the main wave frequencies given earlier. For an electronic positronic plasma, it becomes

$$\Gamma^2\simeq\frac{2\omega_{pe}^2}{\gamma}\left(1-\frac{1}{2\gamma^2}+\cdots\right), \quad (11)$$

where Γ is the instantaneous growth rate. To get the mean growth rate we average over one period. For large γ, neglecting intervals of the period where γ is small, we obtain $\langle\Gamma\rangle=1.41\omega_w$ (ω_w is the frequency of the large-amplitude wave). This value of the growth rate is the one to be compared with the numerical result given by Romeiras[7] $\Gamma_R=1.46\omega_w$ (see Fig. 20, curve C, of Ref. 7). This good agreement between analytical and numerical results reinforces the validity of our treatment. For circulary polarized waves the same instability yields the same growth rate. This case has been recently investigated by Lee and Lerche.[8]

(2) Finite magnetic field. The full dispersion relation (7) has to be considered. In the limit of very large k, we are left with $k^2c^2D+C=0$. Then for the growth rate of instability,

$$\Gamma^2=\tfrac{1}{2}\left\{\left[\left(\frac{\omega_{pi}^2}{\gamma_i^3}+\omega_{ci}^2\right)-\left(\frac{\omega_{pe}^2}{\gamma_e^3}+\omega_{ce}^2\right)\right]^2+\left[\frac{4\omega_{pi}^2\omega_{pe}^2}{\gamma_i\gamma_e}\left(1-\frac{V_+V_-}{c^2}\right)^2\right]^{1/2}-(\Lambda^2+\omega_{ce}^2+\omega_{ci}^2)\right\}. \quad (12)$$

In the case of an electronic positronic plasma,

$$\Gamma^2\simeq2\omega_{pe}^2/\gamma_e-\omega_{ce}^2\,. \quad (13)$$

In both cases the stabilizing rolè of the magnetic field is obvious. Of course we did expect this result from the physical behavior of the plasma. The instability due to pinching of parallel currents is prevented by the magnetic field. These results may give us indications about the effective presence or absence of the large-amplitude wave in the pulsar magnetospheres.[9]

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