Stress relaxation waves in fluids

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The Navier-Stokes equations for incompressible and compressible fluids are generalized by inclusion of viscous stress relaxation, as required by kinetic theory. Two initial-boundary-value problems of the nonlinear generalized Navier-Stokes equations are solved analytically, which describe the propagation of transverse or shear waves due to temporal and spatial velocity pulses $\vec{v}(0,t)$ and $\vec{v}(x,0)$, respectively. It is shown that transverse perturbations propagate in the form of a discontinuous wave with a finite wave speed due to viscous stress relaxation, whereas the conventional Navier-Stokes equations result in nonphysical solutions, suggesting a diffusion process covering the entire fluid with infinite speed.

INTRODUCTION

The nonlinear incompressible and compressible Navier-Stokes equations represent (quasi) parabolic and hyperbolic partial differential equations, respectively. The former propagate signals with infinite speed and the latter propagate certain signals with finite speed in fluids. In an infinite, homogeneous fluid, consider a small (linear) velocity perturbation, which is representable as the Fourier integral

$$\vec{v}(\vec{r},t)=\int_{-\infty}^{+\infty}\vec{v}(\vec{k})e^{i\omega t-i\vec{k}\cdot\vec{r}}d\omega\,,$$

over elementary waves of wavelength $\lambda=2\pi/k$ and frequency $\omega(k)$. If the fluid is compressible so that it sustains both pressure ($\tilde{p}$) and density ($\tilde{\rho}$) perturbations, $\partial\tilde{p}/\partial t=c_s^2\partial\tilde{\rho}/\partial t$, the perturbation can propagate, e.g., in the form of longitudinal sound waves with finite speed $c_s=(\gamma p_0/\rho_0)^{1/2}$ and dispersion

$$\omega^2\doteq c_s^2k^2+i(4\mu/3\rho_0)\omega k^2\,.$$

In a fluid with a viscosity μ, a perturbation may also propagate in the form of a transverse or shear wave. If one applies the curl operation to the Navier-Stokes equation for incompressible fluids, a dispersion law is found for the transverse perturbations which does not represent a wave phenomenon but an aperiodic damping process with dispersion

$$i\omega=-(\mu/\rho_0)k^2\,.$$

As is known, the acoustic dispersion law is derived from a hyperbolic wave equation, whereas the damping relation for the transverse modes follows from the parabolic vorticity equation for incompressible fluids. From experiments, however, it is established that transverse perturbations ($\vec{\nabla}\times\tilde{v}_k=i\vec{k}\times\tilde{v}_k\neq\vec{0}$) propagate as (hyperbolic) shear waves with finite speed.[1] The (incompressible or compressible) Navier-Stokes equations propagate transverse perturbations in the form of a diffusion process with infinite speed; i.e., they do not provide a correct description of shear waves. The discrepancy between the Navier-Stokes equations and the experiments on shear waves is resolved by introducing viscous stress relaxation with a realistic (>0) relaxation time τ, in accordance with first principles of kinetic theory.[2,3] This generalization of the Navier-Stokes equations leads to a hyperbolic transport equation for shear waves in incompressible or compressible fluids. For this reason, the transverse or shear waves represent "stress relaxation waves," which exist only for $\tau>0$.

As an illustration, two hyperbolic initial-boundary-value problems for shear waves with stress relaxation are solved. The solutions of the generalized Navier-Stokes equations with stress relaxation represent transverse waves which are discontinuous at the wave front and have a finite wave speed, $c=(\mu/\rho_0\tau)^{1/2}<\infty$ for $\tau>0$. The first treats the propagation of a shear wave into a semi-infinite fluid space $x\geq 0$, produced by a temporal velocity impulse at the boundary $x=0$ (accelerated wall). The second is concerned with the propagation of a shear wave into an infinite fluid space $-\infty\leq x\leq+\infty$, caused by a spatial velocity pulse in the plane $x=0$ at time $t=0$. Both solutions are valid for nonlinear shear waves.

In principle, the presented generalization of the incompressible parabolic Navier-Stokes equations to hyperbolic equations by means of stress relaxation is analogous to the generalization of the pre-Maxwellian parabolic diffusion equations of the electromagnetic field by Maxwell. His introduction of the electric displacement current $\partial\epsilon\vec{E}/\partial t$ resulted in the correct hyperbolic equations for the electromagnetic field with finite speed of propagation $c=(\epsilon\mu)^{-1/2}$.

PHYSICAL PRINCIPLES

In conventional fluid mechanics, it is assumed that inhomogeneities $\nabla_i v_j$ in the velocity components v_j produce instantaneously viscous stresses Π_{ij}. Mathematically, this is expressed through the phenomenological "flux" ~ "force" relation

$$\Pi_{ij}=-\mu(\nabla_i v_j+\nabla_j v_i-\tfrac{2}{3}\nabla_k v_k \delta_{ij}),$$

where μ is the viscosity and $\vec{\vec{\delta}}$ is the unit tensor. In a real continuum, however, velocity inhomogeneities do not switch on viscous stresses instantaneously but in accordance with a relaxation process of characteristic time τ. By means of the kinetic theory of gases[2] and liquids,[3] one can show that the transport equation for the viscous stresses has the form of a temporal $(\partial/\partial t)$ and spatial $(\vec{v}\cdot\vec{\nabla})$ relaxation equation

$$\frac{\partial}{\partial t}\Pi_{ij}+v_k\nabla_k\Pi_{ij}=-\tau^{-1}\Pi_{ij}-\mu\tau^{-1}(\nabla_i v_j+\nabla_j v_i-\tfrac{2}{3}\nabla_k v_k\delta_{ij}).$$

This equation is approximate insofar as the coupling of heat flows q_i and stresses Π_{ij} and higher-order terms in the derivatives of v_i are neglected.[2,3] It has temporal and spatial derivatives as required for a $\vec{r},t$-dependent field equation and is Galilei covariant. If relaxation effects are disregarded, it reduces to the static stress equation.

Thus, consideration of viscous stress relaxation leads to a reformulation of the conventional Navier-Stokes theory of incompressible and compressible fluids. In place of the Navier-Stokes equations, we have the hydrodynamic equations with viscous stress relaxation:

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v}\right)=-\vec{\nabla}p-\vec{\nabla}\cdot\vec{\vec{\Pi}},\qquad(1)$$

$$\frac{\partial\rho}{\partial t}+\vec{v}\cdot\vec{\nabla}\rho=-\rho\vec{\nabla}\cdot\vec{v},\qquad(2)$$

$$\frac{\partial\vec{\vec{\Pi}}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{\vec{\Pi}}+\frac{\vec{\vec{\Pi}}}{\tau}=-\frac{\mu}{\tau}(\vec{\nabla}\vec{v}+\vec{\nabla}\vec{v}^{*}-\tfrac{2}{3}\vec{\nabla}\cdot\vec{v}\vec{\vec{\delta}}).\qquad(3)$$

Equations (1)–(3) hold for incompressible $(\vec{\nabla}\cdot\vec{v}=0)$ and compressible $(\vec{\nabla}\cdot\vec{v}\neq 0)$ fluids. For nonisothermal processes, e.g., sound waves, the transport equations for thermal energy and heat flux have to be added to Eqs. (1)–(3).[4,5]

If μ and τ can be treated as $\vec{r}$ independent, it is mathematically more convenient to use, instead of the tensor equation (3), the vector equation

$$\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{\vec{\Pi}}+\vec{\nabla}\cdot(\vec{v}\cdot\vec{\nabla}\vec{\vec{\Pi}})+\tau^{-1}\vec{\nabla}\cdot\vec{\vec{\Pi}}=-\mu\tau^{-1}(\vec{\nabla}^2\vec{v}+\tfrac{1}{3}\vec{\nabla}\vec{\nabla}\cdot\vec{v}),\qquad(4)$$

since Eq. (1) contains the force density $\vec{\nabla}\cdot\vec{\vec{\Pi}}$. If temporal and spatial relaxation of the viscous stresses is disregarded, Eqs. (1) and (4) combine to the classical Navier-Stokes equation

$$\rho(\partial\vec{v}/\partial t+\vec{v}\cdot\vec{\nabla}\vec{v})=-\vec{\nabla}p+\mu\vec{\nabla}^2\vec{v}+(\tfrac{1}{3}\mu)\vec{\nabla}\vec{\nabla}\cdot\vec{v}.$$

Equations (1)–(3) represent a hyperbolic system both in the compressible and incompressible cases. On the other hand, the conventional incompressible Navier-Stokes equations are parabolic. The corresponding field equations for incompressible fluids are obtained by setting $\vec{\nabla}\cdot\vec{v}\equiv 0$ in Eqs. (2), (3), and (4). The most general transient solution of Eqs. (1)–(3) is of the form $\vec{v}(\vec{r},t)=\vec{v}_1(\vec{r},t)+\vec{v}_2(\vec{r},t)$, where $\vec{v}_1$ is an irrotational field (longitudinal or sound waves) and $\vec{v}_2$ is a solenoidal field (transverse or shear waves).

By Eq. (3) velocity gradients $\nabla_i v_j$ produce stresses Π_{ij} in accordance with a relaxation process of characteristic time $\tau>0$. Thus, a stress or vorticity perturbation can no longer diffuse with infinite speed as in the parabolic Navier-Stokes theory $(c=\infty,\ \tau=0)$, but propagates with a finite speed, $c=(\mu/\rho\tau)^{1/2}$ by Eqs. (1)–(3) for dimensional reasons. A vorticity perturbation described by the hyperbolic Eqs. (1)–(3) propagates, therefore, in form of a "wave" which has a "wave front", ahead of which the fluid is unexcited because of the finite wave speed $c<\infty$ for $\tau>0$.

INITIAL-BOUNDARY-VALUE PROBLEM FOR $\vec{v}(0,t)$ PULSE

A simple method for the generation of transverse waves in a viscous fluid consists in setting the plane $x=0$ bounding a semi-infinite fluid $(x\geq 0,\ |y|\leq\infty,\ |z|\leq\infty)$ into sudden motion $\vec{v}_w=\hat{v}H(t)\vec{e}_y$, where $H(t)$ is the Heaviside step function. The resulting viscous interaction between the fluid and the accelerated wall produces a curl $\vec{n}\times[\vec{v}]=\vec{e}_z v(x=0,t)$ at the fluid surface which propagates in form of a transverse wave through the fluid in the x direction. In this dynamic process, the fluid velocity is of the form $\vec{v}=[0,v(x,t),0]$ so that $\vec{\nabla}\cdot\vec{v}=\partial v/\partial y=0$ and $\vec{v}\cdot\vec{\nabla}\vec{v}=\vec{0}$; i.e., the fluid motion behaves incompressibly (even if the fluid is compressible) and linearly. Furthermore, $\vec{v}\cdot\vec{\nabla}\vec{\vec{\Pi}}=v\partial\vec{\vec{\Pi}}/\partial y=\vec{\vec{0}}$ since $\vec{\vec{\Pi}}$ has only the components $\Pi_{yx}=\Pi_{xy}=\Pi(x,t)$ by Eq. (3), and $\vec{\nabla}p=\vec{0}$ by Eq. (1).

Thus, Eqs. (1)–(3) lead to the following initial-boundary-value problem for the transverse velocity wave $\vec{v}(x,t)$ in the y direction propagating in the x direction, as a result of the sudden wall motion in the plane $x=0$:

$$\rho_0\frac{\partial v}{\partial t}=-\frac{\partial\Pi}{\partial x},\qquad(5)$$

$$\frac{\partial\Pi}{\partial t}+\frac{\Pi}{\tau}=-\frac{\mu}{\tau}\frac{\partial v}{\partial x}, \tag{6}$$

$$v(x=0,t)=\hat{v}H(t), \quad t\geq 0 \tag{7}$$

$$v(x,t=0)=0, \quad x>0 \tag{8}$$

$$\frac{\partial v(x,t=0)}{\partial t}=0, \quad x>0 \tag{9}$$

$$H(t)\begin{cases}=0, & t\leq -0\\ =1, & t\geq +0.\end{cases}$$

Equations (5) and (6) represent a hyperbolic system, from which one obtains by elimination wave equations for the stress component $\Pi_{xy} \equiv \Pi(x,t)$ and the velocity field $v(x,t)$:

$$\frac{\partial^2\Pi}{\partial t^2}+\frac{1}{\tau}\frac{\partial\Pi}{\partial t}=c^2\frac{\partial^2\Pi}{\partial x^2} \tag{10}$$

and

$$\frac{\partial^2 v}{\partial t^2}+\frac{1}{\tau}\frac{\partial v}{\partial t}=c^2\frac{\partial^2 v}{\partial x^2}, \tag{11}$$

where

$$c=(\mu/\rho_0\tau)^{1/2} \tag{12}$$

is the (maximum) speed of the stress-relaxation wave. Both $\Pi(x,t)$ and $v(x,t)$ satisfy similar (hyperbolic) wave equations with the same wave speed c. In the limit $\tau\to 0$ and $c\to\infty$, with $c^2\tau \to \mu/\rho_0$, Eqs. (10) and (11) reduce to parabolic equations, according to which boundary values of $\Pi(x,t)$ and $v(x,t)$ would diffuse with infinite speed into the fluid (conventional Navier-Stokes theory). Accordingly, only for $\tau>0$ and $c<\infty$, transverse or shear waves exist in the fluid which represent, therefore, stress-relaxation waves.

According to Eq. (11) and Eqs. (7)–(9), the velocity field $v(x,t)=\hat{v}u(\xi,\mathcal{T})$ of the stress-relaxation wave under consideration is described by the dimensionless initial-boundary-value problem

$$\frac{\partial^2 u}{\partial \mathcal{T}^2}+\frac{\partial u}{\partial \mathcal{T}}=\frac{\partial^2 u}{\partial \xi^2}, \tag{13}$$

$$u(\xi=0,\mathcal{T})=H(\mathcal{T}), \quad \mathcal{T}\geq 0 \tag{14}$$

$$u(\xi,\mathcal{T}=0)=0, \quad \xi>0 \tag{15}$$

$$\frac{\partial u(\xi,\mathcal{T}=0)}{\partial \mathcal{T}}=0, \quad \xi>0 \tag{16}$$

where

$$u(\xi,\mathcal{T})=v(x,t)/\hat{v}, \quad \xi=x/c\tau, \quad \mathcal{T}=t/\tau. \tag{17}$$

Equations (13)–(16) are solved by means of the Laplace transform technique[6] which gives

$$\bar{u}(\xi,s)=\mathcal{L}(u(\xi,\mathcal{T}))=\int_0^\infty e^{-s\mathcal{T}}u(\xi,\mathcal{T})d\mathcal{T}, \tag{18}$$

$$\bar{u}(0,s)=\mathcal{L}(u(0,\mathcal{T}))=\int_0^\infty e^{-s\mathcal{T}}H(\mathcal{T})d\mathcal{T}=s^{-1}. \tag{19}$$

Since the initial conditions (15) and (16) vanish, Eqs. (13) and (14) yield for the transformed velocity $\bar{u}(\xi,s)$ the ordinary boundary-value problem

$$\frac{d^2\bar{u}}{d\xi^2}-(s^2+s)\bar{u}=0, \tag{20}$$

$$\bar{u}(\xi=0,s)=s^{-1}. \tag{21}$$

Since $u(\xi,s)$ must be finite for $\xi\to\infty$, the solution of Eqs. (20) and (21) is

$$\bar{u}(\xi,s)=s^{-1}e^{-(s^2+s)^{1/2}\xi}. \tag{22}$$

The inverse Laplace transform gives for the velocity field the complex integral

$$u(\xi,\mathcal{T})=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} s^{-1}e^{-(s^2+s)^{1/2}\xi}e^{s\mathcal{T}}ds. \tag{23}$$

Hence,

$$u(\xi,\mathcal{T})=\frac{\partial\Phi(\xi,\mathcal{T})}{d\xi}, \tag{24}$$

where

$$\Phi(\xi,\mathcal{T})=-\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} s^{-1}\bar{f}(\xi,s)e^{s\mathcal{T}}ds \tag{25}$$

and

$$\bar{f}(\xi,s)\equiv e^{-(s^2+s)^{1/2}\xi}/(s^2+s)^{1/2}. \tag{26}$$

According to a known inversion integral,[6] the inverse transform of Eq. (26) is

$$f(\xi,\mathcal{T})=\mathcal{L}^{-1}(\bar{f}(\xi,s)) = e^{-\mathcal{T}/2}I_0(\tfrac{1}{2}(\mathcal{T}^2-\xi^2)^{1/2})H(\mathcal{T}-\xi), \tag{27}$$

where $I_\nu(\mathcal{T})$ is the modified Bessel function of order ν. By Eq. (25),

$$\Phi(\xi,\mathcal{T})=-\mathcal{L}^{-1}(s^{-1}\bar{f}(\xi,s)) = -\int_0^{\mathcal{T}}\mathcal{L}^{-1}(\bar{f}(\xi,s))d\mathcal{T} = -\int_0^{\mathcal{T}} f(\xi,\mathcal{T})d\mathcal{T}, \tag{28}$$

i.e.,

$$\Phi(\xi,\mathcal{T})=-H(\mathcal{T}-\xi)\int_\xi^{\mathcal{T}} e^{-\alpha/2}I_0(\tfrac{1}{2}(\alpha^2-\xi^2)^{1/2})d\alpha. \tag{29}$$

From this potential, the dimensionless velocity field is obtained as

$$u(\xi,\mathcal{T})=H(\mathcal{T}-\xi)\left(e^{-\xi/2}+\tfrac{1}{2}\xi\int_\xi^{\mathcal{T}} e^{-\alpha/2}(\alpha^2-\xi^2)^{-1/2}I_1(\tfrac{1}{2}(\alpha^2-\xi^2)^{1/2})d\alpha\right) \tag{30}$$

in accordance with Eq. (24). By Eq. (17), the corresponding dimensional solution for the velocity field is

$$v(x,t)=\hat{v}H(ct-x)\left(e^{-x/2c\tau}+(x/2c\tau)\int_{x/c\tau}^{t/\tau}e^{-\alpha/2}[\alpha^2-(x/c\tau)^2]^{-1/2}I_1(\tfrac{1}{2}[\alpha^2-(x/c\tau)^2]^{1/2})d\alpha\right). \tag{31}$$

Equation (31) indicates that the transverse stress relaxation wave is discontinuous at $x=ct$, the position of the wave front at time t. At any time $0\leq t\leq\infty$, only the region $0\leq x\leq ct$ of the fluid is excited by the wave, since $v(x<ct,t)>0$ and $v(x>ct,t)=0$ by Eq. (31). The velocity signal $v(0,t)=\hat{v}H(t)$ generated at the boundary $x=0$ at time t is thus transported with finite speed $c=(\mu/\rho_0\tau)^{1/2}<\infty$ in form of a discontinuous wave into the fluid space $x>0$ as $t>0$ increases.

Application of the asymptotic formula $I_0(z)\sim e^z/(2\pi z)^{1/2}$, $|z|\gg 1$, and expansion of $z=\frac{1}{2}[\alpha^2-(x/c\tau)^2]^{1/2}$ for large α values in Eq. (31) yields, in the limit $\tau\to 0$, $c\tau\to 0$:

$$v(x,t)=\hat{v}(2/\sqrt{\pi})\int_\eta^\infty e^{-\beta^2}d\beta,\qquad \eta\equiv x/2(\mu t/\rho_0)^{1/2}. \tag{32}$$

This is the familiar solution of the parabolic Navier-Stokes equations due to Stokes.[7] Euqation (32) suggests that $v(x,t)>0$ throughout the entire fluid $0\leq x\leq\infty$ for any, no matter how small time $t>0$. Thus, the parabolic Stokes solution gives a completely misleading picture for a shear wave in the form of a diffusion process which spreads with infinite speed.

Figure 1 shows $u(\xi,\mathcal{T})$ versus ξ for $\mathcal{T}=10^0$, 10^1, and 10^2, with wave fronts at $\hat{\xi}=10^0$, 10^1, and 10^2. It is seen how the perturbation $u(0,\mathcal{T})=H(\mathcal{T})$ produced at the wall $\xi=0$ moves in the form of a discontinuous wave into the fluid space $\xi\geq 0$ so that an increasing but finite region $0\leq\xi\leq\hat{\xi}$ of the fluid is set into motion with increasing $\mathcal{T}$. In the limit $\mathcal{T}=\infty$, $u(\xi,\mathcal{T})=1$ throughout the fluid $0\leq\xi\leq\infty$. The corresponding unrealistic parabolic solution with the velocity field extending to infinity for any time $t>0$ is illustrated in Ref. 7.

INITIAL-BOUNDARY-VALUE PROBLEM FOR $\vec{v}(x,0)$ PULSE

Another fundamental method for shear wave generation makes use of a velocity pulse $\vec{v}_0=v(x,0)\vec{e}_y$ generated at time $t=0$ within a limited region $|x|<\Delta x$. The decay of this velocity pulse occurs in form of a shear wave with velocity field $\vec{v}=(0,v(x,t),0)$ in the y direction propagating in the x directions. Accordingly, $\vec{\nabla}\cdot\vec{v}=\partial v/\partial y=0$, $\vec{v}\cdot\vec{\nabla}\vec{v}=\vec{0}$, and $\vec{v}\cdot\vec{\nabla}\vec{\Pi}=v\partial\vec{\Pi}/\partial y=\vec{0}$, since $\vec{\Pi}$ has only the components $\Pi_{yx}=\Pi_{xy}=\Pi(x,t)$. Again, the transverse wave "behaves" incompressibly and linearly and $\vec{\nabla}p=\vec{0}$ by Eq. (1).

As in the previous problem, Eqs. (1)–(3) give the wave Eqs. (10) and (11) for $\Pi(x,t)$ and $v(x,t)$, respectively. Hence, the shear wave produced by the velocity pulse $\vec{v}_0=v(x,0)\vec{e}_y$ is described by the initial-boundary-value problem

$$\frac{\partial^2 v}{\partial t^2}+\frac{1}{\tau}\frac{\partial v}{\partial t}=c^2\frac{\partial^2 v}{\partial x^2}, \tag{33}$$

$$v(x,t=0)=v_0(x),\quad |x|\leq\infty \tag{34}$$

$$\frac{\partial v(x,t=0)}{\partial t}=w_0(x),\quad |x|\leq\infty \tag{35}$$

where $|w_0(x)|\geq 0$ is included for reasons of generality. The solution of Eqs. (33)–(35) is accomplished by means of Riemann's method,[8]

$$v(x,t)=e^{-t/2\tau}\left(\tfrac{1}{2}[v_0(x-ct)+v_0(x+ct)]+\tfrac{1}{2}\int_{x-ct}^{x+ct}\Psi(x,t,\alpha)d\alpha\right), \tag{36}$$

where

$$\Psi(x,t,\alpha)=v_0(\alpha)\left(\frac{t}{2\tau}\right)I_1\left(\frac{1}{2c\tau}[c^2t^2-(\alpha-x)^2]^{1/2}\right)\Big/[c^2t^2-(\alpha-x)^2]^{1/2}+\frac{1}{c}\left(w_0(\alpha)+\frac{1}{2\tau}v_0(\alpha)\right)I_0\left(\frac{1}{2c\tau}[c^2t^2-(\alpha-x)^2]^{1/2}\right). \tag{37}$$

As a concrete example for the initial conditions in Eqs. (34) and (35), an initial velocity distribution of the form of a Dirac pulse is chosen,

$$v_0(x)=\hat{v}_0\delta(x),\quad w_0(x)=0,\quad |x|\leq\infty. \tag{38}$$

In this case, the general solution in Eqs. (36) and (37) becomes, in dimensionless form

$$u(\xi,\mathcal{T})=\begin{cases}\frac{1}{2}e^{-\mathcal{T}}[\delta(\xi-\mathcal{T})+\delta(\xi+\mathcal{T})+I_1((\mathcal{T}^2-\xi^2)^{1/2})/(\mathcal{T}^2-\xi^2)^{1/2}+I_0((\mathcal{T}^2-\xi^2)^{1/2})], & |\xi|\leq\mathcal{T}\\ 0, & |\xi|>\mathcal{T}\end{cases} \tag{39}$$

where

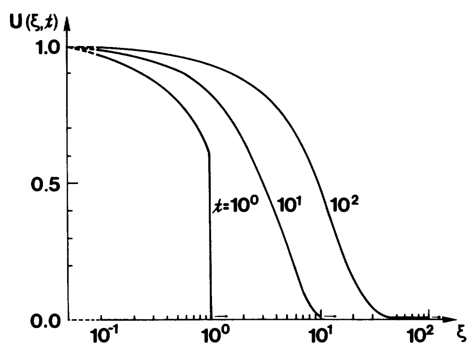

FIG. 1. $u(\xi, \mathcal{T})$ versus ξ for $\mathcal{T}=10^0$, 10^1, and 10^2 with $u(0, \mathcal{T})=H(\mathcal{T})$.

$$u(\xi, \mathcal{T})=v(x,t)/(\hat{v}_0/2c\tau), \quad \xi=x/2c\tau, \quad \mathcal{T}=t/2\tau. \tag{40}$$

Equation (39) indicates that the shear wave spreads in the space $|\xi| \leq \infty$ in form of a symmetrical wave $u(-\xi, \mathcal{T})=u(+\xi, \mathcal{T})$, due to the symmetry of the initial conditions (38). The wave is discontinuous at its fronts $\hat{\xi}=\pm\mathcal{T}$, which propagate with the speed

$$v(\hat{x}, t)=\pm c, \quad \hat{x}=\pm ct. \tag{41}$$

In the limit $\tau \to 0$, $c\tau \to 0$, application of the asymptotic formula $I_\nu(z) \sim e^z/(2\pi z)^{1/2}$, $|z| \gg 1$, and expansion of $z=(\alpha^2-\xi^2)^{1/2}$ for large α values in Eq. (39) yields

$$u(\xi, \mathcal{T})=(2\pi\mathcal{T})^{-1/2}e^{-\xi^2/2\mathcal{T}}, \quad |\xi| \leq \infty. \tag{42}$$

This is the corresponding solution of the parabolic Navier-Stokes equations.[7] Equation (42) would indicate that the shear wave has the form of a Gaussian extending from $\xi=-\infty$ to $\xi=+\infty$ for any, no matter how small time $\mathcal{T}>0$ (corresponding to an infinite speed of propagation). It is obvious that the solution (42) is physically not meaningful.

In Fig. 2, the dimensionless velocity field $u(\xi, \mathcal{T})$ of the shear wave is shown versus ξ for $\mathcal{T}=10^0$, 10^1, and 10^2, the wave fronts being in each case at $\hat{\xi}=\pm\mathcal{T}$. Due to the finite wave speed c, the fluid is not excited in the region $|\xi|>\mathcal{T}$ ahead of the wave fronts. The shape of the wave is flat with relatively steep flanks leading to the discontinuous fronts. Thus, the shear wave does not resemble the Gaussian of the parabolic theory, Eq. (42), which extends over the entire space $|x|<\infty$. The unrealistic parabolic solution is illustrated in Ref. 7.

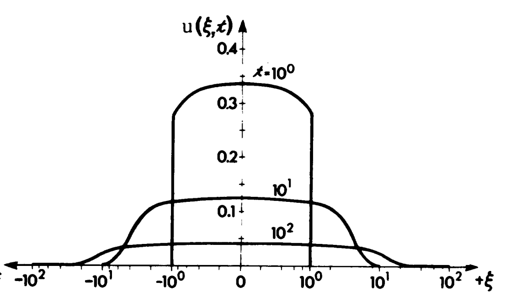

FIG. 2. $u(\xi, \mathcal{T})$ versus ξ for $\mathcal{T}=10^0$, 10^1 and 10^2 with $u(\xi, 0)=\delta(\xi)$.

CONCLUSIONS

A generalization of the Navier-Stokes equations is presented considering viscous stress relaxation, which results in a physically meaningful theory for transverse waves in viscous fluids. The fundamental speed of the stress relaxation waves is given by $c=(\mu/\rho\tau)^{1/2}$, where μ is the viscosity, ρ is the density, and τ is the relaxation time of the stress tensor. For any medium,[1] it is $c \lesssim (\frac{3}{4})^{1/2}c_s$, where c_s is the speed of the longitudinal waves, e.g.,[9] $c=1.2\times10^5$ cm sec^{-1} and $c_s=1.5\times10^5$ cm sec^{-1} for water at $T=20\,^\circ$C and $p_0=1$ atm.

Exact solutions are derived for stress relaxation waves propagating in the x direction due to velocity pulses $\vec{v}(0,t)$ and $\vec{v}(x,0)$ in the y direction, respectively. For the geometry of these transverse waves, the nonlinear generalized Navier-Stokes equations become linear, so that the solutions given hold for waves of large intensity. The solutions are discontinuous at the wave fronts, which is typical for hyperbolic field equations. The corresponding solutions of the conventional Navier-Stokes equations indicate a diffusion process with infinite wave speed and without wave front, i.e., give a qualitatively and quantitatively insufficient picture of the propagation of transverse waves in fluids.

In the simplified stress relaxation equation (3) proposed, the term $\overleftrightarrow{\Pi}\cdot\vec{\nabla}\vec{v}$ is neglected since it is of the order of magnitude of $(\mu/\tau)|\vec{\nabla}\vec{v}|^2$, which is nonlinear in the derivatives. It should be noted that the term $\overleftrightarrow{\Pi}\cdot\vec{\nabla}\vec{v}$ vanishes exactly for the wave problems treated above, $\overleftrightarrow{\Pi}\cdot\vec{\nabla}\vec{v}=\vec{0}$, since $\vec{v}=(0, v(x,t), 0)$ and $\overleftrightarrow{\Pi}$ has only the components $\Pi_{yx}=\Pi_{xy}$. For this reason, the solutions presented are exact solutions of the nonlinear Navier-Stokes equations with viscous stress relaxation.

The introduction of stress relaxation ($\tau>0$, $c<\infty$) as required by kinetic theory[2,3] changes the mathematical type of the hydrodynamic equations for the transverse velocity field from parabolic to hyperbolic, and the nature of vorticity transport from an unrealistic diffusion with infinite speed ($\tau=0$, $c=\infty$) to the physically correct wave propagation with finite speed ($\tau>0$, $c<\infty$). This holds not only for the simplified stress transport equation (3) but for the general stress transport equation.[2,3] The hyperbolic type of the resulting partial differential equation of second order is determined by the coefficients of the second-order derivatives of the linear field terms alone, i.e., not by the terms which are linear or nonlinear in the first-order derivatives.[8]

The parabolic (P) and hyperbolic (H) solutions are approximately equal behind the wave front, i.e., for a symmetrical solution $|v_P(x,t)| \lesssim \lesssim |v_H(x,t)|$ for $|x| < ct$, but this approximate agreement gets worse and worse as $|x|$ approaches $\hat{x} = ct$. The main difference between the P and H solutions occurs ahead of the wavefront, since $|v_P(x,t)| > 0$ for $ct \leq |x| < \infty$, whereas in reality $v_H(x,t) = 0$ for $ct < |x| < \infty$, since physical perturbations or real signals can propagate only with finite speed.

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