Adiabatic theory for a single nonlinear wave in a Vlasov plasma

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A large-amplitude high-frequency modulated wave launched by an external source significantly changes the plasma equilibrium. The adiabatic Vlasov distribution function of the electrons is found for one-dimensional initial- and boundary-value problems. The nonlinear dispersion relation is derived to all orders in the electric-field amplitude. A new nonlinear mode below the electron plasma frequency exists. In this state the plasma is described by a double-humped distribution function.

I. INTRODUCTION

Let us assume that a single wave $E(kx-\omega t, x-v_g t, t)$ is launched into the plasma from an external source. Here $kx-\omega t$ denotes the fast oscillation of the wave, k is the wave vector, ω is the frequency, v_g is the group velocity of the wave packet, and $x-v_g t$ and t stand for the slow space and time modulations, respectively. The scale on which the slow variation takes place is determined by the source and the plasma properties. For example, the spatial scale L is related to the focusing of the wave and the spreading of the wave packet. The time scale T is determined by the adiabatic turn-on of the pump and the wave-plasma interactions. The latter may lead to Landau damping and parametric processes with their own respective time scales.

The plasma equilibrium changes significantly in the presence of a large-amplitude wave. We shall consider waves, whose energy may be comparable to or greater than the thermal energy of the particles. One studies the dynamics of the electrons and considers the ions to form a stationary quasineutral background. In the collisionless regime the plasma state is given by the Vlasov distribution function. There are two distinctly different problems in the Vlasov theory of waves. Firstly, for all particle velocities v such that $|v-v_g| \ll L/T$, the distribution function should be derived from an initial-value problem. In this case the particles move through the wave packet slowly and respond to the time variation of the amplitude. This was the Landau approach,[1] which for $L\to\infty$ is valid for all phase space. Since it corresponds to a homogeneous problem, we shall call the standard distribution function homogeneous. Secondly, for all v such that $v-v_g \gg L/T$, the distribution function is found from a boundary-value problem.[2] The particles move quickly in the modulated wave and adjust to its spatial variations. In a typical laser-pellet experiment the focusing of the beam is $L\approx 10^{-2}$ cm, the time modulation $T\approx 10^{-9}$ sec and $v_t \approx 10^9$ cm/sec. In the coupling of lower hybrid (LH) waves launched by a waveguide array at the edge of a tokamak plasma $L\approx 1$ cm, $T\approx 10^{-6}$ sec, and $v_t\approx 10^8$ cm/sec. In both cases $v_t T/L \approx 100$ and all of phase space can be treated as a boundary-value problem. Only recently[3,4] it was pointed out that the ponderomotive problem belongs to this latter category. The distribution function (f) in both limits is local in the electric-field amplitude nearly everywhere in phase space. This is a great simplification to the nonlinear problem and will allow us to write f explicitly to all orders in E. In general, only parts of the phase space correspond to a local theory.

We construct the Vlasov distribution function from the adiabatic invariant (I) for a system evolving in time or space. When the system changes slowly from one steady state to another, the adiabatic invariant is constant to all orders in the parameter measuring the slow variation.[5] The Hamiltonian of the system is not conserved and the energy of the particles may vary substantially after long periods of time. However, for long times I is approximately constant. The essential difference between our approach and that of Bernstein, Greene, and Kruskal (BGK),[6] is that we evolve the system, while the BGK construction relies on the existence of a steady state, i.e., a conservative Hamiltonian. The BGK approach does not show how the system has evolved to such a state. An infinite variety of BGK states are possible, which is not the way nature operates.

If the Hamiltonian is $H(I,t)$, the nonlinear frequency is $\Omega \equiv \partial H/\partial I$. The initial-value problem I is a constant of motion in phase space when $\Omega \gg T^{-1}$. This is satisfied nearly everywhere, except for a narrow region of phase space near the separatrix (border line in phase space between the bounded and unbounded particle motion). Close to the separatrix one has to keep the whole time history of the problem and the theory is essentially nonlocal. This region corresponds to the nonadiabatic particles which will be called resonant particles. One should remember that the trapped particles are adiabatic, i.e., nonresonant, for sufficiently large amplitude of E. In the common language this means that $(eEk/m)^{1/2} \equiv \omega_B \gg \gamma_L$, where γ_L is the Landau damping. Of course, this is only a simple example of the general relation between the nonlinear frequency and the time scale. For the boundary-value problem one finds a different and, to our knowledge, a new adiabatic invariant J. The corresponding nonlinear frequency should be much larger than L^{-1} and J will be a constant nearly everywhere in phase space.

The nonlinear distribution function, which we find, describes the phenomena that depend on the local value of the electric-field amplitude and is correct to all orders in E. This is essentially a nonresonant distribution in a more general sense, since it includes the trapped particles. The resonant contribution to the distribution function involves the time history of the electric field, i.e., it is a nonlocal function. For a large-amplitude single wave the contributions come from a negligible part of phase space. Since we neglect this part, phenomena like Landau damping will not be included in our study. We must emphasize that our solution represents most of the physics, since it determines the major contributions to the sources in Maxwell's equations and the resulting nonlinear modes.

One finds that the solutions for the initial- and boundary-value problems are very different. The boundary-value problem exhibits more strongly the nonlinear behavior. It includes what in simple terms is called ponderomotive density change. The concept of a ponderomotive potential does not have any basic physical significance. It holds true only in the dipole approximation, where a factorization is known to take place.[4] When $k=0$, the initial- and boundary-value solutions for f differ by a multiplicative factor $\exp(-\phi_P/T_e)$, where ϕ_P is the ponderomotive potential and T_e is the electron temperature. In general, the initial- and boundary-value distribution functions correspond to different dynamical problems. The initial value f leads to the generation of a steady-state current and the boundary value f shows a pondermotive (zero-frequency) density depression. For large-amplitude waves both exhibit a new nonlinear mode below the electron plasma frequency. This mode is associated with a bifurcation of the initial distribution function (Maxwellian) and represents a new state of the plasma.

There is strong evidence that such a mode has been found recently in a magnetized plasma loaded waveguide.[7] The observed solitary structure called an "electron hole" has the qualitative features of the trapped particle mode, described in this paper as a solution of the initial-value problem.

This paper is organized as follows. In Sec. II, we develop an intuitive approach, which is along the lines of the traditional nonresonant perturbation theory. In Sec. III a canonical formulation of the Vlasov equation is used to derive the adiabatic invariants and to write explicitly the distribution functions. Section IV is devoted to the nonlinear dispersion relation and the resulting new mode. A summary of what we believe are the important points and suggestions for further research are discussed in Sec. V.

II. PERTURBATIVE APPROACH TO THE VLASOV EQUATION

The electron distribution function will be given in terms of the electric field $E(kx-\omega t, x-v_g t, t)$. The actual dependence of E on the fast oscillation and the modulations can be found from the Maxwell equations. Before we engage in this general problem one should examine the functional dependence of f on E for a specific example.

Let us consider that the amplitude of the first harmonic of E, excited by an external source, is much larger than the amplitudes of all the higher harmonics. This will allow us to explore certain limits and develop some intuition about the self-consistent solution. One should try to extract from the following example the general features, and the discussion in Sec. III will substantiate them.

The Vlasov equation for electrons in the frame moving with the wave packet is

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}=\omega v_0(x,t)\cos(kx-\omega t)\frac{\partial f}{\partial v}, \tag{1}$$

where $v_0 \equiv eE_0/m\omega$. To simplify the calculation further let us assume that $v_g \ll v_t$, i.e., Eq. (1) is nearly valid in the laboratory frame. We shall use this approximation throughout the paper, since the generalization to arbitrary v_g is trivial. The solution of (1) can be written as

$$f=\sum_{n=-\infty}^{\infty} f_n(v,x,t)\,e^{-in(\omega t-kx)}, \tag{2}$$

where the amplitudes f_n are slowly modulated in time and space. By substituting (2) in (1) one gets the infinite set of equations

$$\frac{1}{\omega}\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)f_0=\tfrac{1}{2}v_0\frac{\partial}{\partial v}(f_1+f_1^*), \tag{3a}$$

$$-in\left(1-\frac{kv}{\omega}\right)f_n+\frac{1}{\omega}\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)f_n =\tfrac{1}{2}v_0\frac{\partial}{\partial v}(f_{n-1}+f_{n+1}),\quad n\geq 1. \tag{3b}$$

If we assume a long scale of spatial modulation ($v_t \ll L/T$) the distribution function can be found from the infinite chain relations (3a) and (3b) as an initial-value problem. In the nonresonant domain $1-kv/\omega \gg (\omega T)^{-1}$, and from (3b) one can write

$$f_n = \frac{1}{-in\left(1-\frac{kv}{\omega}\right)}\left[1-\frac{\frac{i}{\omega}\frac{\partial}{\partial t}}{n\left(1-\frac{kv}{\omega}\right)}\right]\frac{v_0}{2} \times \frac{\partial}{\partial v}(f_{n-1}+f_{n+1}). \tag{4}$$

Note that one considers only the first derivative with respect to the slow time evolution. Equation (3a), then, takes the simplified form

$$\frac{1}{\omega}\frac{\partial f_0}{\partial t} = \tfrac{1}{2}v_0\frac{\partial}{\partial v}(f_1+f_1^*). \tag{5}$$

The Taylor series for f_n in the oscillation velocity v_0 is constructed by including the contributions of the corresponding harmonics from (4) and (5). In general, one can write

$$f_n = v_0^n P_n(v_0, v) + \frac{\partial v_0^n}{\partial t} Q_n(v_0, v), \tag{6}$$

where P_n, Q_n are polynomials in v_0. The first term on the right-hand side (rhs) of (6) is local in the electric field and the second term takes into account the time modulation. The equation for the time average amplitude f_0 includes only the first derivative terms. To find f_0 one has to integrate Eq. (5) $\int_{t_0}^{t} dt'$ and use the appropriate initial value for f. In this paper we use $f(t=0)=f_M$, where f_M is the Maxwellian distribution. This choice simplifies the expressions, but the method is applicable for an arbitrary initial distribution. The resulting formula for f_0 is local in v_0 (i.e., it does not depend on $\partial v_0/\partial t$). One neglects the nonlocal part of the amplitudes f_n and by summing the Fourier series (2) one can find the total local distribution function.

To illustrate some of these steps we find f_1 to first order in v_0 from (4), where we neglect f_2 and use $f_0 \simeq f_M$. The result is substituted in (5):

$$\frac{\partial f_0}{\partial t} = \frac{1}{4}\frac{\partial v_0^2}{\partial t}\nabla_v\left(1-\frac{kv}{\omega}\right)^{-2}\nabla_v f_M . \tag{7}$$

By integrating over time one finds the average distribution function to order v_0^2:

$$f_0^{(2)} = \left[1+\frac{v_0^2}{4}\nabla_v\left(1-\frac{kv}{\omega}\right)^{-2}\nabla_v\right]f_M . \tag{8}$$

This is the usual nonresonant quasilinear distribution function. A steady-state drift $\langle v\rangle \equiv \int v f_0^{(2)} \neq 0$ is generated. It has been shown[4] that in the dipole approximation ($k=0$), the infinite Taylor series for f_0 can be summed. Furthermore, for $k=0$, all the local harmonics were found and the total distribution function simply became

$$f = f_M[v+v_0(t)\sin\omega t] , \tag{9}$$

i.e., it is the Maxwellian in the oscillating frame.

In the case of very slow time modulation ($v_t \gg L/T$) the Vlasov equation should be solved as a boundary-value problem. The chain relations are

$$\frac{v}{\omega}\frac{\partial f_0}{\partial x} = \tfrac{1}{2}v_0\frac{\partial}{\partial v}(f_1+f_1^*), \tag{10}$$

$$f_n = -in\left(1-\frac{kv}{\omega}\right)^{-1}\left[1-\frac{i\frac{v}{\omega}\frac{\partial}{\partial x}}{n\left(1-\frac{kv}{\omega}\right)}\right]\tfrac{1}{2}v_0 \times \frac{\partial}{\partial v}(f_{n-1}+f_{n+1}). \tag{11}$$

The Taylor series in v_0 is constructed as before, but one integrates over space $\int_{x_B}^{x} dx'$, at the boundary point $x_B, f(x_B)=f_M$. To order v_0^2 one finds for f_0

$$f_0^{(2)} = \left(1+\frac{v_0^2}{4v}\nabla_v\frac{v}{(1-kv/\omega)^2}\nabla_v\right)f_M . \tag{12}$$

There is no singularity at $v=0$ since $\nabla_v f \sim v f_M$. By comparing with the result in (8) one can note a basic difference. The average density from (8) is $\langle n^{(2)}\rangle = n_0$. The boundary-value distribution function from (12) leads to what is called a ponderomotive density change $\langle n^{(2)}\rangle \neq n_0$. However, no steady-state current is generated.

In the dipole approximation to all orders in v_0 the total distribution function is[4]

$$f = f_M[v+v_0(x)\sin\omega t]\exp\left(-\frac{v_0^2(x)}{2v_t^2}\right). \tag{13}$$

The solutions in (9) and (13) are remarkably similar. Both correspond to a Maxwellian in the oscillating frame and the ponderomotive effect can be factorized by introducing an effective potential, i.e., the ponderomotive potential. However, we have to warn the reader that this simple picture is true only in the $k=0$ case. In general the dynamics in the two problems is very different.

Before we go further we would like to discuss an approximate scheme which gives rather well the qualitative features of the general solution. Since the oscillation frame plays an important role in finding the distribution function, one should try to generalize this concept. For $k \neq 0$ the parameter of nonlinearity is of the form $v_0/(1-kv/\omega)$. We suggest that in the initial-value problem f^h is approximately

$$f^h \simeq \exp[-(w+S)^2], \tag{14}$$

where h stands for homogeneous and we have omitted the trivial normalization and $w=v/v_t$. S is given by

$$S=\frac{w_0\sin(\omega t-kx)}{1-aw-\frac{1}{2}aS}, \tag{15}$$

where $w_0=v_0/v_t$, $a=kv_t/\omega$. The oscillation frame is written in terms of a simple continued fraction. We solve S from (15) to find

$$S_\pm=\frac{1}{a}-w\pm\left[\left(\frac{1}{a}-w\right)^2-2\,\frac{w_0}{a}\sin(\omega t-kx)\right]^{1/2}. \tag{16}$$

S_- is used for the bulk of the distribution $w<[1-(2aw_0)^{1/2}]/a$, and S_+ represents the tail $w>[1+(2aw_0)^{1/2}]/a$. It should be noted that f^h is given by the Hamiltonian of the system [see (14) and (16)]:

$$f^h=\exp[-(1/a\pm\sqrt{2H})^2], \tag{17a}$$

where

$$H=\tfrac{1}{2}(1/a-w)^2-\frac{w_0(t)}{a}\sin(\omega t-kx). \tag{17b}$$

In the frame moving with the phase velocity ω/k, H still retains an explicit time dependence through the modulation $w_0(t)$. Therefore, H is not conserved and the expression in (14) is not the proper Vlasov distribution, but only an approximation to the true solution. The fact that f is given as a function of H in terms of a continued fraction in w_0 makes it a very good approximation. This is not apparent yet and we shall discuss it in the next sections.

If $w_0(t)=\text{const}$, the solution for f in (17a) will be formally identical to the most widely used form for the distribution function of the untrapped particles in the BGK approach; for example, see Ref. 8. However, here f is not simply written as in the BGK theory. It is constructed from the evolution of a certain initial distribution. So far, this construction is based on intuition and the actual derivation will be presented in the next section. f in (17a) describes the linear theory, the dipole approximation, and represents a number of higher-order terms, as we have checked from the chain relations. However, it does not give all of the terms from the perturbation series generated by (4) and (5). Therefore, an analytic continuation through the resonant domain, which has acquired a proper width, is not possible. The continued fraction solution in (14) cannot be extended to describe the trapped particles. It fails to predict the existence of a nonlinear mode below the electron plasma frequency. In Sec. IV it will be shown that the contributions from the region of trapped particles are responsible for this mode.

Now we write f which corresponds to the boundary-value problem. By using the ideas of the oscillating frame and factorization we find f^p, where p stands for ponderomotive

$$f^p=f^h\exp\left(-\frac{w_0^2(x)}{2|1-aw-aS|^3}\right), \tag{18}$$

where the result for f^h in (14) and (16) is taken for a spatially modulated amplitude $w_0(x)$. In the limit $k\to 0$, (18) gives the distribution of Eq. (13). The effective potential (ponderomotive) in (18) is both velocity and high-frequency dependent. Although not entirely correct, the expression in (18) contains many features of the exact solution. The resulting dispersion relation predicts the existence of a nonlinear mode below the electron plasma frequency. The reason for this is that here the trapped particles do not play as significant a role as in the initial-value problem.

III. ADIABATIC KINETIC THEORY

The Vlasov equation for the initial-value problem is

$$\frac{\partial f}{\partial t}+v\,\frac{\partial f}{\partial x}-\frac{\partial\phi(t,kx-\omega t)}{\partial x}\,\frac{\partial f}{\partial v}=0. \tag{19}$$

In a frame of reference moving with the phase velocity $\tilde{x}=x-\omega/kt$, $\tilde{t}=t$, $\tilde{v}=v-\omega/k$, Eq. (19) becomes

$$\frac{\partial f}{\partial\tilde{t}}+\tilde{v}\,\frac{\partial f}{\partial\tilde{x}}-\frac{\partial\phi(\tilde{t},k\tilde{x})}{\partial\tilde{x}}\,\frac{\partial f}{\partial\tilde{v}}=0. \tag{20}$$

In the canonical formalism the evolution of the system is described by the Hamiltonian

$$H=\tfrac{1}{2}\tilde{v}^2+\phi(\tilde{t},k\tilde{x}), \tag{21}$$

where $\tilde{x}$ and $\tilde{v}$ are the canonical coordinate and momentum, respectively. Equation (20) can then be written in the form

$$\frac{df}{d\tilde{t}}=\frac{\partial f}{\partial\tilde{t}}+\{H,f\}=0, \tag{22}$$

and the Poisson bracket is defined with the convention $\{\tilde{v},\tilde{x}\}=1$. The Vlasov equation requires that f be a constant of the motion. Because of the explicit time dependence in ϕ, the energy is not conserved and f cannot depend only on H. It is well known from classical mechanics that a combination of H and the amplitude of $\phi(\phi_0)$ is conserved to all orders in the slow time variation. That quantity is the adiabatic invariant[5]

$$I=\frac{1}{\sqrt{2\pi}}\int[H-\phi(\tilde{t},k\tilde{x})]^{1/2}d\tilde{x}, \tag{23}$$

where the integral is taken over one cycle of the motion. In the three-parameter space $(\tilde{x}, \tilde{v}, \phi_0)$ $f(I)$ is a solution of the Vlasov equation when $\Omega \equiv \partial H/\partial I \gg \gamma$, where γ is the rate of change of ϕ_0. For large-amplitude waves this is satisfied in all of phase space, except for a negligible region near the separatrix. $f(I)$ is discontinuous at the separatrix. This is not surprising, since the particle motions in the bounded and the unbounded regions are qualitatively different and lead to different adiabatic invariants. For every fixed v the distribution function $f(I(v, x, t))$ is discontinuous at no more than two values of x. If we average over x, the resulting average distribution function $\langle f\rangle_x$ is continuous everywhere in velocity space. On imposing the initial condition $\phi(t_0)=0, f(t_0)=f_M$ and by using $I(t_0)=I(t)$ one finds the following explicit form for the distribution function in the region of unbounded particle motion:

$$f^{os} = \exp\left\{-\left[\frac{1}{a} - \frac{k}{v_t} I^{os}(H, \phi)\,\mathrm{sgn}\left(\frac{\omega}{k} - v\right)\right]^2\right\}. \quad (24)$$

The result is written in the laboratory frame. I^{os} is the action for the unbounded particles (os stands for oscillations). Obviously, the trapped distribution function cannot be found from an initial condition with a vanishing electric field. To determine it we define a norm (N) of the distribution function as the integral of f over phase space. The domain in phase space is periodic in $\tilde{x}$, $0 \leq k\tilde{x} < 2\pi$, and extends over all velocities $-\infty < v < \infty$. For the initial-value problem N is the number of particles in phase space. The norm is a function of the evolution parameter, which in this case is time. The Vlasov equation (22) leads to conservation of the norm $N=$const. Let θ be the canonical angle associated with the action I. Since the transformation $(\tilde{x}, \tilde{v}) \to (I, \theta)$ is canonical, one can write

$$N = \int_{V_{ph}} f(\tilde{x}, \tilde{v})\, d\tilde{x}\, d\tilde{v} = 2\pi \int_{V_I} f(I)\, dI\,. \quad (25)$$

Now let us evaluate I^{os} at the separatrix and denote the result by A. Similarly, let the value of the invariant for bounded motion (I^r) at the separatrix be B. A can be written as a function of B: $A(B)$. From (23) it is easy to determine the proper range for I^{os} and I^r. When $v \to \infty$, $I^{os} \to \infty$, and when one approaches the center of the domain of bounded motion $I^r \to 0$. By using the explicit result in (24) the invariance of the norm leads to a simple equation for the trapped distribution function (Tr stands for trapped)

$$\int_0^B f^{Tr}(I)\,dI = \frac{N}{2\pi}\left(\int_{-\infty}^{-A(B)} + \int_{A(B)}^{\infty}\right) \exp\left[-\left(\frac{1}{a} - \frac{k}{v_t} I\right)^2\right] dI\,. \quad (26)$$

By differentiating with respect to B and by substituting $B = I^r$ one obtains the trapped distribution function

$$f^{Tr}(I^r) = 2\,\frac{dA(I^r)}{dI^r} \exp\left[-\frac{1}{a^2} - \left(\frac{kA(I^r)}{v_t}\right)^2\right] \times \cosh\left(\frac{2kA(I^r)}{av_t}\right). \quad (27)$$

Given the initial condition, the distribution function is determined uniquely by (24) and (27) for all times and everywhere in phase space in terms of the corresponding action. This is in contrast to the BGK approach, where there is an infinite degree of freedom in choosing either $f^{os}(H)$ or $f^{Tr}(H)$.

To be more explicit we shall choose ϕ in the form of a plane wave:

$$\phi = \frac{\omega}{k} v_0(t) \sin(k\tilde{x})\,. \quad (28)$$

The integral in (23) can be evaluated in terms of elliptic functions. We define a parameter p:

$$p^2 = \frac{2(\omega v_0/k)}{H + \omega v_0/k}\,. \quad (29)$$

For the oscillating particles one has[9]

$$I^{os} = \frac{2\sqrt{2}}{\pi k}\left(H + \frac{\omega v_0}{k}\right)^{1/2} E(p)\,, \quad (30)$$

where $p^2 < 1$ and $E(p)$ is the complete elliptic integral of the second kind. I^{os} is the action for the nonlinear oscillator. The bounded (trapped) particles give rise to the action for the nonlinear rotor ($p^2 > 1$):

$$I^r = \frac{4\sqrt{2}}{\pi k}\left(H + \frac{\omega v_0}{k}\right)^{1/2}\left[pE\left(\frac{1}{p}\right) - \frac{p^2-1}{p} K\left(\frac{1}{p}\right)\right]. \quad (31)$$

Here K is the complete elliptic integral of the first kind. A can be found from (30) when $p \to 1$:

$$A = (4/\pi)(v_0\omega/k^3)^{1/2}\,. \quad (32)$$

Similarly, from (31) when $p \to 1$ one gets for B:

$$B = (8/\pi)(v_0\omega/k^3)^{1/2} = 2A\,. \quad (33)$$

The explicit form for the trapped distribution from (27) becomes

$$f^{Tr}(I^r) = \exp\left[-\frac{1}{a^2} - \left(\frac{kI^r}{2v_t}\right)^2\right] \cosh\left(\frac{kI^r}{av_t}\right). \quad (34)$$

Formulas (24) and (34) give the total distribution function for the initial-value problem everywhere in phase space. The adiabatic theory is valid when

$$\Omega = \pi\omega_B/K(p) \gg T^{-1}\,. \quad (35)$$

Near the separatrix[10]

$$K(p) \underset{p\to 1}{\longrightarrow} \tfrac{1}{2}\ln 16/(1-p^2). \tag{36}$$

The conditions (35) and (36) show that large-amplitude waves will be described by an adiabatic theory, i.e., the response of the plasma is local in the electric-field amplitude and is reversible. At the same time (35) sets a limit on the present theory. Obviously, the distribution function from (24) and (34) does not lead to Landau damping. This is in full agreement with Ref. 11, where f is constructed from the exact particle trajectories for a plane-wave electric field. As was shown in Ref. 11, for times t such that $\omega_B t \gg 1, \gamma_L(t) \to 0$, where $\gamma_L(t)$ is the exact Landau damping. If $\omega_B \gg \gamma_L$, then before the amplitude of the wave has changed substantially, the growth rate has phase mixed to zero. For large-amplitude waves the trapped particles are adiabatic. Since the characteristic of nonresonant particles is a reversible interaction, the trapped particles are nonresonant.

The distribution function from (24) and (30) gives the linear theory, the dipole approximation, and reproduces all terms in the perturbation series described in the preceding section. Furthermore, a detailed calculation justifies the approximation of the unbounded motion by an oscillating frame in terms of a continued fraction in v_0. If the region of trapped particles is small (small amplitudes v_0) or out in the tail of the distribution function ($v_t \ll \omega/k$) most of the particles are given by f^{∞} from (24) with the action from (30) approximated by $I^{\infty} \approx (2H)^{1/2}/k$. But this is exactly the solution in (14) and (15). For a comparison between the adiabatic theory and the perturbative approach to fourth order in the electric-field amplitude see Appendix A.

For the boundary-value problem the Vlasov equation is

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{E(x, x-(\omega/k)t)}{m}\frac{\partial f}{\partial v} = 0, \tag{37}$$

where E is the electric field for a spatially modulated wave. Let $\zeta = x - (\omega/k)t$ be the variable of the fast oscillations and let x denote the spatial modulation. We define $\psi(x,\zeta)$ such that $E(x,\xi) = -m[\partial\psi(x,\zeta)/\partial\zeta]$. Note that the quantity ψ is not the actual potential, since the derivative is taken with respect to the fast variable only. We change variables $t, x \to \zeta, x$ and Eq. (37) can be written in the form

$$v\frac{\partial f}{\partial x} + \left(v - \frac{\omega}{k}\right)\frac{\partial f}{\partial \xi} - \frac{\partial\psi(x,\xi)}{\partial\xi}\frac{\partial f}{\partial v} = 0. \tag{37$'$}$$

On making the substitution $u = v^2$ Eq. (37′) becomes

$$\frac{\partial f}{\partial x} + \frac{\sqrt{u} - (\omega/k)}{\sqrt{u}}\frac{\partial f}{\partial \xi} - 2\frac{\partial\psi(x,\xi)}{\partial\xi}\frac{\partial f}{\partial u} = 0. \tag{38}$$

The system, described by f, evolves in space and the corresponding Hamiltonian is

$$G = [(\omega/k) - \sqrt{u}]^2 + 2\psi(x,\xi). \tag{39}$$

Here ξ and u are the canonical coordinate and momentum. To within a factor of $\frac{1}{2}$, H from (21) is similar to G above, except for the difference between ϕ and ψ. The canonical variables, however, are very different, and so is the dynamics. The quantity, which is a constant of motion, is the adiabatic invariant J:

$$J = \frac{1}{2\pi}\int\left[\frac{\omega}{k} - [G - 2\psi(x,\xi)]^{1/2}\,\mathrm{sgn}\left(\frac{\omega}{k} - \sqrt{u}\right)\right]^2 d\xi. \tag{40}$$

Here the integration is over one cycle of the fast oscillation. Note the difference between I from (23) and J as given above. While the integral in (23) is taken over velocity, the action J is expressed in terms of an integral over the kinetic energy. A straightforward integration of (40) for the unbounded particles leads to

$$kJ^{\infty} = (\omega/k)^2 + G - 2\omega I^{\infty}(\tfrac{1}{2}G, \psi)\,\mathrm{sgn}[(\omega/k) - v], \tag{41}$$

where I^{∞} is the action for the free-oscillations from (23). Similarly, the adiabatic invariant for trapped particles is

$$kJ^{\tau} = 2\omega I^{\tau}(\tfrac{1}{2}G, \psi). \tag{42}$$

To within a trivial normalization factor the action for the bounded motion is formally the same as for the initial-value problem. From (42) the nonlinear spatial frequency $\Omega_s \equiv \partial G/\partial J$ is

$$\Omega_s = k\Omega/2\omega. \tag{43}$$

Here Ω is the nonlinear time frequency $\partial G/\partial I$. The requirement for the validity of the adiabatic theory is $\Omega_s \gg L^{-1}$, where L is the length of spatial modulation of the wave packet. Also, the boundary-value problem leads to $v_t \gg L/T$. These two conditions can be summed up in the following inequalities:

$$2\omega/\Omega \ll kL \ll kv_t T. \tag{44}$$

This implies well-focused waves and large electric fields. The general solution for the boundary-value distribution function is $f(J)$. If we impose at the boundary $\psi(x_B,\xi) = 0, f(x_B) = f_M$, the solution for the unbounded particles is

$$f^{\infty} = \exp\left(-\frac{kJ^{\infty}}{v_t^2}\right). \tag{45}$$

To find the trapped distribution function (f^{Tr}) we use again the principle of conservation of the norm. It should be pointed out that the canonical variables here are (v^2, ξ) and the corresponding norm is the current, not the number of particles. This is the essential difference between the two cases. Since the current is zero at $x = x_B$, one can write

$$\left(\int_{\infty}^{A_1(B_1)} + \int_{A_2(B_1)}^{\infty}\right)\exp\left(-\frac{kJ}{v_t^2}\right)dJ + \int_0^{B_1} f(J)\,dJ = 0 . \tag{46}$$

B_1 is the value of the nonlinear rotor from (42) at the separatrix. A_1 and A_2 are the values of J^{os} from (41) taken at the separatrix for $\omega/k > v$ and $\omega/k < v$, respectively. The integrals in (46) are extended over the whole domain of the corresponding adiabatic invariants. This implies that the boundary points A_1, A_2 are monotonic functions of B_1. Let us assume that for a certain B_{1c}, $dA_1/dB_1 = 0$. For $B_1 < B_{1c}$, $A_1(B_1)$ is monotonic and has covered a certain domain of the adiabatic invariant. For $B_1 > B_{1c}$, A_1 will repeat the same values it had before. To avoid the double counting we have to subtract all the contributions that have been taken previously. If for $B_1 > B_{1c}$ there are no new values of A_1 one should replace the first integral in (46) by $\int_{\infty}^{A_1(B_{1c})}$. This procedure can be extended to more complicated cases. We differentiate with respect to B_1 and substitute $B_1 = J^r$. The trapped distribution function then becomes

$$f^{\mathrm{Tr}}(J^r) = -\frac{dA_1(J^r)}{dJ^r}\exp\left(-\frac{kA_1(J^r)}{v_t^2}\right)\theta(J_c^r - J^r) + \frac{dA_2(J^r)}{dJ^r}\exp\left(-\frac{kA_2(J^r)}{v_t^2}\right). \tag{47}$$

Formulas (45) and (47) give the total boundary-value distribution function everywhere in phase space. Again discontinuity occurs at the separatrix, but for a fixed v it is at no more than two points of the fast oscillation variable. The average f is smooth everywhere.

To be explicit, we choose for ψ a plane wave of the form

$$\psi = (\omega/k)\,v_0(x)\sin(k\xi) . \tag{48}$$

Now the integral in (40) can be calculated and J^{os} is given by (41), where I^{os} is explicitly written in (30). Similarly, J^r is given by (42) and (31). At the separatrix the adiabatic invariants are

$$B_1 = \left(\frac{16\omega}{\pi k^2}\right)\left(\frac{v_0\omega}{k}\right)^{1/2}, \tag{49}$$

$$A_1 = \frac{\omega^2}{k^3} + 2\left(\frac{\pi k}{16\omega}\right)^2 kB_1^2 - \tfrac{1}{2}B_1 , \tag{50a}$$

$$A_2 = \frac{\omega^2}{k^3} + 2\left(\frac{\pi k}{16\omega}\right)^2 kB_1^2 + \tfrac{1}{2}B_1 ; \tag{50b}$$

$dA_1/dB_1 = 0$ for $B_{1c} = (1/2k)(8\omega/\pi k)^2$. We substitute (49) and (50) in (47) where $B_1 = J^r$ and the trapped distribution function becomes

$$f^{\mathrm{Tr}}(J^r) = \left[\cosh\left(\frac{kJ^r}{2v_t^2}\right) - 2\left(\frac{\pi a}{8}\right)^2 \frac{kJ^r}{v_t^2}\sinh\left(\frac{kJ^r}{2v_t^2}\right)\right] \times \exp\left[-\frac{1}{a^2} - 2\left(\frac{\pi a k J^r}{16 v_t^2}\right)^2\right], \quad J^r < \frac{1}{2k}\left(\frac{8\omega}{\pi k}\right)^2, \tag{51a}$$

$$f^{\mathrm{Tr}}(J^r) = \left[\tfrac{1}{2} + \left(\frac{\pi a}{8}\right)^2 \frac{kJ^r}{v_t^2}\right] \times \exp\left[-\frac{1}{a^2} - 2\left(\frac{\pi a k J^r}{16 v_t^2}\right)^2 - \frac{kJ^r}{2v_t^2}\right], \quad J^r > \frac{1}{2k}\left(\frac{8\omega}{\pi k}\right)^2 . \tag{51b}$$

From the explicit form for the initial-value distribution function (24) and (34), one can see a certain similarity with the expressions in (45) and (51). The relations between the corresponding adiabatic invariants in (41) and (42) lead to a factorization of the form

$$f(\text{boundary}) = f(\text{initial})\,\eta(\phi^{\mathrm{eff}}) . \tag{52}$$

ϕ^{eff} can be regarded as an effective potential, but the expression is rather involved. Only in the dipole approximation it reduces to $\exp(-\phi_p/T)$, where ϕ_p is the usual ponderomotive potential. In general, ϕ^{eff} is velocity and high-frequency dependent.

The distribution function in (45) and (41) describes linear theory, the $k = 0$ approximation, and gives all terms in the perturbation series. It shows that for a sufficiently small $a = kv_t/\omega$, the oscillation frame result in terms of a continued fraction (18) is a very good approximation. For an agreement between the adiabatic theory and the perturbation theory to fourth order in the electric-field amplitude see Appendix B.

IV. NONLINEAR MODE BELOW THE ELECTRON PLASMA FREQUENCY

The general solution for the distribution function $f(v,E)$ in terms of adiabatic invariants determines the sources in Maxwell's equations. The latter are solved with the appropriate initial or boundary conditions to find the self-consistent E. For the electrons in the one-dimensional case we have

$$\frac{\partial E}{\partial t} = 4\pi e n_0 \int_{-\infty}^{\infty} v f(v,E)\,dv , \tag{53}$$

where n_0 is the initial (unperturbed) density. Equivalently, one can use Poisson's equation. The nonlinear dispersion relation is the condition for existence of a solution for E. Let us assume that the source and the plasma properties allow for the excitation of a finite number of harmonics:

$$E=\sum_{n=1}^{N} E_n(x,t)e^{-in(\omega t-kx)}. \quad (54)$$

$E_n(x,t)$ are the amplitudes modulated in space and time. The corresponding distribution function can be written as

$$f=\sum_{n=1}^{N} f_n(v,\{E_i\})e^{-in(\omega t-kx)}, \quad (55)$$

where $f_n(v,\{E_i\})$ contain the slow space and time variation. By substituting (54) and (55) in (53) one finds a system of N equations:

$$E_n=\varphi_n(E_1,\ldots,E_N),\quad n=1,\ldots,N. \quad (56)$$

Equations (56) are homogeneous in the sense that $E_n=0$ for all $n=1,\ldots,N$ is a solution. A nontrivial solution will exist when a certain condition is satisfied and this is the nonlinear dispersion relation. For small amplitudes it reduces to the determinant of the coefficients of E_i. In practice, the amplitude of a certain harmonic is much larger than the others. We assume that this is the first harmonic and the dispersion relation reduces to

$$-i\omega E_1=4\pi e n_0\int_{-\infty}^{\infty} v f_1(v,E_1)\,dv. \quad (57)$$

We have taken into account the self-consistent response of the pump on itself and have neglected the generation of higher harmonics. The dispersion relation is a two-dimensional surface in the three-dimensional parameter space (ω,k,E_1).

For the initial-value problem the amplitude f_1 is found by averaging $f\exp[i(\omega t-kx)]$ over the fast oscillations. The expressions for f in (24) and (34) are used and I is given by (30) and (31). Both the unbounded and the trapped particles are taken into account. f_1 is a very complicated function and to evaluate the rhs of (57) analytically is a hopeless task. We should add that, in general, any component f_n is more involved than the total distribution f. The reason is that only f contains the dynamical symmetry introduced by the adiabatic invariants. This is illustrated even in the dipole approximation in Ref. 4 when f_0 is compared with the results for f in (9) and (13). The integral in (57) is done numerically, where we approximate the elliptic functions uniformly with an accuracy better than 3×10^{-5} (see Ref. 10).

For small E_1 the only wave present is the Langmuir wave. As the electric-field amplitude increases a new nonlinear mode below ω_{pe} appears. A large E_1 is needed to satisfy the basic condition $\omega_B \gg \gamma_L$, since γ_L for the lower frequencies of the nonlinear mode is large. The dispersion curves for different electric-field amplitudes have been plotted in Fig. 1. Notice the appearance of two branches. The range of k is limited and is smaller for larger E_1. This is a general feature of the nonlinear dispersion relations and is also true in the boundary-value case. The upper branch corresponds to the nonlinear Langmuir wave. The untrapped particles are responsible for this mode and the distribution function in the oscillating frame (14) and (16) is a good approximation. With f from (14) and (16) the nonlinear dispersion relation for the upper branch to all orders in v_0/v_t and to second order in $k\lambda_D$ is

$$x=1+\frac{3}{2}\frac{\lambda}{x}+\frac{5}{8}\frac{\lambda b}{x^2}, \quad (58)$$

where $x=(\omega/\omega_{pe})^2$, $b=(eE_1/m\omega_{pe}v_t)^2$, $\lambda=(k\lambda_D)^2$, with ω_{pe} being the electron plasma frequency. For a detailed calculation see Appendix C. The lower mode corresponds to the excitation of the trapped particles. The dispersion relation is approximately

$$\omega=\omega_B\equiv\left(\frac{ekE_1}{m}\right)^{1/2}. \quad (59)$$

A simple intuitive derivation of (59) is presented in Appendix D, but the formula was initially discovered by a numerical integration. In this case the number of trapped particles is greater than the number of unbounded electrons. The dielectric properties of the medium have changed in the presence of a large-amplitude E_1 and the plasma will admit such a propagating mode. This phenomenon is related to the self-induced transparency[12] and we hope that further research will shed more light on this subject. In the limit $k\lambda_D\to0$ the results for the lower branch should not be trusted, since we

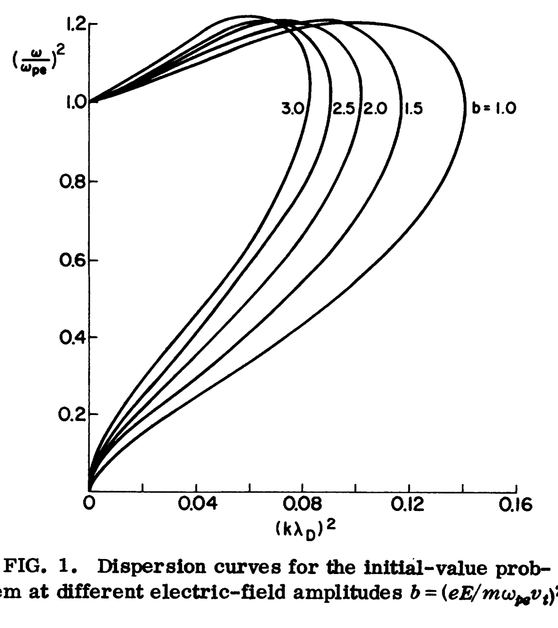

FIG. 1. Dispersion curves for the initial-value problem at different electric-field amplitudes $b=(eE/m\omega_{pe}v_t)^2$.

have neglected the ion dynamics and the analysis is restricted to $\omega \gg \omega_{pi}$.

In the initial-value problem the modes can exist for very large E_1 and there is no constraint to impose an upper limit. For the trapped particle mode there is a critical strength of the electrostatic potential, below which the mode cannot be excited. On the basis of stability considerations (see Appendix D) we determine that the threshold condition is approximately

$$\frac{e\varphi}{T} > 2.7\,, \tag{60}$$

where φ is the amplitude of the self-consistent potential ($E_1 = k\varphi$) and T is the temperature ($\frac{1}{2} m v_t^2 = T$). This is in qualitative agreement with the experimental observations in Ref. 7. One should note that the theory of electron holes[13] is unable to predict the critical potential.

To complete the analysis of the initial-value problem we find the equilibrium to which the normal modes correspond. With values for (ω, k, E_1) from Fig. 1 we calculate f_0, the distribution function averaged over the fast oscillations. We have plotted on Fig. 2, f_0 from the upper branch. Its form is almost Maxwellian, but it corresponds to a higher effective temperature. The result for the lower branch is plotted on Fig. 3. It exhibits a double hump in the bulk of the distribution function[4] and a very wide region of trapped particles. This is a nonlinear plasma state far from the initial equilibrium.

Our theory predicts that from an initial equilibrium with a Maxwellian f and an electric field $E = 0$ the system can evolve adiabatically to only two stable states. One of them corresponds to the Langmuir wave and is present even for very small amplitudes of E. Its distribution function is nearly Maxwellian. The other state is the trapped particle mode and exists only at sufficiently large amplitudes of E when most of the particles are trapped. The corresponding phase velocities of this mode are always smaller than those of the Langmuir wave. The dispersion relation is given approximately by (59) and $\omega/k \sim \sqrt{\varphi}$, where φ is the amplitude of the self-consistent potential. The distribution function is approximately f^{Tr} from (34). One can see that in phase space f^{Tr} forms a ring, peaked at $I_c^r = 2\omega/k^2$. The area of this ring is $\pi/4$ of the area enclosed by the separatrix. To see this use (33) for the boundary value of I^r at the separatrix and the dispersion relation from (59). All of these results have been confirmed experimentally in Ref. 7, at least qualitatively. We are aware that a recent theory[13] explains some of the features of the electron holes. However, it suffers from the general weakness of the BGK approach in which no evolution of the system is possible.

The distribution function for the boundary-value problem is calculated from (45) and (51), where J is given by (41) and (42) and I from (30) and (31). To find the amplitude f_1 we average $f \exp[i(\omega t - kx)]$ over the fast oscillations and the result is substituted in Eq. (57). The spatial modulation exhibits much stronger nonlinear dependence. The dispersion curves are plotted in Fig. 4 for different electric-field amplitudes. By increasing E_1 the plasma mode acquires negative nonlinear dispersion, i.e., it becomes backward. At the same time a new nonlinear mode appears at much lower frequencies. Both modes propagate not only below ω_{pe}, but below the ponderomotively depressed density: $n_p = n_0 \exp[-(v_0^2/2v_t^2)]$. In this sense there is

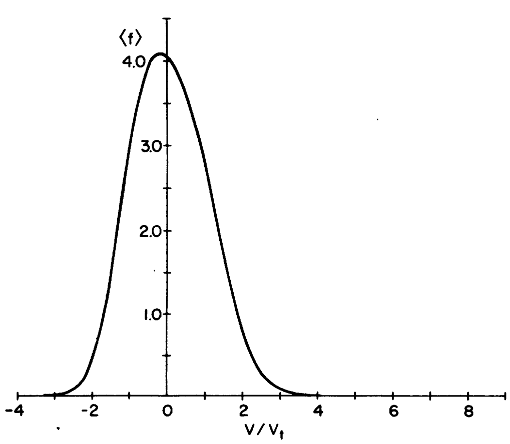

FIG. 2. The average distribution function on the upper branch from Fig. 1. The normalization is $f_M(v=0) = 2\pi$, $b = 1$, $(\omega/\omega_{pe})^2 = 1.05$, and $(k\lambda_D)^2 = 0.02$.

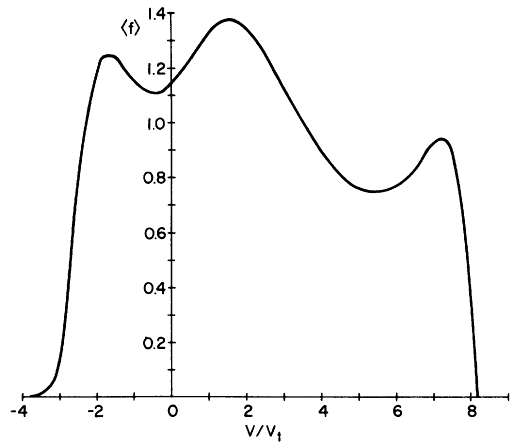

FIG. 3. The average distribution function on the lower branch from Fig. 1. The normalization is $f_M(v=0) = 2\pi$, $b = 1$, $(\omega/\omega_{pe})^2 = 0.16$, and $(k\lambda_D)^2 = 0.02$.

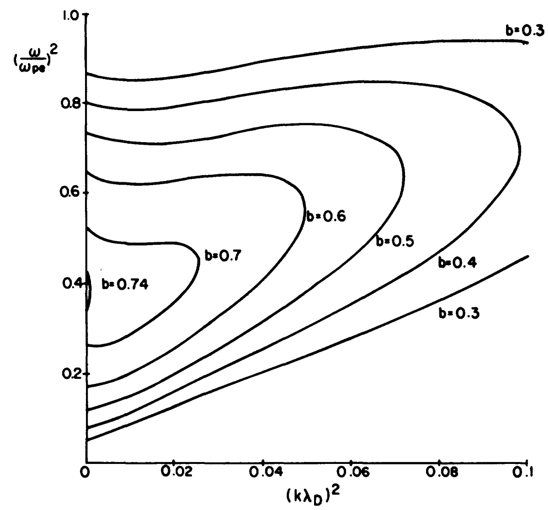

FIG. 4. Dispersion curves for the boundary-value problem for different electric-field amplitudes $b=(eE/m\omega_{pe}v_t)^2$.

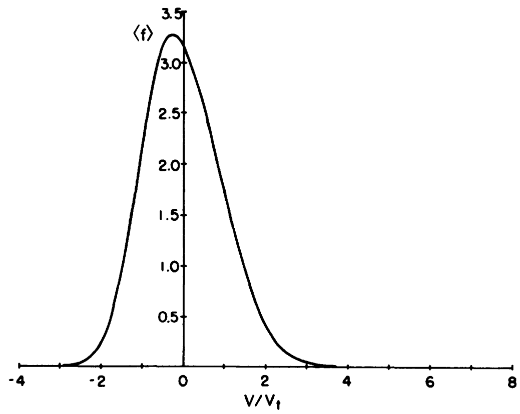

FIG. 5. The average distribution function on the upper branch from Fig. 4. The normalization is $f_M(v=0)=2\pi$, $b=0.5$, $(\omega/\omega_{pe})^2=0.75$, and $(k\lambda_D)^2=0.04$.

an effective tunneling. These nonlinear modes are related to the recently observed anomalous propagation.[14] The experiment was carried out with a CO_2 laser beam, transmitted through an overdense z-pinch plasma. While we have not taken into account an electromagnetic pump wave, and probably mixed couplings, it appears that a nonlinear mode is responsible for the nonlinear propagation. It occurs in the regime $v_0/v_t \simeq 1$, which is the one described here. For very large amplitudes $b \equiv (eE_1/m\omega_{pe}v_t)^2 > 0.74$, there is no propagating mode. For even higher amplitudes new modes at smaller frequencies appear. The region where the lower and upper branch of the dispersion curve are joined corresponds to very large v_g. One should remember, that the distribution function was evaluated explicitly with the assumption $v_g < \omega/k$. In principle, f can be written for any v_g, but the actual calculation of the dispersion relation becomes extremely complicated. By leaving v_g as a parameter in f and the integral in (57), one has to determine it from the dispersion relation, which should be corrected. A converging procedure will indicate self-consistency. At the present time this has not been done, therefore the portion of the graph where the two branches connect should not be trusted.

Once the dispersion relation is known, we determine to what kind of equilibrium the normal modes correspond. By taking particular values for (ω, k, E_1) from Fig. 4 we calculate f_0, the distribution averaged over the fast oscillations. The results are quite striking. For the upper branch f_0 has almost a Maxwellian form (see Fig. 5). This is not surprising, since the upper branch is related to the linear Langmuir wave. The lower branch, however, exhibits a double hump (see Fig. 6). This is a true nonlinear mode, corresponding to a plasma state far from the initial equilibrium.

The basic features of all these phenomena can be described by the oscillating frame approximations (14) and (18). In this case the effect of trapped particles is neglected and the picture is only qualitatively correct. The nonlinear dispersion relation for the boundary-value problem to all orders in v_0/v_t and second order in $k\lambda_D$ is (see Appendix E)

$$x=\exp\left(-\frac{b}{2x}\right)\left[1+\frac{3}{2}\frac{\lambda}{x}-\frac{9}{8}\frac{\lambda b}{x^2}\left(1-\frac{1}{12}\frac{b}{x}\right)\right]. \tag{61}$$

For $b \to 0$ we have the Langmuir wave; $\lambda \to 0$ gives the ponderomotive density depression. Equation (61) predicts correctly the critical E for the ap-

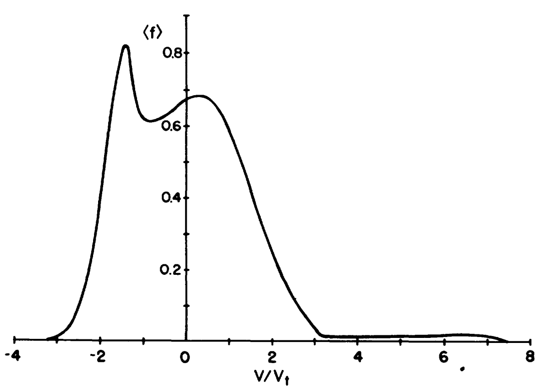

FIG. 6. The average distribution function on the lower branch from Fig. 4. The normalization is $f_M(v=0)=2\pi$, $b=0.5$, $(\omega/\omega_{pe})^2=0.20$, and $(k\lambda_D)^2=0.02$.

pearance of the nonlinear mode (lower branch) and the limit when no propagating mode is present. The boundary-value problem exhibits much stronger nonlinearities even without trapped particles. The electric-field amplitude comes in the dispersive term in (58), while (61) contains an exponential factor.

V. DISCUSSION AND SUMMARY

We have described the kinetic theory for reversible nonlinear processes in a plasma. The particles interact with a single large-amplitude wave. Our study has been limited to electrons in a one-dimensional problem. The extension of the concept of adiabatic invariants to three dimensions, an external magnetic field, and ions is, in principle, straightforward. Serious technical difficulties will most likely stand in the way. From a practical point of view these generalizations are very important. One should review the parametric processes, which in the case of large v_0/v_t have been treated correctly only for a dipole pump.[15] The adiabatic theory will allow us to find the nonlinear steady-state current and the resulting magnetic-field generation. Furthermore, the excitation of higher harmonics can be studied.

Some basic phenomena, however, have been left out. These are the irreversible processes which originate in the domain of phase space where the adiabatic approach fails. The time evolution of the adiabatic invariant must be found. Here we note the interesting relationship between the time asymptotic form of $I(t)$ and potential scattering in quantum mechanics, as studied for the harmonic oscillator in Ref. 16. Also the effect of many small-amplitude waves on the evolution of the nonlinear equilibrium has to be explored. The example in Ref. 17 illustrates the connection with stochastic processes. The particles near the separatrix are most susceptible to perturbations. These may ultimately lead to a change in the plasma state by irreversible transfer of energy and momentum from the wave to the particles.

From the Vlasov equation only the total distribution function f is a constant of the motion. Any averaging procedure will break the inherent dynamical symmetry. Therefore, an equation for $\langle f\rangle$, where $\langle\cdots\rangle$ stands for average, will include some effective terms which describe an interaction. Quantities like the diffusion coefficient will have to be estimated. These bear no basic physical significance, but may prove useful in certain regimes. This problem is similar to the one in fluid theory. An infinite set of equations has to be truncated by a clever approximation.

In our approach we explore the solutions for f in phase space, where a fundamental quantity—the adiabatic invariant—is well defined. For large-amplitude single waves this determines the plasma equilibrium. When we evolve the system from a certain initial state we find the possible stable modes. If at $t=0$, $f=f_M$, and $E=0$ there are only two stable modes in the ω_{pe} range of frequencies to which the system evolves; the Langmuir wave and the trapped particle mode. This is what was observed in the experiment described in Ref. 7. We hope that experiments in the near future will further confirm the existence of nonlinear plasma states and open the door to a vast and rich field of research.

ACKNOWLEDGMENTS

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APPENDIX A

The nonresonant perturbation expansion in powers of v_0 is valid when $v_{\rm ph}\gg v_{\rm Tr}$, where $v_{\rm ph}=\omega/k$ and $v_{\rm Tr}=(2v_0\, v_{\rm ph})^{1/2}$ is the trapping width, and when nearly all of the particles have velocities in the range $v<v_{\rm ph}-v_{\rm Tr}$. This implies that

$$p^2=\frac{4aw_0}{(1-aw)^2+2aw_0(1+\sin\xi)}\ll 1\,, \tag{A1}$$

where $a=kv_t/\omega$, $w=v/v_t$, $w_0=v_0/v_t$, $\xi=kx-\omega t$.

The adiabatic invariant for the free-particles in the initial-value problem

$$\frac{kI^\infty}{v_t}=\frac{2\sqrt{2}}{\pi v_t}\left(H+\frac{\omega v_0}{k}\right)^{1/2}E(p)\,, \tag{A2}$$

when expanded to the fourth order in v_0 gives

$$\begin{aligned}\left(\frac{kI^\infty}{v_t}\right)^{(4)}=&\operatorname{sgn}\left(w-\frac{1}{a}\right)\left(-\frac{1}{a}+w+\frac{w_0}{\alpha}(1-\beta)\right)\\&+\frac{w_0^2a}{4\alpha^3}(2\beta^2-4\beta+3)\\&+\frac{w_0^3a^2}{4\alpha^5}(-2\beta^3+6\beta^2-9\beta+5)\\&+\frac{5w_0^4a^3}{64\alpha^7}(8\beta^4-32\beta^3+72\beta^2-80\beta+35)\,,\end{aligned} \tag{A3}$$

where $\alpha = 1 - aw$ and $\beta = 1 + \sin\xi$.

In the special case of the dipole approximation $k = 0$, it is easy to see that using (A3) in (24) reproduces the oscillating frame result of (9). Substituting (A3) into the distribution function (24) and averaging over ξ terms up to v_0^4, we obtain

$$\langle f^{os}\rangle^{(4)} = \frac{\exp(-w^2)}{\sqrt{\pi}v_t} \times\left[1 + w_0^2\left(\frac{w^2}{\alpha^2} - \frac{aw}{\alpha^3} - \frac{1}{2\alpha^2}\right) + w_0^4\left(\frac{1}{4\alpha^4}(w^4 - 3w^2 + \tfrac{3}{4}) + \frac{5}{4}\frac{aw}{\alpha^5}(-w^2 + \tfrac{3}{2}) + \frac{45}{16}\frac{a^2}{\alpha^6}(w^2 - \tfrac{1}{2}) - \frac{45}{16}\frac{a^3 w}{\alpha^7}\right)\right]. \quad (A4)$$

To compare this with the perturbation expansion of the Vlasov equation for the initial-value case the chain relations (3a) and (3b) have to be expanded to write f_0 to fourth order in v_0. By extending the procedure outlined in Sec. II, we get

$$f_0^{(4)} = \left(1 + \frac{1}{4}w_0^2\nabla_w\frac{1}{\alpha^2}\nabla_w + \frac{3}{32}w_0^4\nabla_w\frac{1}{\alpha^2}\nabla_w^2\frac{1}{\alpha^2}\nabla_w - \frac{3}{64}w_0^4\nabla_w\frac{1}{\alpha^2}\nabla_w\frac{1}{\alpha}\nabla_w\frac{1}{\alpha}\nabla_w - \frac{1}{64}w_0^4\nabla_w\frac{1}{\alpha}\nabla_w\frac{1}{\alpha}\nabla_w\frac{1}{\alpha^2}\nabla_w - \frac{1}{64}w_0^4\nabla_w\frac{1}{\alpha}\nabla_w\frac{1}{\alpha^2}\nabla_w\frac{1}{\alpha}\nabla_w\right)f_M. \quad (A5)$$

It is then a matter of trivial algebra to show that the result one gets from taking all the derivatives in (A5) gives

$$f_0^{(4)} = \langle f^{os}\rangle^{(4)}.$$

APPENDIX B

We can similarly set up the perturbation expansion for the boundary-value problem. The corresponding adiabatic invariant for the free-particles is

$$\frac{kJ^{os}}{v_t^2} = \frac{1}{a^2}\left[1 + (1 - aw)^2 + 2aw_0\sin\xi - 2a\frac{kI^{os}}{v_t}\operatorname{sgn}\left(\frac{1}{a} - w\right)\right], \quad (B1)$$

with kI^{os}/v_t given by (A2). On expanding this to order v_0^4, we obtain

$$\left(\frac{kJ^{os}}{v_t}\right)^{(4)} = w^2 + 2\frac{w_0}{a}\sin\xi\left(1 - \frac{1}{\alpha}\right) + \frac{w_0^2}{\alpha^3}(\beta^2 - 2\beta + \tfrac{3}{2}) + \frac{aw_0^3}{\alpha^5}(-\beta^3 + 3\beta^2 - \tfrac{9}{2}\beta + \tfrac{5}{2}) + \frac{5a^2w_0^4}{\alpha^7}(\tfrac{1}{4}\beta^4 - \beta^3 + \tfrac{9}{4}\beta^2 - \tfrac{5}{2}\beta + \tfrac{35}{32}), \quad (B2)$$

where $\alpha = 1 - aw$ and $\beta = 1 + \sin\xi$. In the dipole limit $k \to 0$, (B2) reduces to

$$(kJ^{os}/v_t)^{(4)} \xrightarrow[k\to 0]{} (w + w_0\sin\omega t)^2 + (w_0^2/2), \quad (B3)$$

which substituted into (45) reproduces the result (13). Substituting (B2) into (45) and averaging over ξ terms up to v_0^4, we find

$$\langle f^{os}\rangle^{(4)} = \frac{\exp(-w^2)}{\sqrt{\pi}v_t}\left[1 + \frac{w_0^2}{\alpha^2}\left(-1 + w^2 - \frac{aw}{\alpha}\right) + \frac{w_0^4}{16}\left(\frac{1}{\alpha^4}(9 - 20w^2 + 4w^4) + \frac{2aw}{\alpha^5}(27 - 10w^2) + \frac{45a^2}{\alpha^6}(w^2 - 1) - \frac{45a^3w}{\alpha^7}\right)\right]. \quad (B4)$$

To compare this with the perturbation expansion of the Vlasov equation for the boundary-value problem, we expand the chain relations (3a) and (3b) to order v_0^4:

$$f_0^{(4)} = \left(1 + \frac{1}{4}w_0^2\frac{1}{w}\nabla_w\frac{w}{\alpha^2}\nabla_w + \frac{3}{32}\frac{w_0^4}{w}\nabla_w\frac{w}{\alpha^2}\nabla_w\frac{1}{w}\nabla_w\frac{w}{\alpha^2}\nabla_w - \frac{3}{64}\frac{w_0^4}{w}\nabla_w\frac{w}{\alpha^2}\nabla_w\frac{1}{\alpha}\nabla_w\frac{1}{\alpha}\nabla_w - \frac{1}{64}\frac{w_0^4}{w}\nabla_w\frac{1}{\alpha}\nabla_w\frac{1}{\alpha}\nabla_w\frac{w}{\alpha^2}\nabla_w - \frac{1}{64}\frac{w_0^4}{w}\nabla_w\frac{1}{\alpha}\nabla_w\frac{w}{\alpha^2}\nabla_w\frac{1}{\alpha}\nabla_w\right)f_M. \quad (B5)$$

Again, it is a matter of algebraic manipulations to show that (B5) is the same as (B4).

APPENDIX C

The dispersion relation (57) can be written as

$$w_0 = \frac{1}{\pi\sqrt{\pi}}\frac{\omega_{pe}^2}{\omega^2}\int_0^{2\pi}d\xi\int_{-\infty}^{\infty}dw\,wf\sin\xi, \quad (C1)$$

where f is the complete distribution function. To derive the nonresonant dispersion relation, we assume that all the electrons are free. Then f in (C1) is replaced by f^{os} of Eq. (24). We derive the initial-value dispersion relation to all orders in the electric-field amplitude v_0, but only to second order in $(k\lambda_D)$. So f^{os} is expanded to second order in $a = kv_t/\omega$. To this order

$$\left(\frac{kI^{\alpha}}{v_t}\right)^{(2)}=\operatorname{sgn}\left(w-\frac{1}{a}\right)\left[\left(-\frac{1}{a}+w-w_0\sin\xi\right)\right.$$
$$+aw_0[\tfrac{1}{2}w_0(\sin^2\xi+\tfrac{1}{2})-w\sin\xi]$$
$$+a^2w_0[-\tfrac{1}{2}w_0^2\sin\xi(\sin^2\xi+\tfrac{3}{2})$$
$$\left.+\tfrac{3}{2}ww_0(\sin^2\xi+\tfrac{1}{2})-w^2\sin\xi]\right]. \tag{C2}$$

Substituting (C2) into (24) and again expanding to order a^2, we put this f^{α} into (C1). We then derive the dispersion relation for the initial-value problem:

$$\left(\frac{\omega}{\omega_{pe}}\right)^2=1+\tfrac{3}{2}a^2+\tfrac{5}{8}a^2w_0^2. \tag{C3}$$

APPENDIX D

The Poisson equation can be written in the form

$$\frac{d^2\phi}{dx^2}=-\omega_{pe}^2\left(\int_{-\infty}^{\infty}f(v,\phi)dv-1\right), \tag{D1}$$

where ϕ is defined in (19) and f is given by (24) and (27). We can treat (D1) as the equation of motion of a particle in a "coordinate" space ϕ and a potential $V(\phi)$:

$$V(\phi)\equiv\omega_{pe}^2\left(\int_0^{\phi}d\psi\int_{-\infty}^{\infty}dv f(v,\psi)-\phi\right). \tag{D2}$$

The solution for ϕ from (D1) will be oscillatory around ϕ_0 which satisfies

$$V'(\phi_0)=0,\quad V''(\phi_0)>0. \tag{D3}$$

On the basis of the analogy to a particle motion in $V(\phi)$ one may call ϕ_0 a point of stable local equilibrium. In terms of the distribution function (D3) leads to

$$\int_{-\infty}^{\infty}dv f(v,\phi_0)=1, \tag{D4}$$

$$\int_{-\infty}^{\infty}dv\left(\frac{\partial f(v,\phi)}{\partial\phi}\right)_{\phi=\phi_0}>0. \tag{D5}$$

For a given f the stable normal modes can be found from (D4) and (D5). As we pointed out before, f is written explicitly in terms of I, not ϕ. The Poisson equation is an integrodifferential equation and little progress can be made without additional assumptions. The procedure again will be to use $f(I)$ with I^{α}, I^r given by (30) and (31), which is justified if ϕ can be approximated by a plane wave for one wavelength. To determine the trapped particle mode we approximate f by f^{Tr} from (34) and expand I^r at the bottom of the potential well: $\phi_0=-\omega v_0/k$ and $p^2>1$ [see (29)]. From (31) one can write

$$I^r\approx\frac{1}{2(\omega k v_0)^{1/2}}\left(v-\frac{\omega}{k}\right)^2. \tag{D6}$$

With the proper normalization the distribution function becomes

$$f=\frac{1}{\sqrt{\pi}v_t}\exp\left[-\frac{1}{a^2}-\frac{a}{16w_0}\left(w-\frac{1}{a}\right)^4\right]$$
$$\times\cosh\left(\frac{(w-1/a)^2}{2\sqrt{aw_0}}\right), \tag{D7}$$

where $w=v/v_t$, $a=kv_t/\omega$, $w_0=v_0/v_t$. Now we take only the exponential growing term in $\cosh(\cdots)$ and use the saddle-point method to calculate the integral in (D4). The saddle points are at

$$w_{1,2}=(1/a)[1\pm2(aw_0)^{1/4}]. \tag{D8}$$

The distribution function f in (D7) can be approximated by

$$f\approx\frac{1}{2\sqrt{\pi}v_t}\left[\exp\left(-\frac{(w-w_1)^2}{\sqrt{aw_0}}\right)+\exp\left(-\frac{(w-w_2)^2}{\sqrt{aw_0}}\right)\right]. \tag{D9}$$

By using f from (D9) in (D4) we find the following relation for the equilibrium point:

$$aw_0=1\quad\text{or}\quad\omega=\omega_B=(ekE/m)^{1/2}, \tag{D10}$$

which is exactly the dispersion relation (59).

Now we turn our attention to (D5) which will establish the condition for stability of the trapped particle mode. From (31) one obtains

$$\frac{\partial I^r}{\partial\phi}=\frac{2}{\pi\omega}K\left(\frac{1}{p}\right), \tag{D11}$$

which is simply the inverse of the nonlinear frequency evaluated at $\omega=\omega_B$. We differentiate f^{Tr} from (34) with respect to ϕ and use I^r from (D6) at $\omega=\omega_B$ and $\partial I^r/\partial\phi$ from (D11). The integrand in (D5) up to a trivial multiplication factor becomes

$$\frac{\partial f}{\partial\phi}\sim\exp\left[-\frac{1}{a^2}-\frac{a^2}{16}\left(w-\frac{1}{a}\right)^4\right]\left\{\sinh\left[\frac{1}{2}\left(w-\frac{1}{a}\right)^2\right]-\frac{a^2(w-1/a)^2}{4}\cosh\left[\frac{1}{2}\left(w-\frac{1}{a}\right)^2\right]\right\}. \tag{D12}$$

We substitute (D12) in (D5) and the numerical integration leads to the following condition for the stability of the trapped particle mode

$$a=kv_t/\omega<0.86. \tag{D13}$$

From the dispersion relation (D10) and the definition of temperature ($\frac{1}{2}mv_t^2=T$) and amplitude of the electrostatic potential ($E=k\varphi$) we find

$$e\varphi/T>2.7. \quad \text{(D14)}$$

This is consistent with the experimental observations in Ref. 7.

APPENDIX E

To derive the nonresonant dispersion relation for the boundary-value problem, we expand kJ^{os}/v_t^2 to order a^2:

$$\left(\frac{kJ^{os}}{v_t^2}\right)^{(2)}=(w^2-2ww_0\sin\xi+w_0^2\sin^2\xi+\tfrac{1}{2}w_0^2)$$
$$+a(-2w^2w_0\sin\xi+3ww_0^2\sin^2\xi+\tfrac{3}{2}ww_0^2-w_0^3\sin^3\xi-\tfrac{3}{2}w_0^3\sin\xi)$$
$$+a^2(-2w^3w_0\sin\xi+6w^2w_0^2\sin^2\xi+3w_0^2w^2-5ww_0^3\sin^3\xi-\tfrac{15}{2}ww_0^3\sin\xi$$
$$+\tfrac{5}{4}w_0^4\sin^4\xi+\tfrac{15}{4}w_0^4\sin^2\xi+\tfrac{15}{32}w_0^4). \quad \text{(E1)}$$

This expression is then substituted into (45) and the resulting distribution function is again expanded to order a^2. This final distribution function is substituted in (C1) to give the ponderomotive dispersion relation:

$$\left(\frac{\omega}{\omega_{pe}}\right)^2=\exp(-\tfrac{1}{2}w_0^2)[1+\tfrac{3}{2}a^2(1-\tfrac{3}{4}w_0^2+\tfrac{1}{16}w_0^4)]\,. \quad \text{(E2)}$$

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