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Analytic approximation of the Lorenz attractor by invariant manifolds

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The strange attractor of the Lorenz model is found to be well approximated by suitably chosen two-dimensional invariant manifolds through the three stationary points of the flow in phase space. The stationary probability density, defined by the two-dimensional flow on the invariant manifolds, is determined in the vicinity of the origin of the phase space in terms of two parameters and compared with the numerically determined stationary distribution on the Lorenz attractor.

I. INTRODUCTION

In recent years there has been much interest in expanding the scope of traditional statistical mechanics to encompass also the statistical behavior of dissipative, nonconservative systems. Such systems may be roughly divided into two classes—those which derive their statistical behavior from a coupling of their degrees of freedom to stochastic random forces, and those whose dynamics is governed by completely deterministic laws but which exhibit apparently random behavior nonetheless, because their trajectories in the steady state lie on strange attractors.[1]

Turbulent hydrodynamic systems almost certainly belong to the second class, which is therefore of great interest. Perhaps the simplest prototype system within that class, which still has some physical relevance, is the Lorenz model.[2] It was designed to describe certain features of Bénard convection[3] in high-Prandtl-number fluids. It is governed by the nonlinear differential equations

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y\,,\\ \dot{y} &= -y + rx - xz\,,\\ \dot{z} &= -bz + xy\,.\end{aligned} \tag{1.1}$$

The three variables x, y, z describe heat convection in a horizontal fluid layer heated from below. x and y are the amplitudes of streaming velocity and temperature, respectively, in a roll pattern in suitable units, while z is the second harmonic amplitude of the temperature profile, which provides for nonlinear feedback for the fundamental amplitudes x, y. The geometrical factor b is usually taken as $b=\frac{8}{3}$. The Prandtl number σ is assumed to satisfy $\sigma > b+1$. The Rayleigh number r is proportional to the total temperature difference across the fluid layer. Its numerical value controls the state of the system.

Besides Bénard convection there are other physical systems which are modelled by the Lorenz equations.[4-6] Within the last few years an abundant literature on the properties of the Lorenz model has grown up.[7-20] We now have a rather detailed knowledge of its surprisingly rich bifurcation behavior if the parameter r is varied.[13,16] Various approximation schemes of statistical mechanics have also been tried with notable success to elucidate some of the statistical features of the dynamics found in numerical studies.[18,19] However, very little progress has so far been made with a more direct statistical mechanical approach to the stationary dynamics on strange attractors. In the present paper we want to report the first steps of such an approach.

It is very tempting to consider the strange attractor in the phase space of a dissipative system in analogy to the energy hypersurface in the phase space of a conservative system, and then to generalize the statistical mechanics on the energy hypersurface to a statistical mechanics on the strange attractor. Unfortunately, such a program meets with severe difficulties. The first difficulty is the rather complicated geometrical structure of the strange attractor, which, for the Lorenz model, is locally the product of a two-dimensional manifold and a Cantor set.[1,2] Fortunately, at least for the Lorenz model, this difficulty may not be really severe from a practical point of view, since numerically the Cantor set structure is not an important geometrical feature of the Lorenz attractor. Hence, for practical purposes, all the different submanifolds of the product may be lumped into one effective two-dimensional manifold. The flow on the strange attractor is then approximated by the two-dimensional flow on that manifold. This flow is only a semiflow, since its trajectories forward in time merge with each other at some places of the manifold. This is the price one must pay for neglecting the Cantor set substructure.

The next difficulty then is that neither the Lorenz attractor as a whole nor any of its two-dimensional submanifolds is known analytically. It is the aim of the present paper to remove this difficulty by computing analytically a two-dimensional surface which approximates the Lorenz attractor found in numerical studies. This surface is constructed locally and piecewise by three differentiable invariant manifolds through the stationary points of Eqs. (1.1) for $r > 1$:

$$P_0 = (0, 0, 0)\,,$$
$$P_\pm = (\pm[b(r-1)]^{1/2}, \pm[b(r-1)]^{1/2}, r-1)\,. \tag{1.2}$$

The knowledge of these manifolds then allows us to determine the two-dimensional flow, which approximates the flow on the strange attractor.

The next step of statistical mechanics would now be the construction of a time-independent probability distribution from the properties of the flow on the hypersurface. The basic ingredients of such a construction for conservative systems are the ergodicity of the flow on the energy hypersurface, which usually has to be assumed, and the incompressibility of the flow in phase space which follows from Liouville's theorem. Both properties together prescribe the unique microcanonical distribution over the energy surface.

For the semiflow approximating the flow on the Lorenz attractor the ergodicity assumption seems also highly reasonable. A unique time-independent probability distribution over that surface must then be generated by that flow. Unfortunately, however, the flow is compressible, which makes the explicit construction of the stationary probability density very difficult. Although it is straightforward to formulate its local properties, i.e., the partial differential equation it satisfies, we have so far not been able to characterize it globally, i.e., to formulate the necessary initial conditions. Thus, in the present paper, we have to content ourselves by computing the stationary probability distribution locally in the vicinity of P_0 in terms of two unknown parameters, which may be taken as the normalization constant and the curvature of the distribution across the z axis for a given value of z. Both parameters could only be computed by global methods. Nevertheless, our results for the probability density near the boundary of the attractor in the vicinity of P_0 are definite. They show that the probability density approaches zero on the boundary of the attractor and rises with the distance from the boundary by a power law. The exponent is expressed simply in terms of the parameters of the model. These analytical results can be checked against those found numerically.

The paper is organized as follows. In Sec. II we determine three two-dimensional invariant manifolds through $P_0, P_\pm$, respectively, and show that they accurately approximate the Lorenz attractor, which is computed numerically. In Sec. III we determine the stationary probability density in the vicinity of P_0. Section IV contains the discussion of the results and our final conclusions.

II. INVARIANT MANIFOLDS CONTAINING THE STATIONARY POINTS

Let us assume that the Rayleigh number r satisfies $r > r_T$ with $r_T = \sigma(\sigma + b + 3)/(\sigma - b - 1)$. The trajectories in the steady state then lie on the Lorenz attractor. In the three-dimensional phase space of the Lorenz model we now look for two-dimensional manifolds which we write in the form

$$x = f(y, z)\,. \tag{2.1}$$

We require that locally these manifolds be left invariant by Eqs. (1.1). Differentiating Eq. (2.1) with respect to time and inserting Eqs. (1.1), we obtain the partial differential equation of f

$$(-y + rf - fz)\frac{\partial f}{\partial y} + (fy - bz)\frac{\partial f}{\partial z} + \sigma f = \sigma y\,. \tag{2.2}$$

We now consider those two-dimensional invariant manifolds which contain the stationary points P_0, $P_\pm$.

By linearizing Eqs. (1.1) in the vicinity of $P_\pm$ and writing the corresponding equation for f, one finds that within this approximation there is only one two-dimensional invariant manifold passing through $P_\pm$. This is the plane containing the two-dimensional flow, which spirals outward from these two unstable stationary points. This manifold is locally attracting and therefore stable.

A similar analysis linearized around the origin P_0 reveals that there are infinitely many two-dimensional invariant manifolds passing through that point. Of these, only three may, in general, be represented by planes in the vicinity of P_0, and two of them are attracting. Of these we chose the one which extends into the half-space $z > 0$, which is known to contain the strange attractor. These considerations now completely fix the special solutions of Eq. (2.2), which we want to determine.

We look for solutions in the form of a power-series expansion

$$f(y_0 + \eta, z_0 + \zeta) = x_0 + \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{mn}\eta^m\zeta^{n-m}\,, \tag{2.3}$$

with

$$a_{00} = 0\,, \quad (x_0, y_0, z_0) \in \{P_0, P_\pm\}\,.$$

Comparing coefficients we obtain for $n = 1$ the two closed nonlinear equations

$$
\begin{aligned}
&a_{01}^2 y_0 + (r - z_0)a_{11}a_{01} + (\sigma - b)a_{01} - x_0 a_{11} = 0\,,\\
&a_{11}^2(r - z_0) + y_0 a_{01}a_{11} + (\sigma - 1)a_{11} + x_0 a_{01} = \sigma\,.
\end{aligned} \tag{2.4}
$$

For $(x_0, y_0, z_0) = P_0$, we obtain

$$
\begin{aligned}
&a_{01} = 0\,,\\
&a_{11} = \frac{-\sigma + 1 + [(\sigma - 1)^2 + 4r\sigma]^{1/2}}{2r}\,,
\end{aligned} \tag{2.5}
$$

where the sign of the square root was determined by choosing the locally attracting invariant manifold through P_0.

The numerical values for the essential coefficients for $\sigma = 10$, $r = 28$, and $b = \frac{8}{3}$ are given in Table I. For $(x_0, y_0, z_0) = P_+$ we obtain a cubic equation

$$
\begin{aligned}
&a_{01}^3 (b(r-1))^{1/2}(br - 2b + 2)\\
&\quad + a_{01}^2 b(2b - 3\sigma + 2 + r(2\sigma - 3 - b))\\
&\qquad + a_{01}(b(r-1))^{1/2}(br + \sigma^2 + \sigma - \sigma b) - \sigma b(r-1) = 0
\end{aligned} \tag{2.6}
$$

which, in the region of interest $(r > r_T,\ \sigma > b + 1)$ has one real solution.

The coefficient a_{11} is determined in terms of a_{01} by

$$
a_{11} = \frac{(\sigma - b)a_{01} + [b(r-1)]^{1/2} a_{01}^2}{[b(r-1)]^{1/2} - a_{01}}\,. \tag{2.7}
$$

The higher-order coefficients of the power-series expansion (2.3) are determined by the linear recursion relations

$$
a_{m-1,n}c_{mn}^{(1)} + a_{mn}c_{mn}^{(2)} + a_{m+1,n}c_{mn}^{(3)} = b_{mn}\,, \tag{2.8}
$$

where

$$
n \geq 2\,, \quad n \geq m \geq 0
$$

and

$$
\begin{aligned}
c_{mn}^{(1)} &= (n - m + 1)(x_0 + y_0 a_{11})\,,\\
c_{mn}^{(2)} &= \sigma - m + (r - z_0)(m+1)a_{11} - b(n - m) + y_0(n - m + 1)a_{01}\,,\\
c_{mn}^{(3)} &= (r - z_0)(m+1)a_{01} - x_0(m+1)\,,\\
b_{mn} &= \sum_{\nu=2}^{n-1}\sum_{\mu=0}^{\nu} [(r - z_0)(\mu - m - 1)a_{m+1-\mu,n+1-\nu}a_{\mu\nu} + y_0(m - \mu + \nu - 1 - n)a_{m-\mu,n-\nu+1}a_{\mu\nu}]\\
&\quad + \sum_{\nu=1}^{n-1}\sum_{\mu=0}^{\nu} [(m + 1 - \mu)a_{m+1-\mu,n-\nu}a_{\mu\nu} + (m - \mu + \nu - 1 - n)a_{m-1-\mu,n-\nu}a_{\mu\nu}]\,.
\end{aligned} \tag{2.9}
$$

We have no information on the convergence properties of these power-series expansions but suspect that they are only asymptotic expansions. We will return to this point briefly in the concluding section. We have solved the recursion relations for the coefficients with $n \leq 10$. It turns out that within an accuracy of ~1%, the manifolds determined by the expansions around P_0 and $P_\pm$ overlap and together form a connected two-dimensional surface.

In Fig. 1 this surface is compared with the strange attractor obtained by numerically solving Eqs. (1.1). The boundary curve in Fig. 1 is formed by the trajectory flowing outward from the point P_0. The comparison shows that the numerical agreement between the three invariant

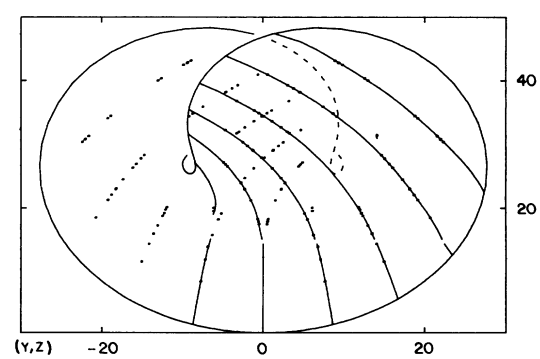

FIG. 1. Comparison of the manifolds [Eqs. (2.1) and (2.3)] for $n \leq 10$ and the Lorenz attractor. The solid lines are the contour lines $f = -4$, 0, 4, 8, 12, and 16 determined from the manifolds through P_0 (lower part) and P_+ (upper part). The dots are determined numerically and indicate the corresponding contour lines of the Lorenz attractor.

TABLE I. Parameters of the invariant manifold and the probability density near P_0 for $b = \frac{8}{3}$, $\sigma = 10$, $r = 28$.

	a_{11}	a_{12}	a_{33}
invariant manifold	4.58×10^{-1}	6.58×10^{-3}	-5.15×10^{-5}
	c_2	u_{20}	Q
probability density	1.74×10^{-2}	2.23×10^{-3}	3.43

manifolds through $P_0, P_\pm$ and the numerically determined attractor is very good. The good agreement is quite surprising in view of the fact that the invariant manifolds have been determined from local properties of Eqs. (1.1) only, while the attractor, which they approximate, is a global property.

We now investigate whether the invariant manifolds are indeed attracting within the boundary curve. To this end we introduce

$$\xi = x - f(y,z), \quad (2.10)$$

and consider

$$\dot{\xi} = \left(-\sigma - (r-z)\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right)\xi \equiv -c(y,z)\xi. \quad (2.11)$$

We have checked that indeed

$$c(y,z) > 0 \quad (2.12)$$

within the boundary curve, wherever f approximates the attractor.

III. STATIONARY PROBABILITY DENSITY

We now consider the flow described by Eqs. (1.1) restricted to the invariant manifold $x = f(y,z)$. It satisfies the equations

$$\dot{y} = -y + rf(y,z) - zf(y,z),$$
$$\dot{z} = -bz + yf(y,z). \quad (3.1)$$

The probability density P on the invariant manifold satisfies the continuity equation

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial y}[(-y + rf - zf)P] + \frac{\partial}{\partial z}[(-bz + fy)P] = 0. \quad (3.2)$$

We are here interested in the time-independent probability density satisfying $\partial P/\partial t = 0$. We assume that the system is ergodic so that the time-independent probability density is unique. Equation (3.2) specifies the variation of P along the characteristics, which are given by Eq. (3.1). Initial conditions for (3.2) would have to specify P along a curve intersecting the characteristics. Unfortunately, no such initial conditions are available here. Instead, the stationary distribution must be determined from a complicated global self-consistency condition, which expresses the uniqueness of the distribution under the flow on the invariant manifold. So far this problem has defied any solution. However, some progress can be made, if we restrict our attention to the vicinity of the point P_0. Our goal is to determine the probability density near the boundary of the attractor given by the curve

$$z = R(y).$$

The function $R(y)$ satisfies the equation

$$\frac{dR}{dy} = \frac{-bR + f(y,R)y}{-y + rf(y,R) - f(y,R)R}. \quad (3.3)$$

Near $y=0$, it is of the form

$$R(y) = c_2 y^2 + O(y^4), \quad (3.4)$$

where c_2 is obtained from our result for f:

$$c_2 = a_{11}/(b + 2ra_{11} - 2). \quad (3.5)$$

We now introduce the new coordinate

$$\zeta = z - R(y) \quad (3.6)$$

in Eq. (3.2). At the same time we take

$$P = \exp(u). \quad (3.7)$$

We obtain for small y and ζ, arbitrary but restricted to that part of the manifold through P_0 which approximates the attractor (cf. Fig. 1):

$$0 = \{[(r-\zeta)f_1 - 1 - b] + y^2[(r-\zeta)f_1' c_2 + 3f_3(r-\zeta) + f_1' - c_2 f_1] + O(y^4)\}$$
$$+ [y(r-\zeta)f_1 - y + O(y^3)]\frac{\partial u}{\partial y} + \{-b\zeta + y^2[-2c_2(r-\zeta)f_1 + 2c_2 - bc_2 + f_1] + O(y^4)\}\frac{\partial u}{\partial \zeta}, \quad (3.8)$$

where $f_1 = f_1(\zeta)$, $f_3 = f_3(\zeta)$ are defined by the power-series expansion for $f(y,z)$ around P_0:

$$f(y,z) = f_1(z)y + f_3(z)y^3 + O(y^5). \quad (3.9)$$

We look for a solution to Eq. (3.8) of the form

$$u(y,\zeta) = u_0(\zeta) + u_2(\zeta)y^2 + O(y^4), \quad (3.10)$$

with the appropriate symmetry $u(y,\zeta) = u(-y,\zeta)$. Inserting in Eq. (3.8) and comparing powers of y^2 we find

$$0 = (r-\zeta)f_1 - 1 - b - b\zeta u_0'(\zeta),$$
$$0 = G(\zeta) + [2(r-\zeta)f_1 - 2]u_2(\zeta) - b\zeta u_2'(\zeta), \quad (3.11)$$

with

$$G(\zeta) = (r-\zeta)f_1' c_2 + 3f_3(r-\zeta) + f_1' - c_2 f_1$$
$$+ [-2c_2(r-\zeta)f_1 + 2c_2 - bc_2 + f_1]$$
$$\times (1/b\zeta)[(r-\zeta)f_1 - 1 - b]. \quad (3.12)$$

The solutions are

$$u_0(\zeta)=\int_1^\zeta \frac{1}{bs}\,[(r-s)f_1(s)-1-b]ds+A\,,$$
$$u_2(\zeta)=E(\zeta)\left(\int_1^\zeta \frac{G(s)}{bsE(s)}\,ds+B\right), \tag{3.13}$$

with

$$E(\zeta)=\exp\left(\int_1^\zeta \frac{2}{bs}\,[(r-s)f_1(s)-1]ds\right), \tag{3.14}$$

where A,B are constants of integration which cannot be determined from our local calculations. A is a normalization constant, B gives the curvature of $u(y,\zeta)$ across the ζ axis for $\zeta=1$, $y=0$. Close to the boundary of the attractor we have explicitly

$$f_1=a_{11}+a_{12}\zeta+O(\zeta^2)\,,$$
$$f_3=a_{33}+O(\zeta)\,, \tag{3.15}$$

and

$$u_0(\zeta)=Q\ln\zeta+\text{const}+O(\zeta)\,,$$
$$u_2(\zeta)=-u_{20}+O(\zeta,\zeta^{2(ra_{11}-1)/b})\,, \tag{3.16}$$

with the constants

$$Q=(ra_{11}-1-b)/b\,,$$
$$u_{20}=\{1/[2(ra_{11}-1)]\}\,[c_2(ra_{12}-a_{11})(1-2Q)+a_{12}(1+Q)+3ra_{33}]\,. \tag{3.17}$$

The probability density close to the boundary of the attractor and for small y has, therefore, the form

$$P(y,z)=\text{const}(z-c_2y^2)^Q\exp(-u_{20}y^2)\,, \tag{3.18}$$

and contains only one unknown parameter, which is the normalization constant. Comparison with numerical results is made in Fig. 2.

Unfortunately, the probability density on the entire attractor cannot be determined in this way, since more and more unknown parameters would have to be introduced while extending the domain of the calculation. In principle these parameters are fixed by self-consistency conditions which express the single-valuedness of the probability density on the attractor. However, in practice, it has so far not been possible to carry through such a completely self-consistent calculation. One simple consequence of the self-consistency of the steady-state distribution is the fact that the stationary probability density P in the vicinity of $P_\pm$ vanishes for $r>r_T$. In other words, the attractor has holes around these two points. This immediately follows from the fact that the flow on the two-dimensional invariant manifolds through $P_\pm$ for $r>r_T$ spirals away from $P_\pm$ without any possibility ever to return. The only self-consistent solution near $P_\pm$ then is $P\equiv 0$, since taking $P=0$ near $P_\pm$ initially, P remains zero self-consistently at all times. Our result that P vanishes on the boundary of the attractor near P_0 implies that P vanishes along the whole boundary of the attractor, as long as this boundary is formed by the trajectory of the semiflow through P_0. This

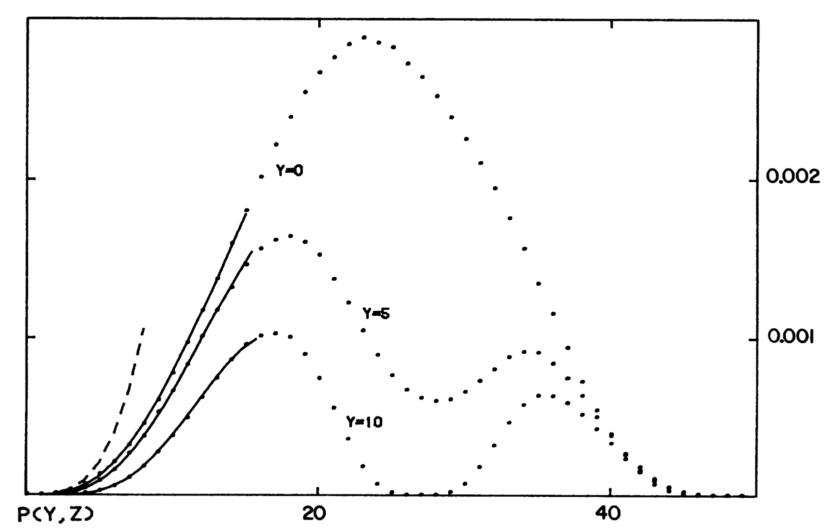

FIG. 2. Comparison of the probability density $P(y,z)$ [Eqs. (3.7), (3.10), and (3.13)] near P_0 with the numerically determined probability density (dots) for three different values of y (solid lines). The dashed line is the asymptotic form (3.18) for $y=0$. The dotted curves are determined by summing over the probability densities on all submanifolds of the attractor for fixed y,z.

result ceases to be valid when the trajectory through P_0 merges with other trajectories which carry a nonzero probability density. This happens on the boundaries encircling the points $P_\pm$.

IV. CONCLUSIONS

In this paper we have investigated the Lorenz model in its three-dimensional phase space. We have formulated a linear recursive scheme by which two-dimensional invariant manifolds through the three stationary points $P_0, P_\pm$ of the model can be determined in the form of power-series expansions. The points $P_\pm$ possess only one such invariant manifold; P_0 has infinitely many, but only one which reduces to a plane near P_0, is locally attracting and passes through the region in phase space occupied by the strange attractor. We have solved the power-series expansion of these three invariant manifolds up to tenth order. The results show that within the accuracy of the calculation, the manifolds are connected with each other and, together, form an attracting connected two-dimensional surface which approximates the Lorenz attractor, without sharing the Cantor-set substructure of the latter. The reason why the manifolds approximate the attractor is not known. It is also not known in a rigorous way whether the power-series expansion of the manifolds is convergent or asymptotic. However, the following rough argument seems to indicate that the expansion can only be asymptotic.

The coefficients of the power-series expansion around P_0 satisfy the relation [cf. Eqs. (2.8) and (2.9)]

$$a_{mn} = b_{mn}/c_{mn}^{(2)}, \tag{4.1}$$

with

$$c_{mn}^{(2)} = \sigma - m + r(m+1)a_{11} - b(n-m).$$

For sufficiently large n, m, it is possible to make $c_{mn}^{(2)}$ arbitrarily small, while b_{mn} remains finite, i.e., the coefficients a_{mn} becomes very large in high orders.

It is interesting to note that the invariant manifolds we have calculated also exist for Rayleigh numbers smaller than the critical one, $1 < r < r_T$. This is in agreement with numerical results[4,20] which indicate that a preform of the attractor for $r > r_T$ already exists in the region $r < r_T$. The principal difference of the results for $r < r_T$ as compared to the case $r > r_T$ is that the points $P_\pm$ are attracting for $r < r_T$. Thus, the steady-state probability density on the manifolds is completely concentrated in the two points $P_\pm$ for $r < r_T$, and the actual attractor consists of only these two points. For $r > r_T$ the points $P_\pm$ are repelling on their surrounding invariant two-dimensional manifolds, which is the reason why these two points are surrounded by holes in the steady-state distribution.

The complete form of the steady-state distribution cannot be obtained from local calculations of the type performed in this paper. However, the form of the distribution near the lower boundary of the attractor near P_0 could be determined locally. The reason that this was possible is as follows. The probability distribution on the z axis for not too large positive z can be calculated exactly, since this part of the phase space is decoupled from the rest, as can be seen from Eqs. (1.1). Furthermore, the form of the probability distribution in the close vicinity of the z axis follows from the assumption of differentiability and symmetry arguments. The trajectories close to the z axis are just the ones which pass by close to the lower boundary of the attractor; therefore, the knowledge of the distribution near the z axis allows us to determine the distribution close to the lower boundary.

We found that the probability distribution at the lower boundary vanishes by a power law. The exponent could be determined in terms of the parameter of the model and agrees with numerically determined values (within the statistical uncertainties of the latter). The calculation of the distribution further away from the lower boundary, but not too far from the z axis, required that we introduce a further undetermined parameter (besides the normalization constant) which would only be determined within a self-consistent calculation. Adjusting this parameter we obtained agreement with numerical calculations for larger distances from the lower boundary.

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