Theory of resonance-radiation pressure

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A general theory of the motion of a two-level atom in a resonant or near-resonant electromagnetic wave of arbitrary amplitude and phase, including effects of radiative relaxation due to interaction with the quantized vacuum field, is developed from first principles. Particular emphasis is placed on the effects of quantum-mechanical fluctuations of the radiation force and on the associated diffusion of atomic momentum due to spontaneous and induced absorption and emission processes. Analytic results and numerical examples are presented for (1) the lower bound on the temperature achievable by radiation cooling in a standing wave tuned below resonance, (2) the heating rate in a strong resonant standing wave, (3) the maximum confinement time for an atom in a Gaussian radiation trap, (4) the deflection and spreading of an atomic beam transversely illuminated by a strong resonant running wave, and (5) the transverse cooling of an atomic beam by a strong running wave tuned below resonance.

I. INTRODUCTION

The subject of atomic motion in resonant radiation is now a rapidly developing field of research. Numerous proposals have been put forward suggesting applications of the radiation force to problems as varied as isotope separation,[1-5] atomic trapping and cooling,[5-9] atomic-beam-deflection spectroscopy,[10-14] and atomic-beam epitaxy.[7] Several experiments have been carried out to verify basic features of the resonance-radiation force[15-21] and to demonstrate certain applications.[1,2,10,12] Other experiments are currently in planning or in progress at a number of laboratories and universities.

Recent developments in the theory of atomic motion in resonant radiation have resulted primarily from the analysis of specific proposed applications[22] and from the study of special problems, such as atomic motion in a plane running wave or a plane standing wave.[23-25] The resulting body of theory consequently lacks the unity of a general theory derived from first principles. A step toward a more unified approach to the theory of atomic motion in resonant radiation was taken in a recent publication[26] in which a general theory of the mean radiation force, based on Ehrenfest's theorem and the optical Bloch equations, was developed and applied to a number of problems of current experimental interest. However, a theory based on Ehrenfest's theorem describes only the mean radiation force and says nothing about the fluctuations of the force about its mean value.

The importance of fluctuations of the radiation force in determining the motion of an atom in electromagnetic radiation was first emphasized by Einstein in 1917.[27] In this early work, Einstein showed that fluctuations due to both spontaneous and induced absorption and emission processes are necessary to account for the Maxwellian distribution of atomic velocity in thermal equilibrium. Although it has been recognized for some time that fluctuations due to the random recoils accompanying spontaneous emission (spontaneous fluctuations) play an important role in certain applications, the fluctuations associated with induced absorption and emission processes (induced fluctuations) have often been ignored. Only recently has it been pointed out that induced fluctuations can be of importance in cooling, trapping, and deflection experiments.[9,28,29]

The purpose of the present paper is (1) to develop a general theory of the motion of a two-level atom in a monochromatic electromagnetic wave, including effects of spontaneous and induced fluctuations, and (2) to emphasize the central importance of induced fluctuations and the resulting diffusion of atomic momentum in a number of applications.

In the model adopted here, the atom is driven by a classically prescribed electromagnetic wave (or coherent state) of arbitrary amplitude and phase and experiences radiative relaxation due to interaction with the quantized vacuum field. A new feature of the present theory is the description of the translational motion of the atom in terms of an operator $\hat{f}(\vec{x},\vec{p})$ whose expectation value, $f(\vec{x},\vec{p}) = \langle \hat{f}(\vec{x},\vec{p})\rangle$, is the Wigner phase-space distribution function.[30] Introduction of the Wigner operator $\hat{f}(\vec{x},\vec{p})$ permits a straightforward derivation of equations for the internal and translational motions of the atom in the Heisenberg picture. This approach is convenient, particularly when the atom interacts with the quantized electromagnetic field, and physical interpretation of the resulting equations is more transparent than in an approach based on the reduced density matrix for atomic motion.

In the following section the properties of the Wigner function for a structureless point particle are briefly reviewed, the Wigner operator is defined, the Heisenberg equation of motion for the Wigner operator is derived, and the classical limit of this equation is examined. The purpose of this section is to introduce the Wigner operator, to fix the notation, and to derive a number of properties of the Wigner operator useful in subsequent calculations.

In Sec. III the theory of atomic motion in a classical electromagnetic wave is worked out neglecting radiative relaxation. It is shown that the motion of a two-level atom is determined by four real functions defined on phase space: (1) the Wigner function $f(\vec{x},\vec{p})$, (2) the distribution $U(\vec{x},\vec{p})$ of the in-phase component of the atomic dipole moment, (3) the distribution $V(\vec{x},\vec{p})$ of the in-quadrature component of the dipole moment, and (4) the distribution $W(\vec{x},\vec{p})$ of population inversion over phase space. The equations of motion for f, U, V, and W are derived, the quasiclassical limit of these equations is calculated, and the quasiclassical equations are applied to the optical Stern-Gerlach effect in this section.

In Sec. IV the semiclassical theory of Sec. III is generalized to include radiative relaxation due to interaction with the quantized electromagnetic field. A Markovian approximation eliminates field operators and leads to general equations of motion for f, U, V and W, now including relaxation terms and terms describing recoil in spontaneous emission. The quasiclassical limit of the atomic equations of motion is then derived, and the connection between the quasiclassical equations and the Ehrenfest-Bloch equations of earlier work[26] is established.

Section V treats the diffusion of atomic momentum associated with induced absorption and emission processes (induced diffusion). It is shown that in the smooth-field approximation U, V, and W can be eliminated from the coupled equations for f, U, V, and W; and the result is a Fokker-Planck equation for $f(\vec{x},\vec{p})$ which clearly displays the coefficients of induced diffusion, as well as the coefficients of spontaneous diffusion and the mean radiation force. In addition, the diffusion coefficients in a plane running wave and in a general standing wave are calculated, physical interpretations of induced diffusion are given, and formulas for the mean energy and momentum transfer to the atom are derived in this section.

In Sec. VI we present a weak-field solution to the quasiclassical equations, and again find that U, V, and W can be eliminated and a single Fokker-Planck equation can be written for $f(\vec{x},\vec{p})$. The weak-field theory is applicable to a number of problems not covered by the smooth-field approximation of Sec. V. Specifically, the damping of atomic motion by a standing wave tuned below resonance and the lower bound on the temperature achievable by radiation cooling are calculated in this section.

In Sec. VII the foregoing theory is further illustrated by application to several simple one-dimensional problems. Analytical results and numerical examples are presented for: (1) the heating rate in a strong resonant standing wave, (2) the maximum confinement time for an atom in a Gaussian radiation trap, (3) the deflection and spreading of an atomic beam transversely illuminated by a strong resonant plane running wave, and (4) the transverse cooling of an atomic beam by a strong running wave tuned below resonance.

The paper concludes in Sec. VIII with a discussion of the limitations of the quasiclassical equations and some remarks about problems remaining to be solved.

II. THE WIGNER OPERATOR

The Wigner function $f(\vec{x},\vec{p})$ describing the quantum-mechanical state of a structureless point particle[30] is traditionally defined by the equation

$$f(\vec{x},\vec{p})=(2\pi\hbar)^{-3}\int d^3s\,\psi(\vec{x}+\tfrac{1}{2}\vec{s})\psi^*(\vec{x}-\tfrac{1}{2}\vec{s})e^{-i\vec{p}\cdot\vec{s}/\hbar}\,, \quad (1)$$

where $\psi(\vec{x})=\langle\vec{x}\,|\psi\rangle$ is the position-representation wave function. A transformation to the momentum representation

$$\psi(\vec{x})=(2\pi\hbar)^{-3/2}\int d^3p\,\phi(\vec{p})e^{i\vec{p}\cdot\vec{x}/\hbar} \quad (2)$$

yields the formula

$$f(\vec{x},\vec{p})=(2\pi\hbar)^{-3}\int d^3q\,\phi(\vec{p}+\tfrac{1}{2}\vec{q})\phi^*(\vec{p}-\tfrac{1}{2}\vec{q})e^{i\vec{x}\cdot\vec{q}/\hbar} \quad (3)$$

for the Wigner function in terms of the momentum-representation wave function $\phi(\vec{p})=\langle\vec{p}\,|\psi\rangle$.

The Wigner function has many of the properties of the phase-space distribution function of classical statistical mechanics. For example, the integral of (1) over momentum space

$$\int d^3p\,f(\vec{x},\vec{p})=\psi(\vec{x})\psi^*(\vec{x})=P(\vec{x}) \quad (4)$$

is the probability density $P(\vec{x})$ for position, the integral of (3) over configuration space,

$$\int d^3x\,f(\vec{x},\vec{p})=\phi(\vec{p})\phi^*(\vec{p})=W(\vec{p}) \quad (5)$$

is the probability density $W(\vec{p})$ for momentum, and the integral of $f(\vec{x},\vec{p})$ over all of phase space is unity. The Wigner function cannot, however, be strictly interpreted as the joint probability density for the position and momentum of the particle because it is not always positive definite.

We now define the Wigner operator in the position representation:

$$\hat{f}(\vec{x},\vec{p})=(2\pi\hbar)^{-3}\int d^3s\,|\vec{x}-\tfrac{1}{2}\vec{s}\rangle\langle\vec{x}+\tfrac{1}{2}\vec{s}|\,e^{-i\vec{p}\cdot\vec{s}/\hbar}. \qquad (6)$$

The expectation value of the Wigner operator is clearly the Wigner function

$$f(\vec{x},\vec{p})=\langle\psi|\hat{f}(\vec{x},\vec{p})|\psi\rangle. \qquad (7)$$

In terms of the momentum basis $|\vec{p}\rangle$

$$\hat{f}(\vec{x},\vec{p})=(2\pi\hbar)^{-3}\int d^3q\,|\vec{p}-\tfrac{1}{2}\vec{q}\rangle\langle\vec{p}+\tfrac{1}{2}\vec{q}|\,e^{i\vec{x}\cdot\vec{q}/\hbar}. \qquad (8)$$

Note that $\hat{f}(\vec{x},\vec{p})$ is Hermitian.

Now Eq. (6) can be inverted to obtain

$$|\vec{x}_1\rangle\langle\vec{x}_2|=\int d^3p\,\hat{f}(\tfrac{1}{2}(\vec{x}_1+\vec{x}_2),\vec{p})e^{i(\vec{x}_2-\vec{x}_1)\cdot\vec{p}/\hbar}, \qquad (9)$$

and an arbitrary operator $\hat{A}$ can be expanded in the position representation as

$$\hat{A}=\iint d^3x_1d^3x_2\,|\vec{x}_1\rangle\langle\vec{x}_1|\hat{A}|\vec{x}_2\rangle\langle\vec{x}_2|. \qquad (10)$$

Using (9) in (10) we find that an arbitrary operator may be expanded in terms of the Wigner operator as

$$\hat{A}=\iint d^3x\,d^3p\,A(\vec{x},\vec{p})\hat{f}(\vec{x},\vec{p}), \qquad (11)$$

where

$$A(\vec{x},\vec{p})=\int d^3s\langle\vec{x}+\tfrac{1}{2}\vec{s}|\hat{A}|\vec{x}-\tfrac{1}{2}\vec{s}\rangle e^{-i\vec{p}\cdot\vec{s}/\hbar}. \qquad (12)$$

or in the momentum representation

$$A(\vec{x},\vec{p})=\int d^3q\langle\vec{p}+\tfrac{1}{2}\vec{q}|\hat{A}|\vec{p}-\tfrac{1}{2}\vec{q}\rangle e^{i\vec{x}\cdot\vec{q}/\hbar}. \qquad (13)$$

When $\hat{A}$ is Hermitian $A(\vec{x},\vec{p})$ is real. Incidentally, the expectation value of Eq. (11)

$$\langle\hat{A}\rangle=\iint d^3x\,d^3p\,A(\vec{x},\vec{p})f(\vec{x},\vec{p}) \qquad (14)$$

expresses the expectation value of the observable $\hat{A}$ in the form of a statistical average, as if $\hat{A}$ were properly represented by the classical observable $A(\vec{x},\vec{p})$ and $f(\vec{x},\vec{p})$ were a valid distribution function in phase space.

If $\hat{A}=F(\hat{\vec{x}})$ is a function of the position operator, then $\langle\vec{x}_1|\hat{A}|\vec{x}_2\rangle=F(\vec{x}_1)\delta(\vec{x}_1-\vec{x}_2)$, and (12) gives $A(\vec{x},\vec{p})=F(\vec{x})$. Similarly, if $\hat{A}=G(\hat{\vec{p}})$, we have $\langle\vec{p}_1|\hat{A}|\vec{p}_2\rangle=G(\vec{p}_1)\delta(\vec{p}_1-\vec{p}_2)$, and (13) gives $A(\vec{x},\vec{p})=G(\vec{p})$. It follows that a Hamiltonian of the form $\hat{H}=\hat{\vec{p}}^2/2m+V(\hat{\vec{x}})$ is expressible in terms of the Wigner operator as

$$\hat{H}=\iint d^3x\,d^3p[p^2/2m+V(\vec{x})]\hat{f}(\vec{x},\vec{p}). \qquad (15)$$

To evaluate the Heisenberg equation of motion for the Wigner operator,

$$\frac{\partial\hat{f}(\vec{x},\vec{p})}{\partial t}=\frac{1}{i\hbar}[\hat{f}(\vec{x},\vec{p}),\hat{H}]$$
$$=(i\hbar)^{-1}\iint d^3x'\,d^3p'[p'^2/2m+V(\vec{x}')]\times[\hat{f}(\vec{x},\vec{p}),\hat{f}(\vec{x}',\vec{p}')], \qquad (16)$$

we need the commutator $[\hat{f}(\vec{x}_1,\vec{p}_1),\hat{f}(\vec{x}_2,\vec{p}_2)]$ for Wigner operators at two distinct points of phase space. The position-representation matrix elements of the product $\hat{f}(\vec{x}_1,\vec{p}_1)\hat{f}(\vec{x}_2,\vec{p}_2)$ are readily evaluated using definition (6). Upon substituting these matrix elements into Eq. (12) and using the result in Eq. (11), we get

$$\hat{f}(\vec{x}_1,\vec{p}_1)\hat{f}(\vec{x}_2,\vec{p}_2)=(\pi\hbar)^{-6}\iint d^3x\,d^3p\,\hat{f}(\vec{x},\vec{p})\exp\{-2i[\vec{\Delta}\cdot(\vec{\xi}-\vec{x})-\vec{\eta}\cdot(\vec{\pi}-\vec{p})]/\hbar\}, \qquad (17)$$

where

$$\vec{\xi}=\tfrac{1}{2}(\vec{x}_1+\vec{x}_2),\quad \vec{\pi}=\tfrac{1}{2}(\vec{p}_1+\vec{p}_2),\quad \vec{\eta}=\vec{x}_1-\vec{x}_2,\quad \vec{\Delta}=\vec{p}_1-\vec{p}_2. \qquad (18)$$

The commutator of $\hat{f}(\vec{x}_1,\vec{p}_1)$ and $\hat{f}(\vec{x}_2,\vec{p}_2)$ then follows directly from (17),

$$[\hat{f}(\vec{x}_1,\vec{p}_1),\hat{f}(\vec{x}_2,\vec{p}_2)]=-2i(\pi\hbar)^{-6}\iint d^3x\,d^3p\,\hat{f}(\vec{x},\vec{p})\sin\{2[\vec{\Delta}\cdot(\vec{\xi}-\vec{x})-\vec{\eta}\cdot(\vec{\pi}-\vec{p})]/\hbar\}. \qquad (19)$$

With the help of this commutation relation, we find that the Heisenberg equation of motion for the Wigner operator, Eq. (16), takes the form

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{m}\cdot\vec{\nabla}\right)\hat{f}(\vec{x},\vec{p})=\int d^3p'\,J(\vec{x},\vec{p}-\vec{p}')\hat{f}(\vec{x},\vec{p}'), \qquad (20)$$

where

$$J(\vec{x},\vec{p})=\frac{1}{i\hbar(2\pi\hbar)^3}\int d^3s[V(\vec{x}+\tfrac{1}{2}\vec{s})-V(\vec{x}-\tfrac{1}{2}\vec{s})]e^{-i\vec{p}\cdot\vec{s}/\hbar}, \qquad (21)$$

and $\vec{\nabla}$ is the gradient with respect to $\vec{x}$. An expectation average of Eq. (20) removes the carets on $\hat{f}(\vec{x},\vec{p})$ and $\hat{f}(\vec{x},\vec{p}')$, and shows that the Wigner function and Wigner operator satisfy the same integrodifferential equation.

The classical limit of Eqs. (20) and (21) is derived by writing $V(\vec{x}\pm\frac{1}{2}\vec{s})$ in (21) as Taylor series in the variable $\vec{s}$, evaluating the integral over $\vec{s}$, and taking the limit of the result as $\hbar\to 0$. Equation (21) becomes

$$J(\vec{x},\vec{p}) = \vec{\nabla}V(\vec{x})\cdot\vec{\nabla}_p\delta(\vec{p}), \tag{22}$$

where $\vec{\nabla}_p$ is the gradient with respect to $\vec{p}$, and (20) reduces to

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{m}\cdot\vec{\nabla}\right)\hat{f}(\vec{x},\vec{p}) = \vec{\nabla}V(\vec{x})\cdot\vec{\nabla}_p\hat{f}(\vec{x},\vec{p}). \tag{23}$$

The expectation value of Eq. (23) is the classical Liouville equation for the distribution $f(\vec{x},\vec{p})$.

We now list a number of relations involving the Wigner operator that are useful in subsequent calculations. First, the commutator of the Wigner operator with the kinetic energy operator is

$$[\hat{f}(\vec{x},\vec{p}),\hat{p}^2/2m] = -i\hbar\vec{p}\cdot\vec{\nabla}\hat{f}(\vec{x},\vec{p})/m. \tag{24}$$

Secondly, if $F(\vec{x})$ is a complex function of position, and if

$$R(\vec{x},\vec{p}) = \frac{1}{i\hbar(2\pi\hbar)^3}\int d^3s\,F(\vec{x}+\tfrac{1}{2}\vec{s})e^{-i\vec{p}\cdot\vec{s}/\hbar}, \tag{25}$$

then

$$\iint d^3x'\,d^3p'\,F(\vec{x}')\hat{f}(\vec{x},\vec{p})\hat{f}(\vec{x}',\vec{p}') = i\hbar\int d^3p'\,R(\vec{x},\vec{p}-\vec{p}')\hat{f}(\vec{x},\vec{p}'), \tag{26}$$

$$\iint d^3x'\,d^3p'\,F^*(\vec{x}')\hat{f}(\vec{x},\vec{p})\hat{f}(\vec{x}',\vec{p}') = -i\hbar\int d^3p'\,R^*(\vec{x},\vec{p}'-\vec{p})\hat{f}(\vec{x},\vec{p}'), \tag{27}$$

$$\iint d^3x'\,d^3p'\,F(\vec{x}')\hat{f}(\vec{x}',\vec{p}')\hat{f}(\vec{x},\vec{p}) = i\hbar\int d^3p'\,R(\vec{x},\vec{p}'-\vec{p})\hat{f}(\vec{x},\vec{p}'), \tag{28}$$

$$\iint d^3x'\,d^3p'\,F^*(\vec{x}')\hat{f}(\vec{x}',\vec{p}')\hat{f}(\vec{x},\vec{p}) = -i\hbar\int d^3p'\,R^*(\vec{x},\vec{p}-\vec{p}')\hat{f}(\vec{x},\vec{p}'). \tag{29}$$

These relations can be proved by straightforward application of Eq. (17) and simple changes of integration variables.

III. ATOMIC MOTION IN A CLASSICAL ELECTROMAGNETIC WAVE

The Hamiltonian for an atom of mass M in a classically prescribed electromagnetic wave in the electric dipole approximation is

$$\hat{H} = \hat{p}^2/2M + \hat{H}_0 - \hat{\vec{\mu}}\cdot\vec{E}_{cl}(\hat{\vec{x}},t), \tag{30}$$

where $\hat{p}^2/2M$ is the kinetic energy associated with the center-of-mass momentum $\hat{\vec{p}}$, $\hat{H}_0$ is the Hamiltonian for the internal motion of the unperturbed atom, $\hat{\vec{\mu}}$ is the electric dipole-moment operator, and $\vec{E}_{cl}(\hat{\vec{x}},t)$ is the classical electric field evaluated at the center-of-mass position $\hat{\vec{x}}$.

For a two-level atom with internal states $|1\rangle$ and $|2\rangle$ of energy $E_1=0$ and $E_2=\hbar\omega_0$, respectively, the internal Hamiltonian and dipole-moment operator can be written as

$$\hat{H}_0 = \hbar\omega_0\hat{S}^+\hat{S}, \tag{31}$$

$$\hat{\vec{\mu}} = \vec{\mu}\hat{S} + \vec{\mu}^*\hat{S}^+, \tag{32}$$

where $\vec{\mu} = \langle 1|\hat{\vec{\mu}}|2\rangle$ and $\hat{S}^+$ and $\hat{S}$ are, respectively, the atomic excitation and deexcitation operators

$$\hat{S}^+ = |2\rangle\langle 1|, \quad \hat{S} = |1\rangle\langle 2|. \tag{33}$$

In addition, we shall need the operator

$$\hat{S}_3 = |2\rangle\langle 2| - |1\rangle\langle 1| = \hat{S}^+\hat{S} - \hat{S}\hat{S}^+ \tag{34}$$

whose expectation value is the population inversion. The internal operators $\hat{S}$, $\hat{S}^+$, and $\hat{S}_3$ satisfy the following commutation and product relations:

$$[\hat{S}^+,\hat{S}] = \hat{S}_3, \tag{35a}$$

$$[\hat{S},\hat{S}^+\hat{S}] = \hat{S}, \tag{35b}$$

$$[\hat{S}^+,\hat{S}^+\hat{S}] = -\hat{S}^+, \tag{35c}$$

$$[\hat{S}_3,\hat{S}^+\hat{S}] = 0, \tag{35d}$$

$$\hat{S}\hat{S}^+ = \tfrac{1}{2}(\hat{I}-\hat{S}_3), \tag{36a}$$

$$\hat{S}^+\hat{S} = \tfrac{1}{2}(\hat{I}+\hat{S}_3), \tag{36b}$$

$$\hat{S}^2 = \hat{S}^{+2} = 0, \tag{36c}$$

$$\hat{S}\hat{S}_3 = -\hat{S}_3\hat{S} = \hat{S}, \tag{36d}$$

$$\hat{S}^+\hat{S}_3 = -\hat{S}_3\hat{S}^+ = -\hat{S}^+, \tag{36e}$$

where $\hat{I} = \hat{S}^+\hat{S} + \hat{S}\hat{S}^+$ is the identity operator.

A general monochromatic electric field may be written as

$$\vec{E}_{cl}(\vec{x},t) = \tfrac{1}{2}[\vec{E}(\vec{x})e^{i\omega t} + \vec{E}^*(\vec{x})e^{-i\omega t}]. \tag{37}$$

In the Heisenberg picture, the dominant time dependence of operators $\hat{S}$ and $\hat{S}^+$, which is due to the internal Hamiltonian (31), is contained in exponential factors $e^{-i\omega_0 t}$ and $e^{i\omega_0 t}$, respectively. Therefore, near resonance ($\omega\approx\omega_0$) substitution of (32) and (37) into the interaction term $-\hat{\vec{\mu}}\cdot\vec{E}_{cl}(\hat{\vec{x}},t)$ yields two slowly varying terms and two terms that vary too rapidly to have any significant influence on atomic motion. Discarding the inessential terms (rotating-wave approximation),[31] the Hamiltonian becomes

$$\hat{H} = \hat{p}^2/2M + \hbar\omega_0\hat{S}^+\hat{S} - \tfrac{1}{2}\hbar[\tilde{\Omega}^*(\hat{\vec{x}})S^+e^{-i\omega t} + \tilde{\Omega}(\hat{\vec{x}})\hat{S}e^{i\omega t}], \tag{38}$$

where $\hbar\tilde{\Omega}(\vec{x}) = \vec{\mu}\cdot\vec{E}(\vec{x}) = \hbar\Omega(\vec{x})e^{i\theta(\vec{x})}$. $\Omega(\vec{x})$ is the on-resonance Rabi flopping frequency of the two-level atom, and $\theta(\vec{x})$ is the effective phase of the applied field [for $\vec{\mu}$ real, $\Omega(\vec{x})$ is the amplitude in frequency units and $\theta(\hat{\bar{x}})$ is the true phase of the component of the applied field in direction $\vec{\mu}$]. The operator $\tilde{\Omega}(\hat{\bar{x}})$ is expressed in terms of the Wigner operator as

$$\tilde{\Omega}(\hat{\bar{x}})=\iint d^3x\, d^3p\, \tilde{\Omega}(\bar{x})\hat{f}(\bar{x},\bar{p}), \tag{39}$$

and hence

$$\hat{H}=\hat{p}^2/2M+\hbar\omega_0\hat{S}^+\hat{S}-\tfrac{1}{2}\hbar\iint d^3x\, d^3p[\tilde{\Omega}^*(\bar{x})\hat{S}^+\hat{f}(\bar{x},\bar{p})e^{-i\omega t}+\tilde{\Omega}(\bar{x})\hat{S}\hat{f}(\bar{x},\bar{p})e^{i\omega t}]. \tag{40}$$

Here the Wigner operator refers to the translational or center-of-mass motion of the atom. Since the internal and translational motions involve different and independent degrees of freedom, each of the internal operators $\hat{S}$, $\hat{S}^+$, and $\hat{S}_3$ commutes with the Wigner operator $\hat{f}(\bar{x},\bar{p})$.

Now consider the Heisenberg equation of motion for $\hat{f}(\bar{x},\bar{p})$

$$\frac{\partial\hat{f}(\bar{x},\bar{p})}{\partial t}=\frac{1}{i\hbar}[\hat{f}(\bar{x},\bar{p}),\hat{H}]. \tag{41}$$

After inserting $\hat{H}$ from (40), a straightforward calculation aided by Eqs. (24)–(29), leads to the result

$$\left(\frac{\partial}{\partial t}+\frac{\bar{p}}{M}\cdot\vec{\nabla}\right)\hat{f}(\bar{x},\bar{p})=\int d^3p'\{[T(\bar{x},\bar{p}'-\bar{p})-T(\bar{x},\bar{p}-\bar{p}')]\hat{S}\hat{f}(\bar{x},\bar{p}')e^{i\omega t}+[T^*(\bar{x},\bar{p}'-\bar{p})-T^*(\bar{x},\bar{p}-\bar{p}')]\hat{S}^+\hat{f}(\bar{x},\bar{p}')e^{-i\omega t}\}, \tag{42}$$

where

$$T(\bar{x},\bar{p})=\frac{1}{2i(2\pi\hbar)^3}\int d^3s\,\tilde{\Omega}(\bar{x}+\tfrac{1}{2}\bar{s})e^{-i\bar{p}\cdot\bar{s}/\hbar}. \tag{43}$$

We see that the equation for $\hat{f}(\bar{x},\bar{p})$ contains the operator

$$\hat{\Pi}(\bar{x},\bar{p})=\hat{S}\hat{f}(\bar{x},\bar{p})e^{i\omega t} \tag{44}$$

and its Hermitian conjugate $\hat{\Pi}^\dagger$. Therefore, in addition to the equation for $\hat{f}(\bar{x},\bar{p})$, we need an equation of motion for $\hat{\Pi}(\bar{x},\bar{p})$. Another straightforward calculation, making use of Eqs. (24)–(29) and Eqs. (35) and (36), shows that the Heisenberg equation for $\hat{\Pi}(\bar{x},\bar{p})$ contains the operator

$$\hat{\Pi}_3(\bar{x},\bar{p})=\hat{S}_3\hat{f}(\bar{x},\bar{p}). \tag{45}$$

So an equation for this operator is required also. Fortunately, the proliferation of operators terminates at this point, and the Heisenberg equation for $\hat{\Pi}_3$ involves only operators $\hat{\Pi}$ and $\hat{\Pi}^\dagger$. The closed system of Heisenberg equations of motion obtained in this way is

$$\left(\frac{\partial}{\partial t}+\frac{\bar{p}}{M}\cdot\vec{\nabla}\right)\hat{f}(\bar{x},\bar{p})=\int d^3p'[C(\bar{x},\bar{p}-\bar{p}')\hat{\Pi}(\bar{x},\bar{p}')+C^*(\bar{x},\bar{p}-\bar{p}')\hat{\Pi}^\dagger(\bar{x},\bar{p}')], \tag{46}$$

$$\left(\frac{\partial}{\partial t}+\frac{\bar{p}}{M}\cdot\vec{\nabla}\right)\hat{\Pi}(\bar{x},\bar{p})=i\Delta\hat{\Pi}(\bar{x},\bar{p})+\tfrac{1}{2}\int d^3p'[C^*(\bar{x},\bar{p}-\bar{p}')\hat{f}(\bar{x},\bar{p}')-B^*(\bar{x},\bar{p}-\bar{p}')\hat{\Pi}_3(\bar{x},\bar{p}')], \tag{47}$$

$$\left(\frac{\partial}{\partial t}+\frac{\bar{p}}{M}\cdot\vec{\nabla}\right)\hat{\Pi}_3(\bar{x},\bar{p})=\int d^3p'[B(\bar{x},\bar{p}-\bar{p}')\hat{\Pi}(\bar{x},\bar{p}')+B^*(\bar{x},\bar{p}-\bar{p}')\hat{\Pi}^\dagger(\bar{x},\bar{p}')], \tag{48}$$

where $\Delta=\omega-\omega_0$ is the detuning frequency and

$$C(\bar{x},\bar{p})=T(\bar{x},-\bar{p})-T(\bar{x},\bar{p}),\quad B(\bar{x},\bar{p})=T(\bar{x},-\bar{p})+T(\bar{x},\bar{p}). \tag{49}$$

Let Π, $\Pi^\dagger$, and Π_3 (without carets) represent the expectation values of $\hat{\Pi}$, $\hat{\Pi}^\dagger$, and $\hat{\Pi}_3$, respectively. Then we can remove the carets in Eqs. (46)–(48), by taking expectation values, to obtain the system of equations describing the time development of the Wigner function $f(\bar{x},\bar{p})$ in a monochromatic field with arbitrary amplitude $\Omega(\bar{x})$ and phase $\theta(\bar{x})$.

The following transformation of Eqs. (46)–(48) is convenient for subsequent work. Let

$$\Pi(\bar{x},\bar{p})=\sigma(\bar{x},\bar{p})e^{-i\theta(\bar{x})},\quad \Pi^\dagger(\bar{x},\bar{p})=\sigma^+(\bar{x},\bar{p})e^{i\theta(\bar{x})}, \tag{50}$$

and

$$U(\bar{x},\bar{p})=\sigma(\bar{x},\bar{p})+\sigma^+(\bar{x},\bar{p}),\quad V(\bar{x},\bar{p})=i[\sigma(\bar{x},\bar{p})-\sigma^+(\bar{x},\bar{p})],\quad W(\bar{x},\bar{p})=\Pi_3(\bar{x},\bar{p}). \tag{51}$$

Then Eqs. (46)–(48), with carets deleted, become

$$\left(\frac{\partial}{\partial t}+\frac{\bar{p}}{M}\cdot\vec{\nabla}\right)f(\bar{x},\bar{p})=\int d^3p'[a_r(\bar{x},\bar{p}-\bar{p}')U(\bar{x},\bar{p}')+a_i(\bar{x},\bar{p}-\bar{p}')V(\bar{x},\bar{p}')], \tag{52}$$

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)W(\vec{x},\vec{p})=\int d^3p'[b_r(\vec{x},\vec{p}-\vec{p}')U(\vec{x},\vec{p}')+b_i(\vec{x},\vec{p}-\vec{p}')V(\vec{x},\vec{p}')]\,, \tag{53}$$

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)U(\vec{x},\vec{p})=\left(\Delta+\frac{\vec{p}}{M}\cdot\vec{\nabla}\theta\right)V(\vec{x},\vec{p})+\int d^3p'[a_r(\vec{x},\vec{p}-\vec{p}')f(\vec{x},\vec{p}')-b_r(\vec{x},\vec{p}-\vec{p}')W(\vec{x},\vec{p}')]\,, \tag{54}$$

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)V(\vec{x},\vec{p})=-\left(\Delta+\frac{\vec{p}}{M}\cdot\vec{\nabla}\theta\right)U(\vec{x},\vec{p})+\int d^3p'[a_i(\vec{x},\vec{p}-\vec{p}')f(\vec{x},\vec{p}')-b_i(\vec{x},\vec{p}-\vec{p}')W(\vec{x},\vec{p}')]\,, \tag{55}$$

where

$$\begin{aligned}
a_r(\vec{x},\vec{p})&=\mathrm{Re}[\tau(\vec{x},-\vec{p})-\tau(\vec{x},\vec{p})]\,,\\
a_i(\vec{x},\vec{p})&=\mathrm{Im}[\tau(\vec{x},-\vec{p})-\tau(\vec{x},\vec{p})]\,,\\
b_r(\vec{x},\vec{p})&=\mathrm{Re}[\tau(\vec{x},-\vec{p})+\tau(\vec{x},\vec{p})]\,,\\
b_i(\vec{x},\vec{p})&=\mathrm{Im}[\tau(\vec{x},-\vec{p})+\tau(\vec{x},\vec{p})]\,,
\end{aligned} \tag{56}$$

and

$$\tau(\vec{x},\vec{p})=\frac{e^{-i\theta(\vec{x})}}{2i(2\pi\hbar)^3}\int d^3s\,\tilde{\Omega}(\vec{x}+\tfrac{1}{2}\vec{s})e^{-i\vec{p}\cdot\vec{s}/\hbar}\,. \tag{57}$$

The variables f, U, V, and W are real.

It is readily shown that the quantities

$$\begin{aligned}
u&=\iint d^3x\,d^3p\,U(\vec{x},\vec{p})\,,\\
v&=\iint d^3x\,d^3p\,V(\vec{x},\vec{p})\,,\\
w&=\iint d^3x\,d^3p\,W(\vec{x},\vec{p})
\end{aligned} \tag{58}$$

are the components of the Bloch vector for the internal motion of the two-level atom. Therefore, roughly speaking, $U(\vec{x},\vec{p})$ and $V(\vec{x},\vec{p})$ represent, respectively, the distributions of the in-phase and in-quadrature components of the atomic dipole moment, and $W(\vec{x},\vec{p})$ represents the distribution of inversion over phase space. For an atom in the ground state, we have $U=V=0$ and $W=-f$.

Next, consider the classical limit of Eqs. (52)–(55). Following the classical-limit argument of Sec. II, we expand $\tilde{\Omega}(\vec{x}+\frac{1}{2}\vec{s})$ in (57) as a Taylor series in $\vec{s}$, and discard terms that vanish as $\hbar\to 0$. We obtain

$$\tau(\vec{x},\vec{p})=-\tfrac{1}{2}i\Omega\delta(\vec{p})+\tfrac{1}{4}\hbar(\vec{\nabla}\Omega+i\Omega\vec{\nabla}\theta)\cdot\vec{\nabla}_p\delta(\vec{p})\,.$$

Then, using the fact that $\vec{\nabla}_p\delta(\vec{p})$ is an odd function of $\vec{p}$, Eqs. (56) become

$$\begin{aligned}
a_r(\vec{x},\vec{p})&=-\tfrac{1}{2}\hbar\vec{\nabla}\Omega\cdot\vec{\nabla}_p\delta(\vec{p})\,,\\
a_i(\vec{x},\vec{p})&=-\tfrac{1}{2}\hbar\Omega\vec{\nabla}\theta\cdot\vec{\nabla}_p\delta(\vec{p})\,,\\
b_r(\vec{x},\vec{p})&=0\,,\\
b_i(\vec{x},\vec{p})&=-\Omega\delta(\vec{p})\,,
\end{aligned} \tag{59}$$

and Eqs. (52)–(55) reduce to

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)f=-\tfrac{1}{2}\hbar[\vec{\nabla}\Omega\cdot\vec{\nabla}_pU+\Omega\vec{\nabla}\theta\cdot\vec{\nabla}_pV]\,, \tag{60}$$

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)U=\left(\Delta+\frac{\vec{p}}{M}\cdot\vec{\nabla}\theta\right)V-\tfrac{1}{2}\hbar\vec{\nabla}\Omega\cdot\vec{\nabla}_pf\,, \tag{61}$$

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)V=-\left(\Delta+\frac{\vec{p}}{M}\cdot\vec{\nabla}\theta\right)U-\tfrac{1}{2}\hbar\Omega\vec{\nabla}\theta\cdot\vec{\nabla}_pf+\Omega W\,, \tag{62}$$

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)W=-\Omega V\,. \tag{63}$$

The term "classical limit" is really inappropriate here, because as $\hbar$ approaches zero the Rabi frequency Ω becomes infinite, indicating that the notion of a Rabi frequency for the internal motion is purely quantum mechanical. We shall refer to Eqs. (60)–(63) as the quasiclassical equations.

An estimate of the range of validity of the quasiclassical approximation is obtained as follows. It is easy to see that the lowest-order neglected term in the expansion of Eq. (57) would have made contributions to Eqs. (60)–(63) involving an additional factor $\hbar$, an additional derivative of $\tilde{\Omega}$ with respect to $\vec{x}$, and an additional derivative of f, U, V, or W with respect to $\vec{p}$, as compared to the highest-order terms that were kept. Therefore, if l is the scale size of variation of $\tilde{\Omega}(\vec{x})$ and Δp is the scale size of variation of f, U, V, and W in momentum space, the neglected terms are smaller by the factor $\hbar/l\Delta p$ than the retained terms. The minimum-scale size of the field $\tilde{\Omega}(\vec{x})$ is on the order of the optical wavelength λ. So the condition for the validity of Eqs. (60)–(63) is $\hbar/\lambda\Delta p\ll 1$ or $\hbar k\ll\Delta p$, where $k=2\pi/\lambda$. In other words, f, U, V, and W must be smooth over a distance in momentum space equal to the momentum $\hbar k$ of one photon of the resonant radiation. This is the case most often encountered in practice.

The quasiclassical equations provide a simple derivation of the optical Stern-Gerlach effect.[32] In a general standing wave $[\theta(\vec{x})=0]$ resonantly tuned ($\Delta=0$), Eqs. (60) and (61) read

$$\begin{aligned}
\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)f&=-\tfrac{1}{2}\hbar\vec{\nabla}\Omega\cdot\vec{\nabla}_pU\,,\\
\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)U&=-\tfrac{1}{2}\hbar\vec{\nabla}\Omega\cdot\vec{\nabla}_pf\,.
\end{aligned} \tag{64}$$

These equations have a solution of the form $f=f_+$, $U=-f_+$ if

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)f_+=\tfrac{1}{2}\hbar\vec{\nabla}\Omega\cdot\vec{\nabla}_p f_+ , \qquad (65)$$

and a solution of the form $f=f_-$, $U=f_-$ if

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)f_-=-\tfrac{1}{2}\hbar\vec{\nabla}\Omega\cdot\vec{\nabla}_p f_- . \qquad (66)$$

The distributions f_+ and f_- each propagate as a classical distribution function of a structureless point particle [see Eq. (23)] but with potential energies $V_+=\frac{1}{2}\hbar\Omega$ and $V_-=-\frac{1}{2}\hbar\Omega$, respectively, of opposite sign. The general solution of Eqs. (64) is

$$f=f_-+f_+ ,$$
$$U=f_--f_+ . \qquad (67)$$

An atom initially in its ground state ($U^0=0$) has $f_+^0=f_-^0=\frac{1}{2}f^0$, and consequently f_+ and f_- start out identical. When $f(\vec{x},\vec{p})$ is initially well localized in phase space, the Wigner function ($f=f_++f_-$) quickly splits into two components, because f_+ and f_- are driven by forces $\vec{F}_+=-\frac{1}{2}\hbar\vec{\nabla}\Omega$ and $\vec{F}_-=\frac{1}{2}\hbar\vec{\nabla}\Omega$ in opposite directions. This is the optical Stern-Gerlach effect which has been discussed recently by several authors.[20,21,32,33]

In the present section we have ignored spontaneous emission. Therefore the equations of motion derived above can be applied with confidence only for an interaction time that is much less than the natural lifetime of the excited state. For transitions in the visible, this limitation can be severe. In the following section we generalize the equations of motion to include effects of spontaneous emission.

IV. ATOMIC MOTION IN THE QUANTIZED ELECTROMAGNETIC FIELD

To take account of the interaction of the atom with the quantized electromagnetic field, in the electric dipole approximation, we must add to the Hamiltonian (30) the Hamiltonian

$$\hat{H}_F=\sum_{\vec{k},\lambda}\hbar\omega_{\vec{k}}\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda} \qquad (68)$$

for the free electromagnetic field and a term $-\hat{\vec{\mu}}\cdot\hat{\vec{E}}(\hat{\vec{x}})$ representing the interaction of the atomic dipole moment $\hat{\vec{\mu}}$ with the quantized electric field

$$\hat{\vec{E}}(\vec{x})=i\sum_{\vec{k},\lambda}\left(\frac{2\pi\hbar\omega_{\vec{k}}}{\mathcal{V}}\right)^{1/2}\vec{e}_{\vec{k},\lambda}(\hat{a}_{\vec{k},\lambda}e^{i\vec{k}\cdot\vec{x}}-\hat{a}^\dagger_{\vec{k},\lambda}e^{-i\vec{k}\cdot\vec{x}}) , \qquad (69)$$

where $\hat{a}^\dagger_{\vec{k},\lambda}$, $\hat{a}_{\vec{k},\lambda}$, $\omega_{\vec{k}}$, and $\vec{e}_{\vec{k},\lambda}$ are, respectively, the creation and annihilation operators, the frequency, and the polarization vector of the field mode of wave vector $\vec{k}$ and polarization index λ ($=1,2$), and $\mathcal{V}$ is the quantization volume. The creation and annihilation operators commute with the atomic operators $\hat{S},\hat{S}^+,\hat{S}_3$ and with the Wigner operator $\hat{f}(\vec{x},\vec{p})$, and between themselves satisfy the usual commutation relations

$$[\hat{a}_{\vec{k},\lambda},\hat{a}^\dagger_{\vec{k}',\lambda'}]=\delta_{\lambda\lambda'}\delta_{\vec{k},\vec{k}'} , \qquad (70)$$

$$[\hat{a}_{\vec{k},\lambda},\hat{a}_{\vec{k}',\lambda'}]=[\hat{a}^\dagger_{\vec{k},\lambda},\hat{a}^\dagger_{\vec{k}',\lambda'}]=0 . \qquad (71)$$

The complete Hamiltonian is

$$\hat{H}=\hat{p}^2/2M+\hat{H}_0+\hat{H}_F-\hat{\vec{\mu}}\cdot[\vec{E}_{cl}(\hat{\vec{x}},t)+\hat{\vec{E}}(\hat{\vec{x}})] . \qquad (72)$$

Note that we have retained the classical applied field $\vec{E}_{cl}(\vec{x},t)$ in the Hamiltonian. In the model used here, the atomic motion is driven by the classical applied field, while the quantized field is treated as a zero-temperature heat bath whose sole purpose is to cause radiative relaxation of the atom. Such a model can be justified on the grounds that a strong monochromatic wave is a state of the radiation field involving large quantum numbers, and hence the correspondence principle ensures that the applied field may be treated classically. Alternatively, it can be shown that the picture of a classical applied field plus a quantized field initially in the vacuum state is related by a canonical transformation to the picture in which the field is fully quantized and the applied field is a coherent state.[34] We choose to work with a classically prescribed applied field because in this picture much of the calculation of equations of motion has already been accomplished in the preceding section.

On substituting $\hat{\vec{\mu}}$ from (32), $\vec{E}_{cl}(\vec{x},t)$ from (37), $\hat{\vec{E}}(\vec{x})$ from (69), $\hat{H}_0$ from (31), and $\hat{H}_F$ from (68) into (72), and keeping only slowly varying terms in the classical interaction and only energy-conserving terms in the interaction with the quantized field, we get

$$\hat{H}=\hat{p}^2/2M+\hbar\omega_0\hat{S}^+\hat{S}+\sum_{\vec{k},\lambda}\hbar\omega_{\vec{k}}\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda}-\tfrac{1}{2}\hbar[\tilde{\Omega}^*(\hat{\vec{x}})\hat{S}^+e^{-i\omega t}+\tilde{\Omega}(\hat{\vec{x}})\hat{S}e^{i\omega t}]$$
$$+i\hbar\sum_{\vec{k},\lambda}(g_{\vec{k},\lambda}\hat{a}^\dagger_{\vec{k},\lambda}\hat{S}e^{-i\vec{k}\cdot\hat{\vec{x}}}-g^*_{\vec{k},\lambda}\hat{S}^+\hat{a}_{\vec{k},\lambda}e^{i\vec{k}\cdot\hat{\vec{x}}}) , \qquad (73)$$

where

$$g_{\vec{k},\lambda} = (2\pi\omega_{\vec{k}}/\hbar\mathcal{V})^{1/2}\vec{\mu}\cdot\hat{e}_{\vec{k},\lambda}. \tag{74}$$

Next we express the interaction terms in (73) in terms of the Wigner operator by use of relations

$$\tilde{\Omega}(\hat{\vec{x}}) = \iint d^3x\, d^3p\, \tilde{\Omega}(\vec{x})\hat{f}(\vec{x},\vec{p}), \tag{75}$$

and

$$e^{\pm i\vec{k}\cdot\hat{\vec{x}}} = \iint d^3x\, d^3p\, e^{\pm i\vec{k}\cdot\vec{x}}\hat{f}(\vec{x},\vec{p}). \tag{76}$$

The Hamiltonian is then

$$\hat{H} = \hat{p}^2/2M + \hbar\omega_0\hat{S}^+\hat{S} + \sum_{\vec{k},\lambda}\hbar\omega_{\vec{k}}\hat{a}^\dagger_{\vec{k},\lambda}\hat{a}_{\vec{k},\lambda} - \tfrac{1}{2}\hbar\iint d^3x\, d^3p[\tilde{\Omega}^*(\vec{x})\hat{S}^+\hat{f}(\vec{x},\vec{p})e^{-i\omega t} + \tilde{\Omega}(\vec{x})\hat{S}\hat{f}(\vec{x},\vec{p})e^{i\omega t}]$$
$$+ i\hbar\sum_{\vec{k},\lambda}\iint d^3x\, d^3p[g_{\vec{k},\lambda}e^{-i\vec{k}\cdot\vec{x}}\hat{a}^\dagger_{\vec{k},\lambda}\hat{S}\hat{f}(\vec{x},\vec{p}) - g^*_{\vec{k},\lambda}e^{i\vec{k}\cdot\vec{x}}\hat{S}^+\hat{a}_{\vec{k},\lambda}\hat{f}(\vec{x},\vec{p})]. \tag{77}$$

As before, we work with the Wigner operator and operators

$$\hat{\Pi}(\vec{x},\vec{p}) = \hat{S}\hat{f}(\vec{x},\vec{p})e^{i\omega t},$$
$$\hat{\Pi}_3(\vec{x},\vec{p}) = \hat{S}_3\hat{f}(\vec{x},\vec{p}), \tag{78}$$

and, in addition, we now make use of field operators

$$\hat{b}_{\vec{k},\lambda} = \hat{a}_{\vec{k},\lambda}e^{i\omega t}, \tag{79}$$

which are slowly varying for field modes near resonance.

With the help of Eqs. (24)–(29), (35), (36), (70), and (71) we find, after some work, the Heisenberg equations of motion for $\hat{f}$, $\hat{\Pi}$, $\hat{\Pi}_3$, and $\hat{b}_{\vec{k},\lambda}$, namely

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{M}\cdot\vec{\nabla}\right)\hat{f}(\vec{x},\vec{p}) = \int d^3p'[C(\vec{x},\vec{p}-\vec{p}')\hat{\Pi}(\vec{x},\vec{p}') + C^*(\vec{x},\vec{p}-\vec{p}')\hat{\Pi}^\dagger(\vec{x},\vec{p}')]$$
$$-\sum_{\vec{k},\lambda}\{g_{\vec{k},\lambda}e^{-i\vec{k}\cdot\vec{x}}\hat{b}^\dagger_{\vec{k},\lambda}[\hat{\Pi}(\vec{x},\vec{p}-\tfrac{1}{2}\hbar\vec{k}) - \hat{\Pi}(\vec{x},\vec{p}+\tfrac{1}{2}\hbar\vec{k})]$$
$$+ g^*_{\vec{k},\lambda}e^{i\vec{k}\cdot\vec{x}}[\hat{\Pi}^\dagger(\vec{x},\vec{p}-\tfrac{1}{2}\hbar\vec{k}) - \hat{\Pi}^\dagger(\vec{x},\vec{p}+\tfrac{1}{2}\hbar\vec{k})]\hat{b}_{\vec{k},\lambda}\}, \tag{80}$$

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{M}\cdot\vec{\nabla}\right)\hat{\Pi}(\vec{x},\vec{p}) = i\Delta\hat{\Pi}(\vec{x},\vec{p}) + \tfrac{1}{2}\int d^3p'[C^*(\vec{x},\vec{p}-\vec{p}')\hat{f}(\vec{x},\vec{p}') - B^*(\vec{x},\vec{p}-\vec{p}')\hat{\Pi}_3(\vec{x},\vec{p}')]$$
$$-\tfrac{1}{2}\sum_{\vec{k},\lambda}g^*_{\vec{k},\lambda}e^{i\vec{k}\cdot\vec{x}}[\hat{f}(\vec{x},\vec{p}-\tfrac{1}{2}\hbar\vec{k}) - \hat{f}(\vec{x},\vec{p}+\tfrac{1}{2}\hbar\vec{k}) - \hat{\Pi}_3(\vec{x},\vec{p}-\tfrac{1}{2}\hbar\vec{k}) - \hat{\Pi}_3(\vec{x},\vec{p}+\tfrac{1}{2}\hbar\vec{k})]\hat{b}_{\vec{k},\lambda}, \tag{81}$$

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{M}\cdot\vec{\nabla}\right)\hat{\Pi}_3(\vec{x},\vec{p}) = \int d^3p'[B(\vec{x},\vec{p}-\vec{p}')\hat{\Pi}(\vec{x},\vec{p}') + B^*(\vec{x},\vec{p}-\vec{p}')\hat{\Pi}^\dagger(\vec{x},\vec{p}')]$$
$$-\sum_{\vec{k},\lambda}\{g_{\vec{k},\lambda}e^{-i\vec{k}\cdot\vec{x}}\hat{b}^\dagger_{\vec{k},\lambda}[\hat{\Pi}(\vec{x},\vec{p}-\tfrac{1}{2}\hbar\vec{k}) + \hat{\Pi}(\vec{x},\vec{p}+\tfrac{1}{2}\hbar\vec{k})]$$
$$+ g^*_{\vec{k},\lambda}e^{i\vec{k}\cdot\vec{x}}[\hat{\Pi}^\dagger(\vec{x},\vec{p}-\tfrac{1}{2}\hbar\vec{k}) + \hat{\Pi}^\dagger(\vec{x},\vec{p}+\tfrac{1}{2}\hbar\vec{k})]\hat{b}_{\vec{k},\lambda}\}, \tag{82}$$

and

$$\dot{\hat{b}}_{\vec{k},\lambda} = i(\omega-\omega_{\vec{k}})\hat{b}_{\vec{k},\lambda} + g_{\vec{k},\lambda}\iint d^3x\, d^3p\, e^{-i\vec{k}\cdot\vec{x}}\hat{\Pi}(\vec{x},\vec{p}). \tag{83}$$

For later convenience, we have written Eqs. (80)–(82) in "normal order" with field creation operators to the left and field annihilation operators to the right of atomic operators.

The field operators are now eliminated from Eqs. (80)–(82) as follows. The solution of Eq. (81), to zeroth order in the interaction, is

$$\hat{\Pi}(\vec{x},\vec{p},t_2) = \hat{\Pi}(\vec{x}-\vec{p}[t_2-t_1]/M,\vec{p},t_1)e^{i\Delta(t_2-t_1)}, \tag{84}$$

and this is an accurate solution of Eq. (81) over a time interval $\Delta t = t_2 - t_1$ that is large compared to the optical period $2\pi/\omega$ but much smaller than the Rabi period $2\pi/\Omega$ or the natural lifetime of the excited state. For a time interval $t_2 - t_1 = 2\pi/\omega_0$ equal to the optical period, the displacement

$\Delta x = p(t_2 - t_1)/M = (v/c)\lambda$ is generally very small compared to the distance over which $\hat{\Pi}$ changes by a significant amount. Therefore we may replace the argument $\vec{x} - \vec{p}[t_2 - t_1]/M$ in (84) by $\vec{x}$, and the result remains accurate for many optical periods. Using the latter form of (84), Eq. (83) is readily integrated:

$$\hat{b}_{\vec{k},\lambda}(t) = \hat{b}_{\vec{k},\lambda}(0)e^{i(\omega-\omega_{\vec{k}})t} + g_{\vec{k},\lambda}\iint d^3x\, d^3p\, \hat{\Pi}(\vec{x},\vec{p},t)e^{-i\vec{k}\cdot\vec{x}} \times \int_0^t e^{i(\omega_0-\omega_{\vec{k}})(t-t')}dt'. \quad (85)$$

Now Eq. (85) and its Hermitian conjugate will eventually be substituted into Eqs. (80)–(82), and expectation values of the resulting equations will be taken with the field in the vacuum state. Because Eqs. (80)–(82) are written in normal order, the contributions to the final equations from the first term in (85) clearly vanish, and therefore this term may be discarded. For t much larger than the optical period, we have for the time integral in (85)

$$\int_0^t e^{i(\omega_0-\omega_{\vec{k}})(t-t')}dt' \approx \pi\delta(\omega_{\vec{k}} - \omega_0) - \frac{i\mathcal{P}}{\omega_{\vec{k}} - \omega_0}. \quad (86)$$

Where $\mathcal{P}$ denotes principal value.[35] It is known that the first term in (86) leads to radiative relaxation, while the second term leads to a divergent frequency shift.[31,36] A proper treatment of the second term requires renormalization theory, and, after renormalization, the effect of this term is quite small and of little interest in the present context. We therefore discard this term also, and assume that the Lamb shift has already been incorporated in the frequency ω_0. Equation (85) becomes, effectively,

$$\hat{b}_{\vec{k},\lambda} = \pi g_{\vec{k},\lambda}\delta(\omega_{\vec{k}} - \omega_0)\iint d^3x\, d^3p\, e^{-i\vec{k}\cdot\vec{x}}\hat{\Pi}(\vec{x},\vec{p}). \quad (87)$$

Upon substituting (87) into Eqs. (80)–(82), using Eqs. (26)–(29), (36), and (78) repeatedly, taking expectation values in the result, and finally passing to the limit of infinite quantization volume ($\mathcal{V} \to \infty$) in the usual way, we obtain

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{M}\cdot\vec{\nabla}\right)f(\vec{x},\vec{p}) = -\tfrac{1}{2}A[f(\vec{x},\vec{p}) + \Pi_3(\vec{x},\vec{p})] + \int d^3p'[C(\vec{x},\vec{p}-\vec{p}')\Pi(\vec{x},\vec{p}') + C^*(\vec{x},\vec{p}-\vec{p}')\Pi^\dagger(\vec{x},\vec{p}')] \times \int d^3k\, Z(\vec{k})[f(\vec{x},\vec{p}+\hbar\vec{k}) + \Pi_3(\vec{x},\vec{p}+\hbar\vec{k})], \quad (88)$$

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{M}\cdot\vec{\nabla}\right)\Pi(\vec{x},\vec{p}) = (i\Delta - \tfrac{1}{2}A)\Pi(\vec{x},\vec{p}) + \tfrac{1}{2}\int d^3p'[C^*(\vec{x},\vec{p}-\vec{p}')f(\vec{x},\vec{p}') - B^*(\vec{x},\vec{p}-\vec{p}')\Pi_3(\vec{x},\vec{p}')], \quad (89)$$

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{M}\cdot\vec{\nabla}\right)\Pi_3(\vec{x},\vec{p}) = -\tfrac{1}{2}A[f(\vec{x},\vec{p}) + \Pi_3(\vec{x},\vec{p})] + \int d^3p'[B(\vec{x},\vec{p}-\vec{p}')\Pi(\vec{x},\vec{p}') + B^*(\vec{x},\vec{p}-\vec{p}')\Pi^\dagger(\vec{x},\vec{p}')] - \int d^3k\, Z(\vec{k})[f(\vec{x},\vec{p}+\hbar\vec{k}) + \Pi_3(\vec{x},\vec{p}+\hbar\vec{k})], \quad (90)$$

where

$$A = (2\pi)^{-2}\mathcal{V}\sum_\lambda \int d^3k\, |g_{\vec{k},\lambda}|^2\delta(\omega_{\vec{k}} - \omega_0) = \frac{4|\vec{\mu}|^2\omega_0^3}{3\hbar c^3} \quad (91)$$

is the Einstein spontaneous emission coefficient, and

$$Z(\vec{k}) = (4\pi\hbar)^{-1}\omega_{\vec{k}}\delta(\omega_{\vec{k}} - \omega_0)\sum_\lambda |\vec{\mu}\cdot\vec{e}_{\vec{k},\lambda}|^2 = (4\pi\hbar)^{-1}\omega_{\vec{k}}\delta(\omega_{\vec{k}} - \omega_0)|\vec{\mu}|^2(1 - \cos^2\theta'), \quad (92)$$

where θ' is the angle between $\vec{k}$ and $\vec{\mu}$.[37] We note for later reference that

$$\int d^3k\, Z(\vec{k}) = \tfrac{1}{2}A. \quad (93)$$

The transformations in Eqs. (50) and (51) convert Eqs. (88)–(90) to the form

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{M}\cdot\vec{\nabla}\right)f(\vec{x},\vec{p}) = -\tfrac{1}{2}A[f(\vec{x},\vec{p}) + W(\vec{x},\vec{p})] + \int d^3p'[a_r(\vec{x},\vec{p}-\vec{p}')U(\vec{x},\vec{p}') + a_i(\vec{x},\vec{p}-\vec{p}')V(\vec{x},\vec{p}')] + \int d^3k\, Z(\vec{k})[f(\vec{x},\vec{p}+\hbar\vec{k}) + W(\vec{x},\vec{p}+\hbar\vec{k})], \quad (94)$$

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{M}\cdot\vec{\nabla}\right)U(\vec{x},\vec{p}) = \left(\Delta + \frac{\vec{p}}{M}\cdot\vec{\nabla}\theta\right)V(\vec{x},\vec{p}) - \tfrac{1}{2}AU(\vec{x},\vec{p}) + \int d^3p'[a_r(\vec{x},\vec{p}-\vec{p}')f(\vec{x},\vec{p}') - b_r(\vec{x},\vec{p}-\vec{p}')W(\vec{x},\vec{p}')], \quad (95)$$

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)V(\vec{x},\vec{p})=-\left(\Delta+\frac{\vec{p}}{M}\cdot\vec{\nabla}\theta\right)U(\vec{x},\vec{p})-\tfrac{1}{2}AV(\vec{x},\vec{p})+\int d^3p'[a_i(\vec{x},\vec{p}-\vec{p}')f(\vec{x},\vec{p}')-b_i(\vec{x},\vec{p}-\vec{p}')W(\vec{x},\vec{p}')]\,, \quad (96)$$

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)W(\vec{x},\vec{p})=-\tfrac{1}{2}A[f(\vec{x},\vec{p})+W(\vec{x},\vec{p})]+\int d^3p'[b_r(\vec{x},\vec{p}-\vec{p}')U(\vec{x},\vec{p}')+b_i(\vec{x},\vec{p}-\vec{p}')V(\vec{x},\vec{p}')]-\int d^3k Z(\vec{k})[f(\vec{x},\vec{p}+\hbar\vec{k})+W(\vec{x},\vec{p}+\hbar\vec{k})]\,, \quad (97)$$

where a_r, a_i, b_r, and b_i are given by Eqs. (56) and (57). Equations (94)–(97) are the general equations describing the time development of the Wigner function f and the distributed Bloch vector (U, V, W) including effects of spontaneous emission.

We now calculate the quasiclassical limit of Eqs. (94)–(97). For the terms in (94)–(97) not involving A or $Z(\vec{k})$, the calculation proceeds exactly as in the preceding section. We obtain from these terms Eqs. (60)–(63). To these equations we must add the quasiclassical contributions from the remaining terms in (94)–(97). The terms containing A are unaffected by the quasiclassical limit. A typical term involving $Z(\vec{k})$ is of the form

$$\int d^3k\, Z(\vec{k})f(\vec{x},\vec{p}+\hbar\vec{k})\,. \quad (98)$$

Recalling that in the quasiclassical approximation $f(\vec{x},\vec{p})$ must be smooth over a distance $\hbar k=\hbar\omega_0/c$ in momentum space, we expand $f(\vec{x},\vec{p}+\hbar\vec{k})$ in powers of $\hbar k^i$. Keeping terms through second order in the expansion, using Eq. (93), and noting that

$$\int d^3k\, Z(\vec{k})k^i=0 \quad (99)$$

because $Z(\vec{k})$ is an even function of k^i, we find that (98) becomes

$$\tfrac{1}{2}Af(\vec{x},\vec{p})+\sum_{i,j}Q^{ij}\frac{\partial^2 f(\vec{x},\vec{p})}{\partial p^i\partial p^j}\,, \quad (100)$$

where

$$Q^{ij}=\tfrac{1}{2}\hbar^2\int d^3k\, Z(\vec{k})k^ik^j\,. \quad (101)$$

Let the dipole transition moment $\vec{\mu}$ be directed along the x^3 axis. Then using $Z(\vec{k})$ from (92) and

$$k^1=k\sin\theta'\cos\phi'\,,\quad k^2=k\sin\theta'\sin\phi'\,,\quad k^3=k\cos\theta'\,, \quad (102)$$

where θ', ϕ' are polar angles in k space and $k=\omega_{\vec{k}}/c$, Eq. (101) yields

$$Q^{ij}=\tfrac{1}{10}(\hbar\omega_0/c)^2Ad^{ij}\,, \quad (103)$$

where d^{ij} is the diagonal matrix

$$d^{ij}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{2}\end{pmatrix}. \quad (104)$$

Other terms involving $Z(\vec{k})$ in Eqs. (94)–(97) are evaluated in the same manner as (98). The full set of quasiclassical equations read

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)f=-\tfrac{1}{2}\hbar(\vec{\nabla}\Omega\cdot\vec{\nabla}_pU+\Omega\vec{\nabla}\theta\cdot\vec{\nabla}_pV)+\sum_{i,j}Q^{ij}\frac{\partial^2(f+W)}{\partial p^i\partial p^j}\,, \quad (105)$$

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)U=\left(\Delta+\frac{\vec{p}\cdot\vec{\nabla}\theta}{M}\right)V-\tfrac{1}{2}AU-\tfrac{1}{2}\hbar\vec{\nabla}\Omega\cdot\vec{\nabla}_pf\,, \quad (106)$$

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)V=-\left(\Delta+\frac{\vec{p}\cdot\vec{\nabla}\theta}{M}\right)U-\tfrac{1}{2}AV+\Omega W-\tfrac{1}{2}\hbar\Omega\vec{\nabla}\theta\cdot\vec{\nabla}_pf\,, \quad (107)$$

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)W=-\Omega V-A(f+W)-\sum_{i,j}Q^{ij}\frac{\partial^2(f+W)}{\partial p^i\partial p^j}\,. \quad (108)$$

The general equations for atomic motion in resonant radiation, Eqs. (94)–(97), and the quasiclassical limit of these equations, Eqs. (105)–(108), are the principal results of this paper. In the calculations which follow we work exclusively with the quasiclassical equations.

Next we establish the connection between the Ehrenfest-Bloch equations of earlier work[26] and the more general quasiclassical equations of the present theory. Let f, U, V, and W be localized in phase space near the point $(\vec{x},\vec{p})=(\vec{r},\vec{\rho})$. Then in an integral over phase space of a product of a smooth function of $(\vec{x},\vec{p})$ and one of the functions functions f, U, V, or W, the smooth function may be evaluated at $(\vec{r},\vec{\rho})$ and taken outside of the integral. In addition, integrals over phase space of terms such as $\vec{p}\cdot\vec{\nabla}f/M$, $-\frac{1}{2}\hbar\vec{\nabla}\Omega\cdot\vec{\nabla}_p f$, and $Q^{ij}\partial^2(f+W)/\partial p^i\partial p^j$ vanish, as can easily be shown by integrating by parts and using the fact that f, U, V, and W vanish at $|\vec{x}|=\infty$ and $|\vec{p}|=\infty$. Using these observations and Eqs. (58), we find that the integrals of Eqs. (106)–(108) over phase space are the optical Bloch equations[31]

$$\dot{u}=(\Delta+\dot{\theta})v-\tfrac{1}{2}Au\,,$$
$$\dot{v}=-(\Delta+\dot{\theta})u+\Omega W-\tfrac{1}{2}Av\,, \qquad (109)$$
$$\dot{w}=-\Omega v-A(w+1)\,,$$

where $\dot{\theta}=\vec{\mathscr{p}}\cdot\vec{\nabla}\theta(\vec{\mathscr{r}})/M$ and $\Omega=\Omega(\vec{\mathscr{r}})$. The Bloch vector (u,v,w) is driven by the field amplitude Ω and phase derivative $\dot{\theta}$ at the position of the moving atom. Next multiply Eq. (105) by $\vec{x}$ and integrate over phase space. An integration by parts gives

$$\iint d^3x\, d^3p\, \vec{x}\left(\frac{\vec{p}\cdot\vec{\nabla}f}{M}\right)=-\vec{\mathscr{p}}/M\,, \qquad (110)$$

and the terms on the right in (105) all vanish on integration. The result is

$$\dot{\vec{\mathscr{r}}}=\vec{\mathscr{p}}/M\,. \qquad (111)$$

Finally, multiply Eq. (105) by $\vec{p}$ and integrate over phase space to obtain the radiation force

$$\vec{F}=\dot{\vec{\mathscr{p}}}=M\ddot{\vec{\mathscr{r}}}=\tfrac{1}{2}\hbar(u\vec{\nabla}\Omega+v\Omega\vec{\nabla}\theta)\,. \qquad (112)$$

Equations (109) and (112) are the Ehrenfest-Bloch equations describing the motion of the centroid of the atomic wave packet.[26]

V. INDUCED MOMENTUM DIFFUSION

It is well known that the momentum of an atom in resonant radiation undergoes a kind of diffusion due to the randomly directed recoils accompanying spontaneous emission. This spontaneous diffusion is described by the last term in Eq. (105). As noted above, induced absorption and emission processes can also give rise to diffusion of atomic momentum.[28] In the present section we treat induced momentum diffusion in the case where the applied field is smooth or the atomic velocity is small.

The goal is to obtain a single equation for the Wigner function in place of the coupled quasiclassical equations for f, U, V, and W. To accomplish this, we solve Eqs. (106)–(108) approximately for U, V, and W in terms of f, and insert the result into Eq. (105). Here the approximate solution of Eqs. (106)–(108) is based on the assumption that the Rabi frequency Ω and the phase derivative $\dot{\theta}=\vec{p}\cdot\vec{\nabla}\theta/M$, at the moving atom, vary by only a small amount during a natural lifetime $\tau=1/A$. This condition is satisfied when the field is sufficiently smooth or when the atomic velocity is sufficiently small. It is shown for this case in the Appendix that the equation for $f(\vec{x},\vec{p})$, accurate through terms of order $\hbar^2$, is the Fokker-Planck equation

$$\left(\frac{\partial}{\partial t}+\frac{p^i}{M}\frac{\partial}{\partial x^i}\right)f=-\frac{\partial}{\partial p^i}(F_e^i f)+\frac{\partial^2}{\partial p^i\partial p^j}(D^{ij}f)\,, \qquad (113)$$

where F_e^i is the effective radiation force, D^{ij} are momentum diffusion coefficients, and we are using the Einstein summation convention; i.e., a repeated index in a term implies summation over that index. The effective force $F_e^i=F^i+\tilde{F}^i$ consists of the radiation force

$$F^i=\frac{-\hbar}{4\bar{\Delta}^2+A^2+2\Omega^2}\left(A\Omega^2\frac{\partial\theta}{\partial x^i}+\bar{\Delta}\frac{\partial\Omega^2}{\partial x^i}\right), \qquad (114)$$

$$\bar{\Delta}=\Delta+\vec{p}\cdot\vec{\nabla}\theta/M\,, \qquad (115)$$

which agrees with that obtained from the Ehrenfest-Bloch equations in the steady-state approximation,[26] plus a correction $\tilde{F}^i$ given in (A26), which is of higher order in $\hbar$ than F^i. It is easy to show that the correction $\tilde{F}^i$ is generally quite negligible compared to F^i, and in the following we shall often set $F_e^i=F^i$ for simplicity. The diffusion tensor $D^{ij}=D_s^{ij}+D_I^{ij}$ consists of coefficients

$$D_s^{ij}=\frac{\hbar^2A\Omega^2k^2d^{ij}}{5(4\bar{\Delta}^2+A^2+2\Omega^2)} \qquad (116)$$

associated with spontaneous emission ($k=\omega_0/c$), plus coefficients

$$D_I^{ij}=\alpha\frac{\partial\Omega}{\partial x^i}\frac{\partial\Omega}{\partial x^j}+\beta\Omega^2\frac{\partial\theta}{\partial x^i}\frac{\partial\theta}{\partial x^j}+\gamma\left(\frac{\partial\Omega^2}{\partial x^i}\frac{\partial\theta}{\partial x^j}+\frac{\partial\theta}{\partial x^i}\frac{\partial\Omega^2}{\partial x^j}\right) \qquad (117)$$

associated with induced absorption and emission processes, where

$$\alpha=\hbar^2[(A^2+2\Omega^2)G^2-8\bar{\Delta}^2\Omega^2(4\bar{\Delta}^2+5A^2+4\Omega^2)]/2AG^3\,, \qquad (118)$$

$$\beta=\hbar^2A[G^2-2\Omega^2(3A^2-4\bar{\Delta}^2)]/2G^3\,, \qquad (119)$$

$$\gamma=-2\hbar^2\bar{\Delta}\Omega^2(2A^2+\Omega^2)/G^3\,, \qquad (120)$$

and

$$G=4\bar{\Delta}^2+A^2+2\Omega^2\,. \qquad (121)$$

For atomic motion in one dimension, Eq. (117) agrees with the result of an earlier calculation[28] in which induced momentum diffusion is attributed to the interaction between the fluctuating atomic dipole moment and the gradient of the applied field.

In view of Eqs. (104) and (116), the contribution to momentum diffusion from spontaneous emission is not isotropic. The anisotropic distribution of dipole radiation leads to a diffusion coefficient D_s^{33}

in the direction parallel to $\vec{\mu}$ that is half as large as the coefficients D_s^{11} and D_s^{22} in directions orthogonal to $\vec{\mu}$. In a strong field $(2\Omega^2 \gg 4\Delta^2 + A^2)$, the coefficients of spontaneous diffusion saturate to $D_s^{ij} = Q^{ij}$.

Consider now two simple examples. In a plane running wave $[\Omega = \text{constant},\ \theta(\vec{x}) = -\vec{k}\cdot\vec{x}]$, the radiation force, Eq. (114) becomes

$$F^i = \frac{A\Omega^2\hbar k^i}{4(\Delta - \vec{k}\cdot\vec{p}/M)^2 + A^2 + 2\Omega^2}, \qquad (122)$$

and the spontaneous and induced diffusion tensors Eqs. (116) and (117) are

$$D_s^{ij} = \frac{\hbar^2 A\Omega^2 k^2 d^{ij}}{5[4(\Delta - \vec{k}\cdot\vec{p}/M)^2 + A^2 + 2\Omega^2]} \qquad (123)$$

and

$$D_I^{ij} = \frac{\hbar^2 A\Omega^2 k^i k^j}{2[4(\Delta - \vec{k}\cdot\vec{p}/M)^2 + A^2 + 2\Omega^2]} \times \left(1 - \frac{2\Omega^2[3A^2 - 4(\Delta - \vec{k}\cdot\vec{p}/M)^2]}{[4(\Delta - \vec{k}\cdot\vec{p}/M)^2 + A^2 + 2\Omega^2]^2}\right), \qquad (124)$$

respectively. Note that in a strong field the coefficients of induced diffusion saturate to $D_I^{ij} = \frac{1}{4}A\hbar^2 k^i k^j$. The form of Eq. (124) indicates that induced diffusion occurs only in directions $\pm\vec{k}$ and not orthogonal to these directions.

In a general standing wave (not necessarily a plane standing wave) we have $\theta(\vec{x}) = 0$. The radiation force, usually called the dipole force in this case, is now

$$F^i = -\frac{\hbar\Delta\, \partial\Omega^2/\partial x^i}{4\Delta^2 + A^2 + 2\Omega^2} \qquad (125)$$

and the diffusion tensors are

$$D_s^{ij} = \frac{\hbar^2 A\Omega^2 k^2 d^{ij}}{5(4\Delta^2 + A^2 + 2\Omega^2)}, \qquad (126)$$

$$D_I^{ij} = \frac{\hbar^2}{2A(4\Delta^2 + A^2 + 2\Omega^2)} \frac{\partial\Omega}{\partial x^i}\frac{\partial\Omega}{\partial x^j} \times \left(A^2 + 2\Omega^2 - \frac{8\Delta^2\Omega^2[4\Delta^2 + 5A^2 + 4\Omega^2]}{(4\Delta^2 + A^2 + 2\Omega^2)^2}\right). \qquad (127)$$

Here the coefficients of induced diffusion do not saturate in a strong field, but continue to increase as

$$D_I^{ij} = \frac{\hbar^2}{2A}\frac{\partial\Omega}{\partial x^i}\frac{\partial\Omega}{\partial x^j} \qquad (128)$$

as the field strength increases. Note that (128) is an exact expression for D_I^{ij} when $\Delta = 0$. In a standing wave induced diffusion proceeds in directions $\pm\vec{\nabla}\Omega$.

A few words concerning the physical interpretation of the diffusion tensors are now in order. It is easy to show that the coefficients of spontaneous diffusion (116) are consistent with the idea that the atom undergoes a random walk in momentum space due to statistically independent recoils $\Delta\vec{p} = \hbar\vec{k}$ occurring at the rate of spontaneous emission and distributed in direction according to the dipole distribution of radiated power. The coefficients of induced diffusion (117), on the other hand, are *not* the result of statistically independent recoils occurring at the rate Ω of induced absorption and emission events. To see this, recall that in a random walk of step size L and step rate R the diffusion constant is of order L^2R. Thus for steps of length $\hbar k$ taken at the saturated rate ($\sim A$) of spontaneous emission, the diffusion constant is $D_s \sim (\hbar k)^2 A$, in agreement as to order of magnitude with the saturated spontaneous coefficients $D_s^{ij} = \frac{1}{10}(\hbar k)^2 A d^{ij}$ derived above. But if we apply the same argument to induced processes, which occur at the rate Ω, we obtain $D_I \sim (\hbar k)^2\Omega$. Comparing this with the on-resonance induced coefficients (128) in a plane standing wave ($\Omega = 2\Omega_0 \cos kx$, $\partial\Omega/\partial x^i \sim k\Omega$), namely $D_I^{ij} \sim (\hbar k)^2\Omega^2/A$, we see that the present theory leads to induced coefficients larger by a factor Ω/A than can be accounted for on the basis of statistically independent recoils, and this factor can be very large. This is an important observation because a number of authors have explicitly assumed that successive induced processes are statistically independent, or else have written rate equations for induced momentum transfer that implicitly assume statistical independence for the underlying induced processes. Predications of such theories will differ greatly from those of the present theory.

The much larger standing-wave induced coefficients predicted by our theory can be understood from the point of view of the optical Stern-Gerlach effect. For exact resonance ($\Delta = 0$) an atom initially in its ground state has equal probability to be in one or the other of the distributions f_+ and f_- in which it experiences forces $\vec{F}_+ = -\frac{1}{2}\hbar\vec{\nabla}\Omega$ and $\vec{F}_- = \frac{1}{2}\hbar\vec{\nabla}\Omega$, respectively (see Sec. III). In time Δt the atom takes, with equal probability, a step $\Delta\vec{p}_+ = \vec{F}_+\Delta t$ or a step $\Delta\vec{p}_- = \vec{F}_-\Delta t$ in momentum space. The coherent acceleration of the atom by one or the other of these forces is terminated by spontaneous emission, which returns the atom to its ground state and initiates a new step $\Delta\vec{p}_+$ or $\Delta\vec{p}_-$ again with equal probability. Thus we again have a random walk of atomic momentum, but now with step size $|\Delta\vec{p}_\pm| = \frac{1}{2}\hbar|\vec{\nabla}\Omega|\Delta t$ determined by the mean time $\Delta t = 4\tau = 4/A$ between spontaneous events from f_+ or f_- and step rate $A/4$ equal to the rate of emission from f_+ or f_- (in each of these distributions the atom has upper state probability $P_2 = \frac{1}{2}$ and the probability that each of the distributions is occupied is also $\frac{1}{2}$). The diffusion coefficient along the direction of $\vec{\nabla}\Omega$ is, therefore, $2D_I = \hbar^2|\vec{\nabla}\Omega|^2/A$, in agreement with the on-resonance induced coefficients (128) for a general standing wave.

The diffusion tensor in a plane running wave, Eq. (124), is also supported by a simple physical argument. If the atom absorbs a photon of wave vector $\vec{k}$ and is then induced to emit a photon (necessarily also of wave vector $\vec{k}$), the net momentum transferred to the atom is zero because the momentum $\hbar\vec{k}$ acquired by the atom in absorption is canceled by the recoil momentum $-\hbar\vec{k}$ of induced emission. In the process of absorption followed by spontaneous emission, on the other hand, the momentum acquired by the atom in absorption is not canceled, on the average, by spontaneous emission because the distribution of spontaneous emission gives equal probability to recoils in opposite directions. Thus the atom gains the momentum of one incident photon for each spontaneous event, and if n spontaneous events occur in time t, the momentum transferred in this time interval is $\vec{p}=n\hbar\vec{k}$. We emphasize that this momentum transfer is due to absorption (an induced processes), and we are ignoring the momentum transfer associated with spontaneous emission. From the above argument we conclude that the mean rate of momentum transfer, i.e., the radiation force, is $\langle\vec{p}\rangle/t=\vec{F}=\hbar\vec{k}\langle n\rangle/t$, and using the well-known expression

$$\langle n\rangle/t=A\Omega^2/[4(\Delta-\vec{k}\cdot\vec{p}/M)^2+A^2+2\Omega^2] \tag{129}$$

for the steady-state rate of spontaneous emission, we obtain the radiation force in Eq. (122). But the number of spontaneous events in a time interval t fluctuates, and the fluctuations of n give rise to a spreading of momentum about the mean momentum,

$$\langle(p^i-\langle p^i\rangle)(p^j-\langle p^j\rangle)\rangle=(\langle n^2\rangle-\langle n\rangle^2)\hbar^2k^ik^j\,. \tag{130}$$

When $\langle(\Delta n)^2\rangle=\langle n^2\rangle-\langle n\rangle^2$ is proportional to t, this spreading of momentum is described by the diffusion tensor

$$\begin{aligned} D_I^{ij}&=\langle(p^i-\langle p^i\rangle)(p^j-\langle p^j\rangle)\rangle/2t\\ &=\hbar^2k^ik^j\langle(\Delta n)^2\rangle/2t\,. \end{aligned} \tag{131}$$

This argument shows that the coefficients of induced diffusion in a plane running wave are determined by the statistics of spontaneous emission. More precisely, the induced coefficients are determined by $\langle(\Delta n)^2\rangle$ in the limit $t\gg\tau=1/A$, for it is only in this limit that $\langle(\Delta n)^2\rangle$ is proportional to t (the momentum statistics are Markovian only for time intervals much longer than τ). Since D_I^{ij} is proportional to $\langle(\Delta n)^2\rangle$, our theory makes a definite prediction concerning the statistics of spontaneous emission in resonance fluorescence.

Equating (124) and (131) and dividing the resulting equation by (129), we obtain

$$\frac{\langle(\Delta n)^2\rangle}{\langle n\rangle}=1-\frac{2\Omega^2[3A^2-4(\Delta-\vec{k}\cdot\vec{p}/M)^2]}{[4(\Delta-\vec{k}\cdot\vec{p}/M)^2+A^2+2\Omega^2]^2}\,. \tag{132}$$

For an atom at rest, this becomes

$$\frac{\langle(\Delta n)^2\rangle-\langle n\rangle}{\langle n\rangle}=-\frac{2\Omega^2(3A^2-4\Delta^2)}{(4\Delta^2+A^2+2\Omega^2)^2}\,. \tag{133}$$

Equation (133) indicates that the statistics of n are *not* Poissonian, since $\langle(\Delta n)^2\rangle\neq\langle n\rangle$. This result disagrees with the conclusion of Picqué that n follows a Poisson Law.[29] The disagreement appears to result from Picqué's implicit assumption that successive photon-scattering processes are statistically independent. Recently Mandel has studied the statistics of spontaneous emission in resonance fluorescence, and has given explicit results for the case of exact resonance.[38] We note that Mandel's rigorous on-resonance result for $[\langle(\Delta n)^2\rangle-\langle n\rangle]/\langle n\rangle$, in the limit $t\to\infty$, is in exact agreement with (133) at $\Delta=0$. We conclude that induced momentum diffusion in a plane running wave is a direct result of the dispersion of the number of spontaneously emitted photons, and that the quasiclassical equations correctly account for the non-Poissonian statistics of spontaneous emission.

Finally, we derive some relations describing the transfer of energy and momentum from the field to the atom. First, multiplying Eq. (113) by p^k, integrating over phase space, and evaluating some integrals by parts (using $f=0$ at $|\vec{x}|=\infty$ and $|\vec{p}|=\infty$), we obtain

$$\frac{d\langle p^k\rangle_{av}}{dt}=\langle F_e^k\rangle_{av}\,, \tag{134}$$

where the $\langle\cdots\rangle_{av}$ indicates an average with respect to the Wigner function, $\langle A\rangle_{av}\equiv\int\int d^3x\,d^3p\,Af$. Hence the average rate of momentum transfer equals the average effective force.

Next we multiply Eq. (113) by the kinetic energy $K=p^kp^k/2M$, integrate over phase space, and again evaluate various integrals by parts to get

$$\frac{d\langle K\rangle_{av}}{dt}=\frac{1}{M}(\langle F_e^kp^k\rangle_{av}+\langle D^{kk}\rangle_{av})\,. \tag{135}$$

This equation states, in particular, that the diffusion term in the Fokker-Planck equation tends to increase the kinetic energy of the atom at the rate $\langle D^{kk}\rangle_{av}/M$. If we are willing to ignore the nonisotropic character of the velocity distribution, we can say that the atom is heated (or cooled) at the rate $dT/dt=2(d\langle K\rangle_{av}/dt)/3k_B$, where k_B is Boltzmann's constant.

When the radiation force $F_e^i=F_v^i+F_D^i$ consists of a part $F_v^i=-\partial V/\partial x^i$ derivable from a potential V

and a part F_D^k not derivable from a potential, say a dissipative force, we have $F_v^k p^k/M = -dV/dt$, and Eq. (135) yields the expression

$$\frac{d\langle E\rangle_{\text{av}}}{dt} = \frac{1}{M}\left(\langle F_D^k p^k\rangle_{\text{av}} + \langle D^{kk}\rangle_{\text{av}}\right) \tag{136}$$

for the rate of change of the total translational energy $\langle E\rangle_{\text{av}} = \langle K\rangle_{\text{av}} + \langle V\rangle_{\text{av}}$. For example, in a radiation trap formed by the dipole force (125) tuned below resonance ($\Delta < 0$), which is derivable from the potential $V = \frac{1}{2}\hbar\Delta \,\text{Ln}\,[1 + 2\Omega^2/(4\Delta^2 + A^2)]$, an atom initially near the minimum of the potential well gains energy at the rate $\langle D^{kk}\rangle_{\text{av}}/M$ and escapes from the trap in a time of order $\Delta t = MV_0/\langle D^{kk}\rangle_{\text{av}}$, where V_0 is the depth of the well. On the other hand, if a dissipative force is present, the atom gains or loses energy at the rate given by (136) until the rate of dissipation $-\langle F_D^k p^k/M\rangle_{\text{av}}$ equals the rate $\langle D^{kk}\rangle_{\text{av}}/M$ of energy input due to fluctuations. It is this condition that determines the temperature achievable by radiation cooling. Specific examples are discussed in the following.

VI. WEAK-FIELD THEORY

In the preceding section we considered the special case in which the Rabi frequency Ω and the phase derivative $\dot\theta = \vec{p}\cdot\vec\nabla\theta/M$, at the moving atom, vary by only a small amount during a natural lifetime $\tau = 1/A$. These conditions are often satisfied in practice. For example, they are satisfied in a plane running wave and in the experiment of Bjorkholm *et al.*[18] in which an atomic beam copropagates with a Gaussian laser beam. The conditions are not satisfied, however, when an atom moves with typical thermal velocity v across the nodes and antinodes of a plane standing wave ($\Omega \propto \cos kx$) of visible light. Here Ω varies with frequency kv, and usually this is not small compared to A. So a different approach must be used in this and related problems in which the field is not smooth or the atom moves rapidly. In this section we derive a Fokker-Planck equation for the Wigner function without placing any constraint on the smoothness of the field or the speed of the atom, but we require that the field be weak.

In a weak field ($\Omega \ll A$), the distributed inversion W does not deviate much from the ground-state value $W = -f$. For this value of W, Eqs. (106) and (107) read

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{M}\cdot\vec\nabla\right)U = \left(\Delta + \frac{\vec{p}}{M}\cdot\vec\nabla\theta\right)V - \tfrac{1}{2}AU - \tfrac{1}{2}\hbar\vec\nabla\Omega\cdot\vec\nabla_p f\,, \tag{137}$$

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{M}\cdot\vec\nabla\right)V = -\left(\Delta + \frac{\vec{p}}{M}\cdot\vec\nabla\theta\right)U - \tfrac{1}{2}AV - \tfrac{1}{2}\hbar\Omega\vec\nabla\theta\cdot\vec\nabla_p f - \Omega f\,. \tag{138}$$

Let

$$\Sigma = \tfrac{1}{2}(U - iV)e^{-i(\theta+\Delta t)}\,. \tag{139}$$

Then Eqs. (137) and (138) may be written together as

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{M}\cdot\vec\nabla + \tfrac{1}{2}A\right)\Sigma = \Gamma\,, \tag{140}$$

where

$$\Gamma = -\tfrac{1}{2}\left(\tfrac{1}{2}\hbar\vec\nabla\tilde\Omega^*\cdot\vec\nabla_p f - i\tilde\Omega^* f\right)e^{-i\Delta t}\,, \tag{141}$$

and, as before, $\tilde\Omega = \Omega e^{i\theta}$. The solution of Eq. (140) is

$$\Sigma(\vec{x},\vec{p},t) = \int_{-\infty}^{t}\Gamma(\vec{x} - \vec{p}(t-s)/M, \vec{p}, s)e^{-A(t-s)/2}\,ds\,. \tag{142}$$

Insertion of (139) and (141) into (142) and a change of integration variables, $t' = t - s$, yields

$$U - iV = e^{i\theta}\int_0^\infty dt'\left[i\tilde\Omega^*(\vec{x} - \vec{p}t'/M)f(\vec{x} - \vec{p}t'/M, \vec{p}, t - t') - \tfrac{1}{2}\hbar\vec\nabla\tilde\Omega^*(\vec{x} - \vec{p}t'/M)\cdot\vec\nabla_p f(\vec{x} - \vec{p}t'/M, \vec{p}, t - t')\right]e^{(i\Delta - A/2)t'}\,. \tag{143}$$

Because of the exponential factor $\exp(-At'/2)$ in (143), $f(\vec{x},\vec{p},t-t')$ makes a significant contribution to $U(\vec{x},\vec{p},t)$ and $V(\vec{x},\vec{p},t)$ only for $t' \lesssim 2/A = 2\tau$. During this short time interval, $f(\vec{x},\vec{p},t)$ behaves very nearly as if the atom were free,

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{M}\cdot\vec\nabla\right)f = 0\,. \tag{144}$$

The solution of this equation is $f(\vec{x},\vec{p},t) = f(\vec{x} - \vec{p}(t-t_0)/M, \vec{p}, t_0)$. So in Eq. (143) we have $f(\vec{x} - \vec{p}t'/M, \vec{p}, t - t') = f(\vec{x},\vec{p},t)$, and hence the quantities f and $\vec\nabla_p f$ can be taken outside of the integral,

$$U - iV = -\tfrac{1}{2}e^{i\theta}\left(\hbar\vec\nabla J'\cdot\vec\nabla_p f - 2iJ'f\right)\,, \tag{145}$$

where

$$J'(\vec{x},\vec{p}) = \int_0^\infty dt\,\tilde\Omega^*(\vec{x} - \vec{p}t/M)e^{(i\Delta - A/2)t}\,. \tag{146}$$

The identity

$$e^{i\theta}\vec\nabla J' = \vec\nabla(e^{i\theta}J') - i\vec\nabla\theta(e^{i\theta}J') \tag{147}$$

converts (145) to the more convenient form

$$U-iV=-\tfrac{1}{2}[\hbar(\vec{\nabla}J-iJ\vec{\nabla}\theta)\cdot\vec{\nabla}_p f-2iJf]\,, \qquad (148)$$

where

$$J(\vec{x},\vec{p})=e^{i\theta(\vec{x})}\int_0^\infty dt\,\tilde{\Omega}^*(\vec{x}-\vec{p}t/M)e^{(i\Delta-A/2)t}\,, \qquad (149)$$

or

$$U=-J_I f-\tfrac{1}{2}\hbar(\vec{\nabla}J_R+J_I\vec{\nabla}\theta)\cdot\vec{\nabla}_p f\,, \qquad (150)$$

$$V=-J_R f+\tfrac{1}{2}\hbar(\vec{\nabla}J_I-J_R\vec{\nabla}\theta)\cdot\vec{\nabla}_p f\,, \qquad (151)$$

where J_R and J_I are the real and imaginary parts of J, respectively. This completes the solution for U and V.

Next consider the function $f+W$ appearing in the last term of Eq. (105). Adding Eqs. (105) and (108) we get the equation

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}+A\right)(f+W)=-\Omega V-\tfrac{1}{2}\hbar(\vec{\nabla}\Omega\cdot\vec{\nabla}_p U+\Omega\vec{\nabla}\theta\cdot\vec{\nabla}_p V) \qquad (152)$$

for this function. Using V from (151) in (152), and keeping only the term on the right of lowest order in $\hbar$, we have

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}+A\right)(f+W)=\Omega J_R f\,. \qquad (153)$$

Equation (153) is solved in the same manner as Eq. (140). We find

$$f+W=Nf\,, \qquad (154)$$

where

$$N(\vec{x},\vec{p})=\int_0^\infty dt\,\Omega(\vec{x}-\vec{p}t/M)J_R(\vec{x}-\vec{p}t/M,\vec{p})e^{-At}\,. \qquad (155)$$

Finally, on substituting (150), (151), and (154) into Eq. (105), we arrive at the Fokker-Planck equation

$$\left(\frac{\partial}{\partial t}+\frac{p^i}{M}\frac{\partial}{\partial x^i}\right)f=-\frac{\partial}{\partial p^i}(F_e^i f)+\frac{\partial^2}{\partial p^i\partial p^j}(D^{ij}f)\,, \qquad (156)$$

where

$$F_e^i=F^i+\frac{\partial D_I^{ij}}{\partial x^j}\,, \qquad (157)$$

$$D^{ij}=D_s^{ij}+D_I^{ij}\,, \qquad (158)$$

$$F^i=-\frac{\hbar}{2}\left(J_I\frac{\partial\Omega}{\partial x^i}+\Omega J_R\frac{\partial\theta}{\partial x^i}\right), \qquad (159)$$

$$D_s^{ij}=NQ^{ij}\,, \qquad (160)$$

$$D_I^{ij}=\frac{\hbar^2}{4}\left[\frac{\partial\Omega}{\partial x^i}\left(\frac{\partial J_R}{\partial x^j}+J_I\frac{\partial\theta}{\partial x^j}\right)-\Omega\frac{\partial\theta}{\partial x^i}\left(\frac{\partial J_I}{\partial x^j}-J_R\frac{\partial\theta}{\partial x^j}\right)\right]. \qquad (161)$$

We now look at two examples. For a plane running wave [$\Omega=$ constant, $\theta(\vec{x})=-\vec{k}\cdot\vec{x}$], the weak-field theory gives

$$F^i=\frac{\hbar A\Omega^2k^i}{4(\Delta-\vec{k}\cdot\vec{p}/M)^2+A^2}\,, \qquad (162)$$

$$D_s^{ij}=\frac{\hbar^2A\Omega^2k^2d^{ij}}{5[4(\Delta-\vec{k}\cdot\vec{p}/M)^2+A^2]}\,, \qquad (163)$$

$$D_I^{ij}=\frac{\hbar^2A\Omega^2k^ik^j}{2[4(\Delta-\vec{k}\cdot\vec{p}/M)^2+A^2]}\,. \qquad (164)$$

These results agree with the smooth-field results, Eqs. (122)–(124), in the limit of small Ω, as they should.

In a plane standing wave, we have $\Omega(\vec{x})=2\Omega_0\cos\vec{k}\cdot\vec{x}$ and $\theta(\vec{x})=0$, where Ω_0 is the Rabi frequency in one of the two counterpropatating running waves that comprise the standing wave. The weak-field theory yields

$$F^i=2\Omega_0^2\hbar k^i(AF_-\sin^2\vec{k}\cdot\vec{x}+2G_+\sin\vec{k}\cdot\vec{x}\cos\vec{k}\cdot\vec{x})\,, \qquad (165)$$

$$D_I^{ij}=\hbar^2\Omega_0^2k^ik^j(AF_+\sin^2\vec{k}\cdot\vec{x}+2G_-\sin\vec{k}\cdot\vec{x}\cos\vec{k}\cdot\vec{x})\,, \qquad (166)$$

$$D_s^{ij}=\tfrac{1}{5}\hbar^2A\Omega_0^2k^2d^{ij}(F_++(4\omega_D^2+A^2)^{-1}\{[(A^2-4\omega_D^2)F_++4\Delta\omega_DF_-]\cos2\vec{k}\cdot\vec{x}+2A(2\omega_DF_+-\Delta F_-)\sin2\vec{k}\cdot\vec{x}\})\,, \qquad (167)$$

where

$$F_\pm=\frac{1}{4(\Delta-\omega_D)^2+A^2}\pm\frac{1}{4(\Delta+\omega_D)^2+A^2}\,, \qquad (168)$$

$$G_\pm=\frac{\Delta-\omega_D}{4(\Delta-\omega_D)^2+A^2}\pm\frac{\Delta+\omega_D}{4(\Delta+\omega_D)^2+A^2}\,, \qquad (169)$$

and $\omega_D=\vec{k}\cdot\vec{p}/M$ is the "Doppler shift." It is easy to show that F_- is equal to a positive quantity times $\Delta\omega_D=\Delta kv$, where v is the component of velocity in direction $\vec{k}$. So, for $\Delta<0$, the first term in (165) is a dissipative force with a strong position dependence due to the factor $\sin^2\vec{k}\cdot\vec{x}$. The second term in (165) is the dipole force associated with the amplitude gradient of the standing wave.

These results are a little cumbersome. Simpler results are obtained for the time-averaged radiation force and diffusion coefficients, on the assumption that the atom moves nearly uniformly in direction $\vec{k}$,

$$\langle F^i\rangle_{av} = \hbar A\Omega_0^2 k^i \left(\frac{1}{4(\Delta-\omega_D)^2+A^2} - \frac{1}{4(\Delta+\omega_D)^2+A^2}\right), \tag{170}$$

$$\langle D_I^{ij}\rangle_{av} = \tfrac{1}{2}\hbar^2 A\Omega_0^2 k^i k^j \left(\frac{1}{4(\Delta-\omega_D)^2+A^2} + \frac{1}{4(\Delta+\omega_D)^2+A^2}\right), \tag{171}$$

$$\langle D_s^{ij}\rangle_{av} = \tfrac{1}{5}\hbar^2 A\Omega_0^2 k^2 d^{ij} \left(\frac{1}{4(\Delta-\omega_D)^2+A^2} + \frac{1}{4(\Delta+\omega_D)^2+A^2}\right). \tag{172}$$

The radiation force (170) agrees with an earlier result based on the Ehrenfest-Bloch equations.[26] The time-averaged radiation force and diffusion coefficients are the sums, respectively, of the radiation forces and diffusion coefficients of the two counterpropagating running waves, as if those waves acted independently.

We can now estimate the temperature achievable by radiation cooling. For small $\omega_D = \vec{k}\cdot\vec{p}/M$, Eqs. (170)–(172) reduce to

$$\langle F^i\rangle_{av} = \frac{16\hbar A\Omega_0^2 \Delta k^i k^j p^j}{M(4\Delta^2+A^2)^2}\ , \tag{173}$$

$$\langle D_I^{ij}\rangle_{av} = \frac{\hbar^2 A\Omega_0^2 k^i k^j}{4\Delta^2+A^2}\ , \tag{174}$$

$$\langle D_s^{ij}\rangle_{av} = \frac{2\hbar^2 A\Omega_0^2 k^2 d^{ij}}{5(4\Delta^2+A^2)}\ . \tag{175}$$

When $\Delta < 0$ the radiation force Eq. (173) damps the component of atomic velocity in direction $\vec{k}$. The maximum damping force

$$\langle F^i\rangle_{av} = -3^{3/2}\hbar\Omega_0^2 k^i k^j p^j / 2MA^2 \tag{176}$$

obtains for $\Delta = -A/\sqrt{12}$. In this case, $\langle D_I^{ij}\rangle_{av} = 3\hbar^2\Omega_0^2 k^i k^j/4A$ and $\langle D_s^{ij}\rangle_{av} = 3\hbar^2\Omega_0^2 k^2 d^{ij}/10A$. The components of atomic motion transverse to $\vec{k}$ are not cooled, but rather are heated by spontaneous fluctuations. These components might also be cooled by application of additional standing waves with $\vec{k}$'s in these directions. Looking only at the component of atomic motion parallel to $\vec{k}$, we read from the equation of motion $\dot{p} = -3^{3/2}\hbar\Omega_0^2 k^2 p/2MA^2$, the damping time $\tau_D = 2MA^2/3^{3/2}\hbar\Omega_0^2 k^2$, and for $\vec{k}$ in the x^1 direction (orthogonal to $\vec{\mu}$) the rate of dissipation $-\langle F\rangle_{av} p/M$ equals the rate of energy input $\langle D_I^{11} + D_s^{11}\rangle_{av}/M$ when

$$p^2/2M = k_B T/2 \approx 0.2\hbar A\ . \tag{177}$$

Thus for typical atomic parameters $A = 10^8$ sec^{-1}, $M = 4\times10^{-23}$ g, $k = 10^5$ cm^{-1}, and $\Omega_0 = 10^7$ sec^{-1}; this component of the motion is cooled to temperature $T \approx 3\times10^{-4}$ K in time $\Delta t \sim \tau_D \approx 10^{-4}$ sec. Of course, the initial velocity p/M must be farily small to be in resonance with the field. Roughly speaking, an atom can be cooled, at the above rate, from the resonance condition $|kp/M| \lesssim A$ to the energy condition $p^2/2M \sim \hbar A$.

It should be noted that in both the smooth-field and weak-field approximations we have tacitly assumed that the interaction time is long compared to the atomic natural lifetime in order to derive the Fokker-Planck equations. Therefore the Fokker-Planck equations derived here are not valid for short interaction time, and, in fact, no Fokker-Planck equation is valid for interaction time $t \lesssim \tau = 1/A$ because the momentum statistics are not Markovian on such a short time scale. For example, a Fokker-Planck equation is incapable of describing the optical Stern-Garlach effect. For this reason all of the Fokker-Planck type descriptions of atomic motion in monochromatic radiation that have appeared in the literature are of limited validity, and only the quasiclassical equations or Eqs. (94)–(97) provide a truly general description of atomic motion in coherent radiation.

VII. ONE-DIMENSIONAL EXAMPLES

First let us estimate the heating rate in a strong ($\Omega_0 \gg A$) resonant ($\Delta = 0$) plane standing wave $\Omega(x) = 2\Omega_0 \cos kx$. The diffusion coefficient (128), averaged over the wave, is $\bar{D} = (\hbar k)^2\Omega_0^2/A$, and the rate of change of kinetic energy (135), for small atomic velocity, is $\dot{K} = k_B\dot{T}/2 = \bar{D}/M$ ($F_e \approx F = 0$ for $\Delta = 0$). So for typical atomic parameters $A = 10^8$ sec^{-1}, $k = 2\pi/\lambda = 10^5$ cm^{-1}, $M = 25$ amu $= 4\times10^{-23}$ g, and a moderately strong field $\Omega_0 = 10^9$ sec^{-1}, we have $\dot{T} = 4\times10^4$ K sec^{-1}. Here induced diffusion leads to a heating rate about 300 times larger than in a plane running wave of the same intensity. Of course, a heating rate of this magnitude persists only until the atom is driven out of resonance with the field. We should note in this connection that, since the diffusion tensor (128) is valid also off resonance in a sufficiently strong field, induced diffusion in a *strong* standing wave tends to inhibit the cooling process considered in Sec. VI.

Consider next atomic trapping by the dipole force (125) in a field of amplitude $\Omega(x) = \Omega_0 \exp(-x^2/w_0^2)$, e.g., transverse trapping on a diameter of a Gaussian laser beam. We shall treat the case of large detuning ($4\Delta^2 \gg A^2 + 2\Omega^2$) below resonance ($\Delta < 0$). In this case, the potential energy $V = \frac{1}{2}\hbar\Delta \ln[1 + 2\Omega^2/(4\Delta^2 + A^2)]$ simplifies to $V = \hbar\Omega^2/4\Delta$, and the well depth is

$$V_0 = -\hbar\Omega_0^2/4\Delta\ . \tag{178}$$

In the limit of large Δ^2, the coefficients of spontaneous and induced diffusion, Eqs. (126) and

(127), reduce to

$$D_s = \hbar^2 A\Omega^2 k^2/20\Delta^2 , \tag{179}$$

$$D_I = \frac{\hbar^2 A}{8\Delta^2}\left(\frac{\partial\Omega}{\partial x}\right)^2 . \tag{180}$$

Since $D = D_s + D_I$ is proportional to $1/\Delta^2$ and V_0 is proportional to $1/\Delta$, it is clear that the confinement time $\Delta t = MV_0/\langle D\rangle_{av} \propto \Delta$ can be made arbitrarily long by detuning far below resonance. However, this conclusion is based on the assumption that the initial kinetic energy of the atom is exactly zero. In practice, the initial kinetic energy cannot be made arbitrarily small. A reasonable lower bound on the initial kinetic energy is the value $p^2/2M \sim \hbar A$ that might be achieved by radiation cooling. For this kinetic energy the atom is strongly trapped initially only if $V_0 \gg \hbar A$, and this yields the constraint $|\Delta| \ll \Omega_0^2/4A$ on the detuning. A conservative estimate of the confinement time is then obtained by replacing Ω by Ω_0 in (179) and $(\partial\Omega/\partial x)^2$ by Ω_0^2/w_0^2 in (180). If w_0 is much larger than the wavelength λ ($k^2 \gg 1/w_0^2$), the induced coefficient $D_I \approx \hbar^2 A\Omega_0^2/8\Delta^2 w_0^2$ is negligible compared to $D_s \approx \hbar^2 A\Omega_0^2 k^2/20\Delta^2$, and the confinement time becomes

$$\Delta t \approx MV_0/D_s \approx -5M\Delta/\hbar A k^2 . \tag{181}$$

If we push $|\Delta|$ to the upper limit $\Omega_0^2/4A$ of the constraint on $|\Delta|$, we obtain, for $k = 10^5$ cm^{-1}, $M = 4\times 10^{-23}$ g, $A = 10^7$ sec^{-1}, and $\Omega_0 = 10^{10}$ sec^{-1}, the value $\Delta t = 5M\Omega_0^2/4\hbar A^2 k^2 = 5$ sec. This value is in accord with the estimate of Gordon and Ashkin for the achievable confinement time in the potential well at the focus of a Gaussian laser beam in the absence of damping.[39] For a beam of radius $w_0 = 10$ μm the power required in the present example is 50 mW. It remains to be seen whether application of additional damping fields will lead to stable radiation trapping.

We look now at some problems for which the atomic motion is described by the Fokker-Planck equation

$$\frac{\partial f}{\partial t} + \frac{p}{M}\frac{\partial f}{\partial x} = -\frac{\partial}{\partial p}\left[(F_0 - \beta p) f\right] + D\frac{\partial^2 f}{\partial p^2} \tag{182}$$

in which F_0, β, and D are constants. Integration of Eq. (182) over x yields the equation

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial p}\left[(F_0 - \beta p) W\right] + D\frac{\partial^2 W}{\partial p^2} \tag{183}$$

for the momentum distribution $W(p) = \int f(x,p)\,dx$. The solution of (183) for the initial condition $W(p,0) = \delta(p - p_0)$ is

$$G(p,p_0;t) = \left(\frac{\beta}{2\pi D(1-e^{-2\beta t})}\right)^{1/2} \exp\left(-\frac{\beta[p - p_0 e^{-\beta t} - F_0(1-e^{-\beta t})/\beta]^2}{2D(1-e^{-2\beta t})}\right), \tag{184}$$

and hence the general solution of (183) is

$$W(p,t) = \int dp_0\, G(p,p_0;t)\, W(p_0,0) . \tag{185}$$

Let the initial distribution of p be a Gaussian of variance σ_0^2 centered at $p = 0$,

$$W(p,0) = (2\pi\sigma_0^2)^{-1/2}\exp(-p^2/2\sigma_0^2) . \tag{186}$$

Then (185) yields the solution

$$W(p,t) = (2\pi\sigma^2)^{-1/2}\exp[-(p-\bar{p})^2/2\sigma^2] , \tag{187}$$

where

$$\bar{p}(t) = F_0(1 - e^{-\beta t})/\beta , \tag{188}$$

$$\sigma^2(t) = \sigma_0^2 e^{-2\beta t} + D(1 - e^{-2\beta t})/\beta . \tag{189}$$

The mean momentum acquires the value $\bar{p} = F_0/\beta$ in time $t \approx 1/\beta$, and the variance of momentum decays from σ_0^2 to $\sigma^2 = D/\beta$ in time $t \approx 1/2\beta$. In the limit $\beta \to 0$,

$$\bar{p}(t) = F_0 t , \tag{190}$$

$$\sigma^2(t) = \sigma_0^2 + 2Dt . \tag{191}$$

The one-dimensional temperature of the distribution (187) is $T = \sigma^2/Mk_B$.

In a strong $[2\Omega^2 \gg A^2 + (kp/M)^2]$ resonant ($\Delta = 0$) running wave, the radiation force, Eq. (122), and total diffusion coefficient, Eq. (123) plus Eq. (124), saturate to values $F_0 = \frac{1}{2}A\hbar k$ and $D = \frac{7}{20}A(\hbar k)^2$ which are independent of p, i.e., $\beta = 0$. Thus, in this case

$$\bar{p} = \tfrac{1}{2}A\hbar k t , \tag{192}$$

$$\sigma^2 = \sigma_0^2 + \tfrac{7}{10}A(\hbar k)^2 t . \tag{193}$$

If the radiation is applied transversely to an atomic beam, (192) is the transverse momentum acquired by the beam, and (193) indicates that the transverse temperature of the beam increases at the rate $\dot{T} = \frac{7}{10}A(\hbar k)^2/Mk_B$. Note that because of induced diffusion $\dot{T}$ is 3.5 times larger than if only spontaneous recoils had been considered.

Suppose the running wave is tuned to $\Delta = -\Omega/\sqrt{2}$. Then, for $\sigma_0^2 \lesssim (M\Omega/k)^2/2$, the strong-field radiation force

$$F = \frac{A\Omega^2\hbar k}{4(\Delta - kp/M)^2 + 2\Omega^2} \tag{194}$$

may be expanded to first order in p as $F=F_0-\beta p$, where

$$F_0=\tfrac{1}{4}A\hbar k\,, \tag{195}$$

$$\beta=A\hbar k^2/2^{3/2}M\Omega\,, \tag{196}$$

and the total diffusion coefficient, at $p=0$, is

$$D=\tfrac{7}{40}A(\hbar k)^2\,. \tag{197}$$

According to Eq. (189), the transverse temperature of an atomic beam crossing the wave is now cooled from an initial value as large as $T_0\approx M\Omega^2/2k_Bk^2$ to a final value $T_F=D/\beta Mk_B\approx\hbar\Omega/2k_B$ in time $\Delta t\approx\frac{1}{2}\beta=2^{1/2}M\Omega/A\hbar k^2$, and the deflection of the beam during this time interval, $\Delta\bar{p}=F_0\Delta t\approx 0.3M\Omega/k$, does not violate the approximation. The fractional decrease of transverse temperature $T_F/T_0=\hbar k^2/M\Omega$ for $k=10^5\ \text{cm}^{-1}$, $M=4\times10^{-23}$ g, and $\Omega=10^9\ \text{sec}^{-1}$, is $\approx\frac{1}{4000}$, and the divergence angle of the beam is correspondingly decreased by the factor $\frac{1}{62}$. The required thickness of interaction region is $L\approx 2.5$ cm for typical beam velocity $v=5\times10^4$ cm/sec and $A=10^8\ \text{sec}^{-1}$. A related calculation was carried out by Krasnov and Shaparev neglecting radiative diffusion.[40]

VIII. CONCLUSION

Although the general equations of motion for a two-level atom in a monochromatic applied field were written down in Eqs. (94)–(97), only the quasiclassical equations Eqs. (105)–(108) were used in the analysis of specific problems. It is important, therefore, to understand the principle limitation of the quasiclassical equations.

The quasiclassical equations describe a continuous flow of momentum from the field to the atom, while the "exact" equations, Eqs. (94)–(97), describe momentum transfer in discrete units of magnitude $\hbar k$ (the convolutions over momentum space in these equations give rise to displacements of atomic momentum of magnitude $\hbar k=\hbar\omega_0/c$). Thus the quasiclassical approximation replaces the true discontinuous momentum-transfer process by a smoothed continuous transfer of momentum, and hence the quasiclassical equations are valid only when the fine-grain quantum-mechanical aspect of momentum transfer is unimportant. It is clear from the derivation of the quasiclassical equations that the discontinuous character of momentum transfer is not important when f, U, V, and W are smooth functions of p over a distance $\hbar k$ in momentum space. Discrete momentum transfer is also unimportant in experiments in which the resolution of momentum measurement is larger than $\hbar k$.[41] In addition, if f, U, V, and W are not initially smooth but interact with the radiation for a time that is long compared to the natural atomic lifetime, then it appears likely that after a few spontaneous events these functions are smoothed sufficiently by spontaneous diffusion to again permit application of the quasiclassical equations.[42] These cases cover a wide range of experimental conditions. However, if the distributions f, U, V, and W are initially well localized in momentum space, as in a well collimated atomic beam, if the atom-field interaction time is less than or comparable to the time between spontaneous events, and if the resolution of momentum measurement is better than $\hbar k$, then the quasiclassical equations fail and the more cumbersome Eqs. (94)–(97) must be used.

It was shown in Secs. V and VI that in the smooth-field and weak-field limits the distributed Bloch vector (U, V, W) can be eliminated from the coupled quasiclassical equations, and a single equation of motion can be written for the Wigner function. In the case where the field is strong $(\Omega\gtrsim A)$ and the Rabi frequency Ω or phase derivative $\dot{\theta}$, at the moving atom, is not a slowly varying function of time, we have not yet found a simple reduction of the quasiclassical equations to a Fokker-Planck equation for f. A number of problems of current experimental interest fall into this category, and have not yet received completely satisfactory theoretical treatments. The problem of cooling an atomic vapor by a strong standing wave[43] and the question whether additional weak damping fields lead to stable trapping in a deep potential well of the dipole force are of this type. We hope to address these and related problems in a future publication.

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APPENDIX: DERIVATION OF THE FOKKER-PLANCK EQUATION IN THE SMOOTH-FIELD APPROXIMATION

Here we show how the coupled quasiclassical equations for f, U, V, and W lead to a single

Fokker-Planck equation for f when the applied field is sufficiently smooth or when the atom moves sufficiently slowly. The Fokker-Planck equation for f, accurate through terms of order $\hbar^2$, is obtained by first solving Eqs. (106)–(108) for U, V, and W in terms of f, to first order in $\hbar$, and then using the result in Eq. (105).

Since $Q^{ij}=(\hbar\omega_0/c)^2 d^{ij}/10$ is of second order in $\hbar$, the last term in Eq. (108) makes no first-order contribution to U, V, or W and may therefore be discarded. Equations (106)–(108) become

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)B^i=M^{ij}B^j+Z^i\,, \tag{A1}$$

where $(B^1,B^2,B^3)=(U,V,W)$,

$$\begin{aligned} Z^1&=-\tfrac{1}{2}\hbar\vec{\nabla}\Omega\cdot\vec{\nabla}_p f\,,\\ Z^2&=-\tfrac{1}{2}\hbar\Omega\vec{\nabla}\theta\cdot\vec{\nabla}_p f\,,\\ Z^3&=-Af\,,\end{aligned} \tag{A2}$$

$$M^{ij}=\begin{pmatrix} -\tfrac{1}{2}A & \bar{\Delta} & 0\\ -\bar{\Delta} & -\tfrac{1}{2}A & \Omega\\ 0 & -\Omega & -A\end{pmatrix}, \tag{A3}$$

$$\bar{\Delta}=\Delta+\vec{p}\cdot\vec{\nabla}\theta/M\,, \tag{A4}$$

and we are using the summation convention in (A1). Apart from transients, which depend on initial conditions and which decay to zero in a time of order $\tau=1/A$, the solution of (A1) is

$$B^i(\vec{x},\vec{p},t)=\int_{-\infty}^{t}dt'\,v^{ij}(\vec{x}-\vec{p}(t-t')/M,\vec{p},t-t')Z^j(\vec{x}-\vec{p}(t-t')/M,\vec{p},t')\,, \tag{A5}$$

where $v^{ij}(\vec{x},\vec{p},t)$ is a solution of

$$\frac{\partial v^{ij}(\vec{x},\vec{p},t)}{\partial t}=M^{ik}(\vec{x}+\vec{p}t/M,\vec{p})v^{kj}(\vec{x},\vec{p},t) \tag{A6}$$

satisfying the initial condition $v^{ij}(\vec{x},\vec{p},0)=\delta^{ij}$. Because of the relaxation terms in (A6), $v^{ij}(t)$ decays with time constant of order $\tau=1/A$, and so the integrand in (A5) is exponentially small except for $t-t'\lesssim\tau$.

Now the matrix $M^{ij}(\vec{x},\vec{p})$ depends on $\vec{x}$ through the functions $\vec{\nabla}\theta(\vec{x})$ and $\Omega(\vec{x})$. We are interested in the case in which the phase gradient and the Rabi frequency, at the moving atom, are nearly constant during the relaxation time, i.e., we assume that $|\Delta\vec{x}|=|\vec{p}|\tau/M$ is small compared to the distance over which M^{ij} changes by a significant amount for all momenta entering the problem. In this case, it is easy to show that the second term of the first argument of M^{ij} in (A6) and the second term of the first argument of v^{ij} in (A5) may be set to zero with negligible loss of accuracy. We shall call this the smooth-field approximation. A change of integration variable then gives (A5) the form

$$B^i(\vec{x},\vec{p},t)=\int_0^{\infty}ds\,v^{ij}(\vec{x},\vec{p},s)Z^j(\vec{x}-\vec{p}s/M,\vec{p},t-s) \tag{A7}$$

and (A6) reads

$$\frac{\partial v^{ij}(\vec{x},\vec{p},s)}{\partial s}=M^{ik}(\vec{x},\vec{p})v^{kj}(\vec{x},\vec{p},s)\,. \tag{A8}$$

In terms of B^i, Eq. (105) becomes

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)f=-\tfrac{1}{2}\hbar(\vec{\nabla}\Omega\cdot\vec{\nabla}_pB^1+\Omega\vec{\nabla}\theta\cdot\vec{\nabla}_pB^2)+Q^{ij}\frac{\partial^2(f+B^3)}{\partial p^i\partial p^j}\,. \tag{A9}$$

When Eqs. (A2) are inserted in (A7) and the result is used in (A9), we obtain a single integrodifferential equation for $f(\vec{x},\vec{p})$.

Let us first evaluate this equation to order $\hbar$. To do so we need B^1 and B^2 to order $\hbar^0$ [note that the last term in (A9) does not contribute because Q^{ij} is of order $\hbar^2$]. From (A2) we see that to order $\hbar^0$, $Z^1=Z^2=0$ and $Z^3=-Af$. So (A7) yields

$$B^i(\vec{x},\vec{p},t)=-A\int_0^{\infty}ds\,v^{i3}(\vec{x},\vec{p},s)f(\vec{x}-\vec{p}s/M,\vec{p},t-s) \tag{A10}$$

to this order in $\hbar$. But to order $\hbar^0$ (A9) is

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)f=0 \tag{A11}$$

which implies $f(\vec{x}-\vec{p}s/M,\vec{p},t-s)=f(\vec{x},\vec{p},t)$. Hence (A10) becomes $B^i=-AI^{i3}f$, where

$$I^{ij}(\vec{x},\vec{p})=\int_0^{\infty}ds\,v^{ij}(\vec{x},\vec{p},s)\,. \tag{A12}$$

It follows that the equation for f to first order in $\hbar$ is

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{M}\cdot\vec{\nabla}\right)f=\tfrac{1}{2}\hbar A[\vec{\nabla}\Omega\cdot\vec{\nabla}_p(I^{13}f)+\Omega\vec{\nabla}\theta\cdot\vec{\nabla}_p(I^{23}f)]\,. \tag{A13}$$

Using $v^{ij}(0)=\delta^{ij}$ and $v^{ij}(\infty)=0$, the integral of (A8) from $s=0$ to $s=\infty$ becomes

$$-\delta^{ij}=M^{ik}\int_0^{\infty}ds\,v^{kj}(s)$$

or

$$I^{ij}=-(M^{-1})^{ij}\,, \tag{A14}$$

and the inverse of M^{ij} is readily shown to be

$$(M^{-1})^{ij} = \frac{-1}{A(4\bar{\Delta}^2 + A^2 + 2\Omega^2)} \begin{bmatrix} 2(A^2+2\Omega^2) & 4A\bar{\Delta} & 4\bar{\Delta}\Omega \\ -4A\bar{\Delta} & 2A^2 & 2A\Omega \\ 4\bar{\Delta}\Omega & -2A\Omega & A^2+4\bar{\Delta}^2 \end{bmatrix} . \tag{A15}$$

Thus (A13) assumes the form

$$\left(\frac{\partial}{\partial t} + \frac{p^i}{M}\frac{\partial}{\partial x^i}\right) f = -\frac{\partial}{\partial p^i}(F^i f) , \tag{A16}$$

where the radiation force F^i is

$$F^i = \frac{-\hbar}{4\bar{\Delta}^2 + A^2 + 2\Omega^2}\left(A\Omega^2 \frac{\partial\theta}{\partial x^i} + \bar{\Delta}\frac{\partial\Omega^2}{\partial x^i}\right). \tag{A17}$$

We also obtain from this calculation the result

$$B^3 = -AI^{33}f = -\frac{(A^2+4\bar{\Delta}^2)f}{4\bar{\Delta}^2 + A^2 + 2\Omega^2} \tag{A18}$$

to zeroth order in $\hbar$.

Next we evaluate the integrodifferential equation for f to second order in $\hbar$. This leads to the desired Fokker-Planck equation for f. The quantities $Z^j(\vec{x} - \vec{p}s/M, \vec{p}, t-s)$ in (A7) are calculated using Eqs. (A2). We first note that, in the smooth-field approximation, the factors $\vec{\nabla}\Omega$ and $\Omega\vec{\nabla}\theta$ in (A2) may be evaluated at $\vec{x}$ instead of $\vec{x} - \vec{p}s/M$. Secondly, the distribution $f(\vec{x} - \vec{p}s/M, \vec{p}, t-s)$ in (A2) may be expanded in powers of s,

$$f(\vec{x} - \vec{p}s/M, \vec{p}, t-s) = f(\vec{x}, \vec{p}, t) - s\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{M}\cdot\vec{\nabla}\right) f(\vec{x}, \vec{p}, t) + \cdots . \tag{A19}$$

Expressions for B^1 and B^2 in terms of f to order $\hbar$ are then obtained by substituting, in turn, (A16) into (A19), (A19) into (A2), and (A2) into (A7). Discarding higher-order terms, the result is

$$B^1 = -AI^{13}f - AJ^{13}\vec{\nabla}_p\cdot(\vec{F}f) - \tfrac{1}{2}\hbar(I^{11}\vec{\nabla}\Omega + I^{12}\Omega\vec{\nabla}\theta)\cdot\vec{\nabla}_p f , \tag{A20}$$

$$B^2 = -AI^{23}f - AJ^{23}\vec{\nabla}_p\cdot(\vec{F}f) - \tfrac{1}{2}\hbar(I^{21}\vec{\nabla}\Omega + I^{22}\Omega\vec{\nabla}\theta)\cdot\vec{\nabla}_p f , \tag{A21}$$

where

$$J^{ij}(\vec{x}, \vec{p}) = \int_0^\infty ds\, s v^{ij}(\vec{x}, \vec{p}, s) . \tag{A22}$$

We now multiply (A8) by s and integrate from $s = 0$ to $s = \infty$. An integration by parts on the left-hand side yields

$$-\int_0^\infty ds\, v^{ij}(s) = M^{ik}\int_0^\infty ds\, s v^{kj}(s) , \tag{A23}$$

and using (A12) and (A14) we find

$$J^{ij} = -(M^{-1})^{ik} I^{kj} = (M^{-1})^{ik}(M^{-1})^{kj} . \tag{A24}$$

Finally, upon substituting (A18), (A20), and (A21) into (A9) and making use of (A14), (A15), (A17), and (A24), we obtain, after some straightforward manipulations, the Fokker-Planck equation

$$\left(\frac{\partial}{\partial t} + \frac{p^i}{M}\frac{\partial}{\partial x^i}\right) f = -\frac{\partial}{\partial p^i}(F_e^i f) + \frac{\partial^2}{\partial p^i \partial p^j}[(D_s^{ij} + D_I^{ij})f] , \tag{A25}$$

where D_s^{ij} and D_I^{ij} are diffusion coefficients associated with spontaneous and induced processes, respectively [D_s^{ij} and D_I^{ij} are given explicitly in Eqs. (116) and (117)], and $F_e^i = F^i + \tilde{F}^i$ is an effective radiation force consisting of the force F^i of Eq. (A17) plus the correction

$$\tilde{F}^i = \frac{\hbar}{2}\left\{\frac{\partial\Omega}{\partial x^i}\left[AF^j\frac{\partial J^{13}}{\partial p^j} + \frac{\hbar}{2}\left(\frac{\partial\Omega}{\partial x^j}\frac{\partial I^{11}}{\partial p^j} + \Omega\frac{\partial\theta}{\partial x^j}\frac{\partial I^{12}}{\partial p^j}\right)\right] + \Omega\frac{\partial\theta}{\partial x^i}\left[AF^j\frac{\partial J^{23}}{\partial p^j} + \frac{\hbar}{2}\left(\frac{\partial\Omega}{\partial x^j}\frac{\partial I^{21}}{\partial p^j} + \Omega\frac{\partial\theta}{\partial x^j}\frac{\partial I^{22}}{\partial p^j}\right)\right]\right\} , \tag{A26}$$

which is of higher order in $\hbar$ than F^i. The correction $\tilde{F}^i$ is generally quite small compared to F^i and may be ignored in most applications.

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[41]Here the term "momentum measurement" is to be understood as any measurement from which the momentum transfer can be inferred. For example, in a measurement of the deflection of an atomic beam the resolution of momentum measurement is larger than $\hbar k$ when the deflection produced by absorption of a single photon is not resolvable.
[42]See, for example, L. Mandel, J. Optics (Paris) 10, 51 (1979), in which the smoothing of the atomic momentum distribution by spontaneous emission is clearly shown.
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