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Stability criteria for high-intensity lasers

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The output characteristics of a high-intensity laser oscillator are examined by means of a semiclassical density-matrix approach. Unlike previous investigations the time derivatives of the field and material quantities are retained, and a basic instability in the semiclassical laser equations is revealed. This instability accounts for the recently reported spontaneous pulsations in xenon laser oscillators. The pulsations can be interpreted as a consequence of spectral holeburning in inhomogeneously broadened lasers, and stability criteria are derived for both standing-wave and traveling-wave oscillators. In simplest terms the pulsations should occur in any inhomogeneously broadened laser where the product of the homogeneous linewidth and the cavity lifetime is less than unity.

I. INTRODUCTION

The starting point for most rigorous treatments of laser oscillation involves the semiclassical formalism introduced by Lamb.[1,2] The results of Lamb's analysis provide at least a qualitative explanation of most laser phenomena that have been reported. Since the calculations in Ref. 1 were only carried through to third order in the electric field, the results tend to be inaccurate at high levels of intensity. This limitation can be remedied by retaining higher-order terms in the calculations,[3,4] but much more useful methods have been developed by Stenholm and Lamb[5] and by Feldman and Feld[6] for treating the properties of high-intensity lasers. In fact, it might be inferred from these and more recent works that all of the characteristics of at least one-dimensional single-mode lasers can now be predicted in detail.

Based on the semiclassical equations (or the simpler rate equation concepts) one would normally expect that a laser with continuous-wave (cw) pumping should produce its output in the form of a cw beam of light. The only exceptions would be lasers which incorporate inside the resonator either active or passive modulation media. Recently, however, an instability has been reported in which an ordinary cw laser yields a pulsed output.[7] This instability has been observed experimentally in xenon lasers and has also been reproduced in numerical solutions of the underlying semiclassical equations. The pulsations had not been noted in previous semiclassical studies of high-intensity lasers because of the assumption that with cw excitation the polarization should be an instantaneous function of the field and population inversion. The general numerical solutions of the time-dependent laser equations have proven to be costly, and it is worthwhile also to study analytically the conditions under which this instability might manifest itself. In the present work the appropriate stability criteria are derived for a high-intensity laser, and using these criteria one can readily determine whether or not a given laser will pulse without actually calculating the output waveforms.

The basic stability criteria for ordinary standing-wave lasers are derived in Sec. II. In numerical and laboratory experiments it has been found that the instability causes periodic fluctuations of all of the field, polarization, and inversion parameters. Consequently the analysis begins in Sec. II A with the expansion of these parameters in a harmonic series. The procedure that we have adopted to test for the pulsation instability is to assume first of all that only a single intense field component is present. The amplitude and frequency of this component are derived in Sec. II B. We then test in Sec. II C whether any other frequency components can exist in the resonator having the same wavelength as the saturating component and experiencing net gain. The single-mode solution found initially is unstable against pulsations if and only if such additional sidebands exist. The resulting stability criteria are put into a simpler form in Sec. II D and graphical solutions are suggested. The important result here is that spontaneous pulsations can occur in many practical laser systems. Physically, it is the severe distortion of the dispersion curve caused by the saturating field which makes possible the existence of sideband frequencies having the same wavelength as the saturating field. The results are illustrated with reference to the 3.51-μm xenon laser systems where the pulsations have been observed experimentally. In Sec. II E it is proved that homogeneously broadened lasers can not exhibit this instability. A similar treatment for one-directional ring lasers is developed in

Sec. III. In most respects the results for ring lasers are qualitatively identical to the standing-wave laser behavior.

II. STANDING-WAVE LASER

The most convenient approach for studying the high-frequency behavior of laser oscillators involves the familiar density-matrix equations coupled with Maxwell's wave equation. This is the approach that has been used in Ref. 1 and in most subsequent rigorous treatments of lasers. Mathematically it is helpful to solve directly the differential equations of motion governing the ensemble-averaged density matrix rather than starting from the response of a single atom. Thus the elements of the matrix are governed by the equations[6]

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right)\rho_{ab}(v,z,t)=-(i\omega_0+\gamma)\rho_{ab}(v,z,t)-\frac{i\mu}{\hbar}E(z,t)[\rho_{aa}(v,z,t)-\rho_{bb}(v,z,t)], \quad (1)$$

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right)\rho_{aa}(v,z,t)=\lambda_a(v,z,t)-\gamma_a\rho_{aa}(v,z,t)+\left(\frac{i\mu}{\hbar}E(z,t)\rho_{ba}(v,z,t)+\text{c.c.}\right), \quad (2)$$

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right)\rho_{bb}(v,z,t)=\lambda_b(v,z,t)-\gamma_b\rho_{bb}(v,z,t)-\left(\frac{i\mu}{\hbar}E(z,t)\rho_{ba}(v,z,t)+\text{c.c.}\right), \quad (3)$$

$$\rho_{ba}=\rho_{ab}^{*} . \quad (4)$$

In these equations γ_a and γ_b represent the decay rates of the diagonal matrix elements, $\gamma=\frac{1}{2}(\gamma_a+\gamma_b)+\gamma_{ph}$ is the decay rate for the off-diagonal elements, λ_a and λ_b are the pumping terms, and ω_0 is the center frequency of the laser transition.

In reducing Eqs. (1)–(4) it is useful to first factor out the rapid time variations of the electric field and the off-diagonal matrix elements. For a single longitudinal mode this factorization can be accomplished with the substitutions

$$E(z,t)=\tfrac{1}{2}\sin(kz)E'(t)\exp(-i\omega t)+\text{c.c.}, \quad (5)$$

$$\rho_{ab}(v,z,t)=P'(v,z,t)\exp(-i\omega t)/2\mu . \quad (6)$$

The amplitudes $E'(t)$ and $P'(v,z,t)$ are generally complex to account for the time varying phases of the field and polarization. With these substitutions and the standard rotating-wave approximation, Eqs. (1)–(4) reduce to

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right)P'(v,z,t)=i(\omega-\omega_0)P'(v,z,t)-\gamma P'(v,z,t)-\frac{i\mu^2}{\hbar}\sin(kz)E'(t)D(v,z,t), \quad (7)$$

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right)D(v,z,t)=\lambda_a(v)-\lambda_b(v)-\frac{\gamma_a+\gamma_b}{2}D(v,z,t)-\frac{\gamma_a-\gamma_b}{2}M(v,z,t)+\frac{\sin(kz)}{2\hbar}[iE'(t)P'^{*}(v,z,t)+\text{c.c.}], \quad (8)$$

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}\right)M(v,z,t)=\lambda_a(v)+\lambda_b(v)-\frac{\gamma_a+\gamma_b}{2}M(v,z,t)-\frac{\gamma_a-\gamma_b}{2}D(v,z,t), \quad (9)$$

where the density difference

$$D(v,z,t)=\rho_{aa}(v,z,t)-\rho_{bb}(v,z,t)$$

and sum

$$M(v,z,t)=\rho_{aa}(v,z,t)+\rho_{bb}(v,z,t)$$

have also been introduced.

A. Harmonic analysis of the laser equations

Equations equivalent to Eqs. (7)–(9), together with the wave equation have been solved numerically to obtain the transient characteristics of a quite general single-mode laser oscillator.[7] In that study it was found that under some conditions the laser is unstable, and the output consists of periodic bursts of energy. Since the output is periodic in time, it is reasonable to introduce the Fourier expansions

$$E'(t)=\sum_n E_n\exp(-in\Delta\omega t), \quad (10)$$

$$P'(v,z,t)=\sum_n P_n(v,z)\exp(-in\Delta\omega t), \quad (11)$$

$$D(v,z,t)=\sum_n D_n(v,z)\exp(-in\Delta\omega t), \quad (12)$$

$$M(v,z,t)=\sum_n M_n(v,z)\exp(-in\Delta\omega t), \quad (13)$$

where $\Delta\omega$ is the fundamental pulsation frequency. Using these substitutions in Eqs. (7)–(9) and equating the coefficients of the nth frequency harmonics, one obtains

$$v\,\frac{\partial P_n(v,z)}{\partial z}=i(\omega+n\Delta\omega-\omega_0)P_n(v,z)-\gamma P_n(v,z)-\frac{i\mu^2}{\hbar}\sin(kz)\sum_j E_{n-j}D_j(v,z)\,, \tag{14}$$

$$v\,\frac{\partial D_n(v,z)}{\partial z}=[\lambda_a(v)-\lambda_b(v)]\,\delta_{n0}+in\Delta\omega D_n(v,z)-\frac{\gamma_a+\gamma_b}{2}D_n(v,z)-\frac{\gamma_a-\gamma_b}{2}M_n(v,z)+\frac{i\sin(kz)}{2\hbar}\sum_j[E_{j+n}P_j^*(v,z)-E_{j-n}^*P_j(v,z)]\,, \tag{15}$$

$$v\,\frac{\partial M_n(v,z)}{\partial z}=[\lambda_a(v)+\lambda_b(v)]\,\delta_{n0}+in\Delta\omega M_n(v,z)-\frac{\gamma_a+\gamma_b}{2}M_n(v,z)-\frac{\gamma_a-\gamma_b}{2}D_n(v,z)\,. \tag{16}$$

With the same substitutions in the wave equation, use of the rotating-wave approximation, isolation of the nth harmonic, multiplication by $\sin(kz)$, and integration over z, one obtains

$$\left(\frac{i}{2t_c}+(\omega+n\Delta\omega-\Omega)\right)E_n=-\frac{\omega_0}{\epsilon_0 L}\int_{-\infty}^{\infty}\int_0^l \sin(kz)P_n(v,z)\,dz\,dv\,. \tag{17}$$

In this equation the frequency factor $\omega+n\Delta\omega$ on the right-hand side has been replaced by the approximate value ω_0, and to be specific the amplifying medium is assumed to extend from $z=0$ to $z=l$ in a cavity of length L.

Equations (14)–(17) completely characterize the harmonic components of the limit cycles corresponding to the spontaneous coherent pulsations. The first-order differential equations can be converted to a large set of algebraic equations by means of a second Fourier expansion in the spatial coordinate z. The solutions of this set would be equivalent to the limit cycles obtained by numerical integration of the differential equations. For the present stability analysis, however, the general solutions of this set are not required. Instead we assume initially that only a single frequency component is oscillating strongly. We then examine the situations in which a sideband of infinitesimal amplitude and frequency displacement $\Delta\omega$ can also satisfy the laser oscillation conditions. These conditions require in effect that the round-trip phase delay of the sideband must also be a multiple of 2π and the round-trip gain must be greater than the loss. If any sideband satisfies these conditions, the assumption of cw single-frequency oscillation is incorrect. Instead the laser mode is unstable and the output consists of undamped intensity pulsations with more than one frequency component. On the other hand, if all possible sidebands of infinitesimal amplitude have more loss than gain, then the assumed single-mode solution is stable and provides a correct description of the laser oscillation.

It is perhaps worth noting that the pulsation phenomena of interest here require a more careful definition of the concept of a laser mode. Ordinarily one considers that a particular longitudinal mode of a laser can be characterized by either its frequency or by the number of wavelengths between the mirrors. Now, however, one finds that for a given number of wavelengths between the mirrors several frequency components can exist simultaneously. Thus it would not be inappropriate to regard each of the frequency components of the periodically pulsing output as one of a set of phase-locked modes specified by both its frequency and wavelength. In any case, the present study is aimed solely at lasers having a single fixed number of wavelengths between the mirrors.

B. Single-frequency oscillation

In line with the preceding comments, we consider first that the laser is oscillating strongly at a single frequency, and components at other frequencies if present at all have amplitudes which are too small to affect the population sum or difference. From Eqs. (14)–(17) this fundamental frequency component must be described by the equations

$$v\,\frac{\partial P_0(v,z)}{\partial z}=i(\omega-\omega_0)P_0(v,z)-\gamma P_0(v,z)-\frac{i\mu^2}{\hbar}\sin(kz)E_0D_0(v,z)\,, \tag{18}$$

$$v\,\frac{\partial D_0(v,z)}{\partial z}=\lambda_a(v)-\lambda_b(v)-\frac{\gamma_a+\gamma_b}{2}D_0(v,z)-\frac{\gamma_a-\gamma_b}{2}M_0(v,z)+\frac{i\sin(kz)}{2\hbar}[E_0P_0^*(v,z)-E_0^*P_0(v,z)]\,, \tag{19}$$

$$\frac{\partial M_0(v,z)}{\partial z}=\lambda_a(v)+\lambda_b(v)-\frac{\gamma_a+\gamma_b}{2}M_0(v,z)-\frac{\gamma_a-\gamma_b}{2}D_0(v,z)\,, \tag{20}$$

$$\left(\frac{i}{2t_c}+\omega-\Omega\right)E_0 = -\frac{\omega_0}{\epsilon_0 L}\int_{-\infty}^{\infty}\int_0^l \sin(kz)P_0(v,z)\,dz\,dv\ . \qquad (21)$$

If the phase of the electric field is set equal to zero, Eqs. (18)–(21) can be replaced by a set of real equations by means of the substitution

$$P_0(v,z)=C_0(v,z)+i\,S_0(v,z)\ . \qquad (22)$$

The results are

$$v\,\frac{\partial S_0(v,z)}{\partial z}=(\omega-\omega_0)C_0(v,z)-\gamma S_0(v,z) -\frac{\mu^2}{\hbar}\sin(kz)E_0 D_0(v,z)\ , \qquad (23)$$

$$v\,\frac{\partial C_0(v,z)}{\partial z}=-(\omega-\omega_0)S_0(v,z)-\gamma C_0(v,z), \qquad (24)$$

$$v\,\frac{\partial D_0(v,z)}{\partial z}=\lambda_a(v)-\lambda_b(v)-\frac{\gamma_a+\gamma_b}{2}D_0(v,z) -\frac{\gamma_a-\gamma_b}{2}M_0(v,z)+\frac{\sin(kz)}{\hbar}E_0 S_0\ , \qquad (25)$$

$$v\,\frac{\partial M_0(v,z)}{\partial z}=\lambda_a(v)+\lambda_b(v)-\frac{\gamma_a+\gamma_b}{2}M_0(v,z) -\frac{\gamma_a-\gamma_b}{2}D_0(v,z)\ , \qquad (26)$$

$$\frac{E_0}{2t_c}=-\frac{\omega_0}{\epsilon_0 L}\int_{-\infty}^{\infty}\int_0^l \sin(kz)S_0(v,z)\,dz\,dv\ , \qquad (27)$$

$$(\omega-\Omega)E_0=-\frac{\omega_0}{\epsilon_0 L}\int_{-\infty}^{\infty}\int_0^l \sin(kz)C_0(v,z)\,dz\,dv. \qquad (28)$$

Next, it is helpful to eliminate the z derivatives by expanding the polarization and population elements in series of spatial harmonics according to

$$S_0(v,z)=\sum_{j=-\infty}^{\infty} S_{0,\,2j+1}(v)\exp[(2j+1)ikz]\,, \qquad (29)$$

$$C_0(v,z)=\sum_{j=-\infty}^{\infty} C_{0,\,2j+1}(v)\exp[(2j+1)ikz]\,, \qquad (30)$$

$$D_0(v,z)=\sum_{j=-\infty}^{\infty} D_{0,\,2j}(v)\exp[(2j)ikz]\,, \qquad (31)$$

$$M_0(v,z)=\sum_{j=-\infty}^{\infty} M_{0,\,2j}(v)\exp[(2j)ikz]\,, \qquad (32)$$

subject to the constraints

$$S_{0,\,\alpha}(v)=S^*_{0,-\alpha}(v),\quad C_{0,\,\alpha}(v)=C^*_{0,-\alpha}(v)\,,$$

etc. With these substitutions Eqs. (23)–(28) can be written

$$0=-[(2j+1)ikv+\gamma]\,S_{0,2j+1}(v)+(\omega-\omega_0)C_{0,\,2j+1}(v)+\frac{i\mu^2E_0}{2\hbar}[D_{0,\,2j}(v)-D_{0,\,2j+2}(v)]\,, \qquad (33)$$

$$0=-[(2j+1)ikv+\gamma]C_{0,\,2j+1}(v)-(\omega-\omega_0)S_{0,\,2j+1}(v)\ , \qquad (34)$$

$$0=[\lambda_a(v)-\lambda_b(v)]\,\delta_{j0}-\left((2j)ikv+\frac{\gamma_a+\gamma_b}{2}\right)D_{0,\,2j}(v)-\frac{\gamma_a-\gamma_b}{2}M_{0,\,2j}(v)-\frac{iE_0}{2\hbar}[S_{0,\,2j-1}(v)-S_{0,\,2j+1}(v)]\,, \qquad (35)$$

$$0=[\lambda_a(v)+\lambda_b(v)]\,\delta_{j0}-\left((2j)ikv+\frac{\gamma_a+\gamma_b}{2}\right)M_{0,\,2j}(v)-\frac{\gamma_a-\gamma_b}{2}D_{0,\,2j}(v)\ , \qquad (36)$$

$$\frac{E_0}{2t_c}=\frac{\omega_0 l}{\epsilon_0 L}\int_{-\infty}^{\infty} S_{0,\,1i}(v)\,dv\ , \qquad (37)$$

$$(\omega-\Omega)E_0=\frac{\omega_0 l}{\epsilon_0 L}\int_{-\infty}^{\infty} C_{0,\,1i}(v)\,dv\ . \qquad (38)$$

Equations (33)–(38) may be combined to obtain two coupled equations for the oscillation amplitude E_0 and frequency ω, and similar calculations have been performed previously for steady-state lasers.[5,6] First, Eqs. (33) and (34) are combined, yielding

$$S_{0,\,2j+1}(v)=\frac{i\mu^2E_0}{2\hbar\gamma}\,\alpha_j(v)[D_{0,\,2j}(v)-D_{0,\,2j+2}(v)]\,, \qquad (39)$$

where α_j is defined by

$$\alpha_j(v)=\frac{\gamma/2}{(2j+1)ikv+i(\omega-\omega_0)+\gamma}+\frac{\gamma/2}{(2j+1)ikv-i(\omega-\omega_0)+\gamma}\ . \qquad (40)$$

Similarly Eqs. (35) and (36) may be combined, yielding

$$D_{0,2j}(v)=-\frac{iE_0}{4\hbar}\frac{\gamma_a+\gamma_b}{\gamma_a\gamma_b}\beta_j(v)[S_{0,2j-1}(v)-S_{0,2j+1}(v)]+\left(\frac{\lambda_a(v)}{\gamma_a}-\frac{\lambda_b(v)}{\gamma_b}\right)\delta_{j0}\ , \qquad (41)$$

where $\beta_j(v)$ is defined by

$$\beta_j(v)=\frac{\gamma_a\gamma_b}{\gamma_a+\gamma_b}\left(\frac{1}{(2j)ikv+\gamma_a}+\frac{1}{(2j)ikv+\gamma_b}\right)\ . \qquad (42)$$

Now Eqs. (39) and (41) produce

$$S_{0,1}(v)=\frac{i4\hbar}{E_0}\frac{\gamma_a\gamma_b}{\gamma_a+\gamma_b}D_{0,0}(v)W(v)sI\ , \qquad (43)$$

where $W(v)$ is the continued fraction

$$W(v)=\cfrac{\alpha_0(v)}{1+\cfrac{\alpha_0(v)\beta_1(v)sI}{1+\cfrac{\alpha_1(v)\beta_1(v)sI}{1+\cfrac{\alpha_1(v)\beta_2(v)sI}{1+\cdots}}}}\ . \qquad (44)$$

In this result sI is a normalized intensity given by

$$sI=\frac{\mu^2E_0^2}{8\hbar^2}\frac{\gamma_a+\gamma_b}{\gamma\gamma_a\gamma_b}\ . \qquad (45)$$

With Eq. (41) for $D_{0,0}(v)$ and the condition $S_{0,1}(v)=S^*_{0,-1}(v)$, the imaginary part of $S_{0,1}(v)$ is

$$S_{0,1i}(v)=\frac{4\hbar}{E_0}\frac{\gamma_a\gamma_b}{\gamma_a+\gamma_b}\frac{N(v)\,sI\,W_r(v)}{1+2W_r(v)sI}\ , \qquad (46)$$

where $N(v)$ is the unsaturated population difference

$$N(v)=\frac{\lambda_a(v)}{\gamma_a}-\frac{\lambda_b(v)}{\gamma_b} \qquad (47)$$

and the subscripts i and r denote, respectively, the imaginary and real parts of a quantity. With Eq. (34) it follows that the imaginary part of $C_{0,1}(v)$ is

$$C_{0,1i}(v)=\frac{4\hbar}{E_0}\frac{\gamma_a\gamma_b}{\gamma_a+\gamma_b}N(v)sI\times\mathrm{Re}\left(\frac{\omega_0-\omega}{ikv+\gamma}\frac{W(v)}{1+2W_r(v)sI}\right)\ . \qquad (48)$$

Equations (37) and (46) may be combined to yield the unsaturated intensity gain coefficient.

$$g=\frac{2\omega_0}{c\epsilon_0E_0}\int_{-\infty}^{\infty}S_{0,1i}(v)\,dv=\frac{\mu^2\omega_0}{c\epsilon_0\gamma\hbar}\int_{-\infty}^{\infty}N(v)\alpha_{0r}(v)\,dv\ . \qquad (49)$$

In terms of this gain coefficient, Eq. (37) can be written

$$\frac{1}{t_c}=\frac{gcl}{L}\int_{-\infty}^{\infty}\frac{N(v)W_r(v)\,dv}{1+2W_r(v)sI}\Big/\int_{-\infty}^{\infty}N(v)\alpha_{0r}(v)\,dv. \qquad (50)$$

Similarly Eqs. (38), (48), and (49) may be combined to obtain

$$\omega-\Omega=\frac{gcl}{2L}\int_{-\infty}^{\infty}N(v)\,\mathrm{Re}\left(\frac{\omega_0-\omega}{ikv+\gamma}\frac{W(v)}{1+2W_r(v)sI}\right)dv\times\left(\int_{-\infty}^{\infty}N(v)\alpha_{0r}(v)\,dv\right)^{-1}\ . \qquad (51)$$

Equations (50) and (51) are a coupled set which may be solved to obtain the frequency ω and the intensity sI of the cw oscillating mode.

C. Sideband stability

The stability of the single-mode laser oscillation is assessed by determining whether any frequency component in addition to ω can also satisfy the oscillation conditions. Thus we now examine the polarization P_1. From Eq. (14) this polarization component is governed by

$$v\frac{\partial P_1(v,z)}{\partial z}=i(\omega+\Delta\omega-\omega_0)P_1(v,z)-\gamma P_1(v,z)-\frac{i\mu^2}{\hbar}\sin(kz)\sum_j E_{1-j}D_j\ . \qquad (52)$$

By definition all frequency and polarization sidebands are infinitesimal in comparison to E_0 and P_0. Therefore, the only population difference is D_0 and Eq. (52) is

$$v\frac{\partial P_1(v,z)}{\partial z}=i(\omega+\Delta\omega-\omega_0)P_1(v,z)-\gamma P_1(v,z)-\frac{i\mu^2}{\hbar}\sin(kz)E_1D_0\ . \qquad (53)$$

As long as we are considering only a single frequency sideband, there is no reason for not assuming that at some time in the past this frequency component was exactly in phase with the fundamental frequency. Thus the amplitude E_1 may be taken to be a real number. It is then convenient to separate Eq. (53) into its real and imaginary parts using the substitution

$$P_1(v,z)=C_1(v,z)+iS_1(v,z) .$$

The results are

$$v\frac{\partial S_1(v,z)}{\partial z}=(\omega+\Delta\omega-\omega_0)C_1(v,z)-\gamma S_1(v,z)-\frac{\mu^2}{\hbar}\sin(kz)E_1D_0(v,z) , \tag{54}$$

$$v\frac{\partial C_1(v,z)}{\partial z}=-(\omega+\Delta\omega-\omega_0)S_1(v,z)-\gamma C_1(v,z) . \tag{55}$$

Using substitutions analogous to Eqs. (29)–(31), the spatial harmonics of S_1 and C_1 must satisfy

$$0=-[(2j+1)ikv+\gamma]S_{1,2j+1}(v)+(\omega+\Delta\omega-\omega_0)C_{1,2j+1}(v)+\frac{i\mu^2E_1}{2\hbar}[D_{0,2j}(v)-D_{0,2j+2}(v)], \tag{56}$$

$$0=-[(2j+1)ikv+\gamma]C_{1,2j+1}(v)-(\omega+\Delta\omega-\omega_0)S_{1,2j+1}(v) . \tag{57}$$

Combining these formulas yields

$$S_{1,2j+1}(v)=\frac{i\mu^2E_1}{2\hbar\gamma}\left(\frac{\gamma/2}{(2j+1)ikv+i(\omega+\Delta\omega-\omega_0)+\gamma}+\frac{\gamma/2}{(2j+1)ikv-i(\omega+\Delta\omega-\omega_0)+\gamma}\right)\times[D_{0,2j}(v)-D_{0,2j+2}(v)]. \tag{58}$$

This equation resembles Eq. (39), and one obtains finally

$$S_{1,2j+1}(v)=\frac{[(2j+1)ikv+\gamma]^2+(\omega-\omega_0)^2}{[(2j+1)ikv+\gamma]^2+(\omega+\Delta\omega-\omega_0)^2}\frac{E_1S_{0,2j+1}(v)}{E_0}. \tag{59}$$

In a similar manner one finds that the first-sideband frequency components of C_1 are related to the fundamental components by

$$C_{1,2j+1}(v)=\frac{\omega+\Delta\omega-\omega_0}{\omega-\omega_0}\frac{[(2j+1)ikv+\gamma]^2+(\omega-\omega_0)^2}{[(2j+1)ikv+\gamma]^2+(\omega+\Delta\omega-\omega_0)^2}\times\frac{E_1C_{0,2j+1}(v)}{E_0}. \tag{60}$$

Based on Eq. (17) the oscillation conditions for the the first sideband may be written

$$\frac{E_1}{2t_c}=\frac{\omega_0 l}{\epsilon_0 L}\int_{-\infty}^{\infty}S_{1,1i}(v)\,dv , \tag{61}$$

$$(\omega+\Delta\omega-\Omega)E_1=\frac{\omega_0 l}{\epsilon_0 L}\int_{-\infty}^{\infty}C_{1,1i}(v)\,dv , \tag{62}$$

and these formulas are similar to Eqs. (37) and (38). With Eqs. (59) and (60) and our previous results for $S_{0,1}$ and $C_{0,1}$, these conditions become

$$\frac{1}{t_c}=\frac{gcl}{L}\mathrm{Re}\int_{-\infty}^{\infty}\frac{(ikv+\gamma)^2+(\omega-\omega_0)^2}{(ikv+\gamma)^2+(\omega+\Delta\omega-\omega_0)^2}\frac{N(v)W(v)}{1+2W_r(v)sI}dv\Big/\int_{-\infty}^{\infty}N(v)\alpha_{or}(v)\,dv , \tag{63}$$

$$\omega+\Delta\omega-\Omega=\frac{gcl}{2L}\mathrm{Re}\int_{-\infty}^{\infty}\frac{\omega_0-\omega-\Delta\omega}{ikv+\gamma}\frac{(ikv+\gamma)^2+(\omega-\omega_0)^2}{(ikv+\gamma)^2+(\omega+\Delta\omega-\omega_0)^2}\frac{N(v)W(v)}{1+2W_r(v)sI}dv\Big/\int_{-\infty}^{\infty}N(v)\alpha_{or}(v)\,dv . \tag{64}$$

The procedure now is to first solve Eq. (64) for $\Delta\omega$ using the values of ω and sI already obtained from Eqs. (50) and (51). The result is substituted into Eq. (63) and the equality is tested. If the right-hand side of Eq. (63) is larger than the left-hand side, one concludes that in the presence of the saturating cw mode characterized by frequency ω and intensity sI there is still net gain for the infinitesimal sideband at frequency offset $\Delta\omega$. In other words, the mode is unstable. However, if the right-hand side is smaller than the left-hand side, the sideband decays away and the cw mode is stable. In short then, the question of mode stability is answered by solving four equations—(50), (51), (63), and (64). While these equations may appear to be a bit complicated, the stability analysis described here requires much less computer time than actual numerical solutions of the intensity waveforms described previously.[7] Also, for many practical lasers these equations may be greatly simplified, and analytic stability tests are sometimes possible.

D. Simplified stability criteria

The first approximation that one should consider concerns the continued fraction $W(v)$. From Eq. (42) it follows that the peak value of the function $\beta_1(v)$ at velocity $v=0$ is always unity. The width of this function, however, is characterized by the smaller of $\gamma_a/2k$ and $\gamma_b/2k$. The function $\alpha_0(v)$, on the other hand, has one or two peaks (depending on the value of $\omega-\omega_0$) which are characterized by the width γ/k. It then follows from the form of $W(v)$ in Eq. (44) that in any velocity integration $\beta_1(v)$ may simply be replaced by zero as long as γ_a or γ_b or both are much smaller than γ. The same result applies if the laser is tuned away from line center such that $\omega-\omega_0$ is greater than the smaller of γ_a and γ_b. But these cases include the vast majority of practical lasers. In the xenon system considered previously, for example, the spontaneous decay rates are about $\gamma_a = 0.741\times10^6\ \text{sec}^{-1}$ and $\gamma_b = 22.7\times10^6\ \text{sec}^{-1}$. Thus γ_a is smaller than γ_b by a factor of 31, and the approximation described here is valid even when the laser is tuned to line center. The only time that the full continued-fraction form of $W(v)$ would be necessary would be in a laser tuned near line center with $\gamma_a \sim \gamma_b$ and negligible pressure broadening ($\gamma \gg \gamma_{ph}$). Such a system would not often by encountered, and one is quite safe in replacing $W(v)$ by $\alpha_0(v)$. Thus the basic stability equations (50), (51), (63), and (64) may be replaced by the set

$$\frac{1}{r}=\int_{-\infty}^{\infty}\frac{N(v)\alpha_{or}(v)\,dv}{1+2\alpha_{or}(v)sI}\Big/\int_{-\infty}^{\infty}N(v)\alpha_{or}(v)\,dv\,, \tag{65}$$

$$\frac{(\omega-\Omega)t_c}{r}=\text{Re}\int_{-\infty}^{\infty}\frac{\omega_0-\omega}{2(ikv+\gamma)}\,\frac{N(v)\alpha_0(v)\,dv}{1+2\alpha_{or}(v)sI}\times\left(\int_{-\infty}^{\infty}N(v)\alpha_{0r}(v)\,dv\right)^{-1}, \tag{66}$$

$$\frac{1}{r}=\text{Re}\int_{-\infty}^{\infty}\frac{(ikv+\gamma)^2+(\omega-\omega_0)^2}{(ikv+\gamma)^2+(\omega+\Delta\omega-\omega_0)^2}\,\frac{N(v)\alpha_0(v)}{1+2\alpha_{0r}(v)sI}dv\times\left(\int_{-\infty}^{\infty}N(v)\alpha_{0r}(v)\,dv\right)^{-1}, \tag{67}$$

$$\frac{(\omega+\Delta\omega-\Omega)t_c}{r}=\text{Re}\int_{-\infty}^{\infty}\frac{\omega_0-\omega-\Delta\omega}{2(ikv+\gamma)}\,\frac{(ikv+\gamma)^2+(\omega-\omega_0)^2}{(ikv+\gamma)^2+(\omega+\Delta\omega-\omega_0)^2}\times\frac{N(v)\alpha_0(v)}{1+2\alpha_{0r}(v)sI}\,dv\left(\int_{-\infty}^{\infty}N(v)\alpha_{0r}(v)\,dv\right)^{-1}, \tag{68}$$

where the threshold parameter $r = gct_c l/L$ has been introduced.

Equations (65)–(68) are still quite complex and contain too many variables for specific solutions to have general utility. Accordingly we now specialize to the case of a laser in which the cavity resonance is tuned to the atomic center frequency ($\Omega=\omega_0$). It follows immediately from Eq. (66) that the laser frequency is also at the atomic resonance ($\omega=\omega_0$), and from Eq. (40) that

$$\alpha_0(v)=[1+i(kv/\gamma)]^{-1}\,. \tag{69}$$

Thus the stability equations (65), (67), and (68) reduce to

$$\frac{1}{r}=\int_{-\infty}^{\infty}\frac{N(v)\,dv}{1+(kv/\gamma)^2+2sI}\Big/\int_{-\infty}^{\infty}\frac{N(v)\,dv}{1+(kv/\gamma)^2}\,, \tag{70}$$

$$\frac{1}{r}=\int_{-\infty}^{\infty}\frac{1+(kv/\gamma)^2}{2}\left(\frac{1}{1+[(kv+\Delta\omega)/\gamma]^2}+\frac{1}{1+[(kv-\Delta\omega)/\gamma]^2}\right)\frac{N(v)\,dv}{1+(kv/\gamma)^2+2sI}\Big/\int_{-\infty}^{\infty}\frac{N(v)\,dv}{1+(kv/\gamma)^2}\,, \tag{71}$$

$$\frac{\Delta\omega t_c}{r}=-\frac{\Delta\omega}{4\gamma}\int_{-\infty}^{\infty}\left(\frac{1-[kv(kv+\Delta\omega)/\gamma^2]}{1+[(kv+\Delta\omega)/\gamma]^2}+\frac{1-[kv(kv-\Delta\omega)/\gamma^2]}{1+[(kv-\Delta\omega)/\gamma]^2}\right)\frac{N(v)\,dv}{1+(kv/\gamma)^2+2sI}\Big/\int_{-\infty}^{\infty}\frac{N(v)\,dv}{1+(kv/\gamma)^2}\,. \tag{72}$$

One would now first solve Eq. (70) for sI. This result would be substituted into Eq. (72) to obtain $\Delta\omega$. Finally the values of sI and $\Delta\omega$ would be used to test the equality in Eq. (71) to see whether the sideband would experience a net gain.

In the xenon laser considered previously and in many other gas lasers the Doppler width $\Delta\nu_D$ is much greater than the homogeneous linewidth $\Delta\nu_h=\gamma/\pi$. In this case $N(v)$ may be replaced by its line center value and Eqs. (70)–(72) simplify further. The integrals in Eq. (70) can be performed exactly, and one obtains the usual expression for the intensity in an inhomogeneously broadened laser:

$$sI=\tfrac{1}{2}(r^2-1)\,. \tag{73}$$

With this result Eqs. (71) and (72) can be written

$$\frac{1}{r}=\int_{-\infty}^{\infty}\frac{1+V^2}{2\pi}\left(\frac{1}{1+(V+U)^2}+\frac{1}{1+(V-U)^2}\right)\frac{dV}{r^2+V^2}\,, \tag{74}$$

$$\frac{\gamma Ut_c}{r}=-\frac{U}{4\pi}\int_{-\infty}^{\infty}\left(\frac{1-V^2-VU}{1+(V+U)^2}+\frac{1-V^2+VU}{1+(V-U)^2}\right)\frac{dV}{r^2+V^2}\,, \tag{75}$$

where $U=\Delta\omega/\gamma$ and $V=kv/\gamma$. A useful formula for performing the integrations is

$$\int_{-\infty}^{\infty} \frac{(g+hx+ix^2)\,dx}{(a+bx+cx^2)(d+ex+fx^2)}$$

$$= \frac{2\pi(A-bB/2c)}{(4ac-b^2)^{1/2}} + \frac{2\pi(C-eD/2f)}{(4df-e^2)^{1/2}}, \quad (76)$$

with

$$A = \frac{(cd-af)(ai-cg)-(ce-bf)(ah-bg)}{(cd-af)(af-cd)-(ce-bf)(ae-bd)}, \quad (77)$$

$$B = \left(\frac{c}{a}\right)\frac{(ae-bd)(ai-cg)-(af-cd)(ah-bg)}{(ae-bd)(ce-bf)-(af-cd)(cd-af)}, \quad (78)$$

$$C = \frac{(af-cd)(di-fg)-(bf-ce)(dh-eg)}{(af-cd)(cd-af)-(bf-ce)(bd-ae)}, \quad (79)$$

$$D = \left(\frac{f}{d}\right)\frac{(bd-ae)(di-fg)-(cd-af)(dh-eg)}{(bd-ae)(bf-ce)-(cd-af)(af-cd)}. \quad (80)$$

Thus, Eqs. (74) and (75) reduce to

$$1 = \frac{(r^2-1)^2+(r^3-r^2+3r+1)U^2+rU^4}{(r^2-1)^2+2(r^2+1)U^2+U^4}, \quad (81)$$

$$2\gamma t_c = \frac{(r^2-1)[(r-1)^2+U^2]}{(r^2-1)^2+2(r^2+1)U^2+U^4}. \quad (82)$$

Equation (82) is simply a quadratic equation in U^2, and the physically interesting solution simplifies to

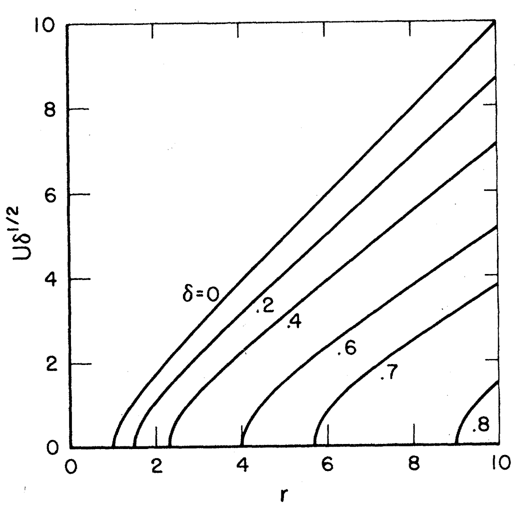

FIG. 1. Normalized frequency shift $U\delta^{1/2} = \Delta\omega(2t_c/\gamma)^{1/2}$ of small-amplitude sidebands having the same number of wavelengths between the mirrors as the dominant saturating mode. For small values of the parameter $\delta = 2\gamma t_c$ the pulsations occur with excitation levels close to threshold ($r=1$).

$$\frac{\Delta\omega}{\gamma} = U = \left(\frac{r^2-1-(r+1)^2\delta}{\delta}\right)^{1/2}, \quad (83)$$

where $\delta \equiv 2\gamma t_c$. If δ is much less than unity, Eq. (83) is

$$U = [(r^2-1)\delta^{-1}]^{1/2}. \quad (84)$$

In the 3.51-μ xenon lasers, for example, we have typically $\gamma = 12.8\times10^6$ sec^{-1} and $t_c = 1.0\times10^{-9}$ sec.[7] Thus $\delta = 0.026 \ll 1$ so that Eq. (84) should be approximately valid. In a 0.6328-μ helium-neon laser, on the other hand, one might expect the values $\gamma \simeq 10^9$ sec^{-1} and $t_c \simeq 10^{-7}$ sec.[8] Thus $\delta = 100$ and the last approximation would certainly not be usable.

Equation (83) is plotted in Fig. 1 as a family of curves of $U\delta^{1/2} = \Delta\omega(2t_c/\gamma)^{1/2}$ vs the threshold parameter r. The fact that real solutions of Eq. (83) exist means that there do exist sidebands of the dominant oscillation mode which also satisfy the oscillation phase condition. The frequency displacement of these sidebands evidently increases rapidly with pumping (threshold parameter), in agreement with the reported xenon laser data. It should be emphasized, however, that these frequencies are calculated under the assumption that the sidebands have infinitesimal amplitude. When strong pulsing occurs, the frequencies determined in this manner are inaccurate.

It is also important to inquire as to the overall conditions under which Eq. (83) has real solutions. From Fig. 1 it is clear that as δ increases, real solutions require very large values of the threshold parameter r. In the limit of large r, Eq. (83) reduces to

$$U \simeq r[(1-\delta)/\delta]^{1/2}. \quad (85)$$

Thus no sideband can satisfy the phase condition unless the parameter $\delta = 2\gamma t_c$ is less than unity. This condition sharply restricts the number of lasers that might be candidates for the spontaneous pulsation effect. The 6328-Å helium-neon laser mentioned previously, for example, is certainly ineligible. The high-gain 3.39-μ helium-neon laser, on the other hand, can be operated with a very short cavity lifetime and might be a good candidate for the pulsation effect.[9] This helium-neon laser is also predominantly inhomogeneously broadened,[10] which is essential for the pulsation effect. Similarly, low-pressure chemical lasers are also Doppler broadened and may have short cavity lifetimes.

It only remains to be determined whether the sidebands that have been derived can exhibit a gain in excess of the cavity losses. For this pur-

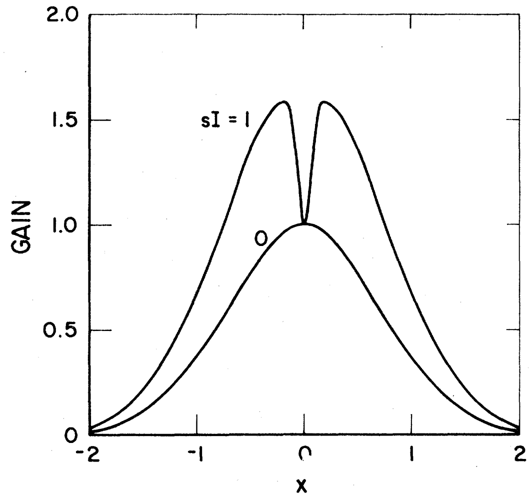

FIG. 2. Gain profiles [right-hand side of Eq. (89)] seen by a small-amplitude sideband propagating in a monoisotopic xenon laser medium which is saturated by a line center signal of intensity sI. As the excitation level is increased the intensity increases and a hole is burned in the gain spectrum.

pose we now examine Eq. (81). The condition that the right-hand side be greater than unity leads immediately to

$$(r-1)U^4+(r-1)^3U^2>0. \tag{86}$$

Above threshold ($r>1$) this condition is satisfied for any real value of U, i.e., whenever Eq. (83) has a real root. Thus when a sideband satisfies the phase condition one can conclude immediately that the dominant oscillation mode is unstable. This instability is always possible for sufficiently large r if δ is less than unity.

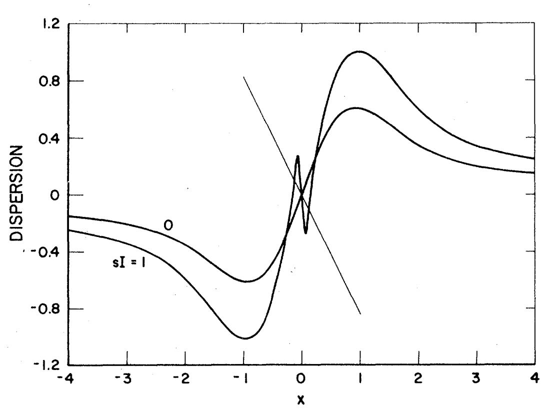

FIG. 3. Dispersion profiles [right-hand side of Eq. (90)] seen by a small-amplitude sideband propagating in a xenon laser which is saturated by a line center signal of intensity sI. The possible sideband frequencies having the same number of wavelengths as the saturating mode are found as the intersections of the dispersion curves and a straight line of slope $-\delta/\epsilon$.

The pulsation instability can be understood physically as a consequence of spectral holeburning. To show this, we first present some solutions of Eqs. (71) and (72). It is convenient numerically to regard the normalized intensity sI as an adjustable parameter rather than the threshold coefficient r. The two quantities are related by Eq. (70), and with this substitution Eqs. (71) and (72) can be written

$$1=\int_{-\infty}^{\infty}\frac{1+V^2}{2}\left(\frac{1}{1+(V+U)^2}+\frac{1}{1+(V-U)^2}\right)\frac{N(V)\,dV}{1+V^2+2sI}\times\left(\int_{-\infty}^{\infty}\frac{N(V)\,dV}{1+V^2+2sI}\right)^{-1}, \tag{87}$$

$$-U\delta=\frac{U}{2}\int_{-\infty}^{\infty}\left(\frac{1-V(V+U)}{1+(V+U)^2}+\frac{1-V(V-U)}{1+(V-U)^2}\right)\frac{N(V)\,dV}{1+V^2+2sI}\times\left(\int_{-\infty}^{\infty}\frac{N(V)\,dV}{1+V^2+2sI}\right)^{-1}. \tag{88}$$

In most gas lasers the atoms have a Maxwell distribution of velocities and in the limit of inhomogeneous broadening the gain profile is a Gaussian. Therefore, it is appropriate to employ the more conventional notation in which the frequency difference is normalized to the full Doppler width at half maximum $\Delta\nu_D$ by the relation $x=2(\nu-\nu_0)(\ln 2)^{1/2}/\Delta\nu_D$ and the natural damping ratio is $\epsilon=\Delta\nu_h(\ln 2)^{1/2}/\Delta\nu_D$.[9] With these definitions Eqs. (87) and (88) can be written in the forms

$$1=\int_{-\infty}^{\infty}\frac{1+V^2}{2}\left(\frac{1}{1+(V+x/\epsilon)^2}+\frac{1}{1+(V-x/\epsilon)^2}\right)\times\frac{\exp(-\epsilon^2V^2)\,dV}{1+V^2+2sI}\left(\int_{-\infty}^{\infty}\frac{\exp(-\epsilon^2V^2)\,dV}{1+V^2+2sI}\right)^{-1}, \tag{89}$$

$$-\frac{x\delta}{\epsilon}=\frac{x}{2\epsilon}\int_{-\infty}^{\infty}\left(\frac{1-V(V+x/\epsilon)}{1+(V+x/\epsilon)^2}+\frac{1-V(V-x/\epsilon)}{1+(V-x/\epsilon)^2}\right)\times\frac{\exp(-\epsilon^2V^2)\,dV}{1+V^2+2sI}\left(\int_{-\infty}^{\infty}\frac{\exp(-\epsilon^2V^2)\,dV}{1+V^2+2sI}\right)^{-1}. \tag{90}$$

The right-hand side of Eq. (89) is plotted in Fig. 2 as a function of x for the intensity values $sI=0$ and $sI=1$ using the natural damping ratio $\epsilon=0.031$ which is appropriate for a xenon laser with a Doppler width of 110 MHz. These curves show the ratio of net gain to loss that would be observed by an infinitesimal standing-wave field with a frequency detuning of x. The value of ϵ is so small for xenon that the unsaturated ($sI=0$) curve in the figure is indistinguishable from the inhomogeneous limit $\exp(-x^2)$. With higher levels of gain and intensity a deep hole is burned in the gain curve. The bottom of this hole always occurs at unity since the gain must equal the loss for the oscillating mode.

The right-hand side of Eq. (90) is plotted as a function of x in Fig. 3 using the same conditions as employed in Fig. 2. The unsaturated $sI=0$ curve is indistinguishable from the inhomogeneous limit $2\pi^{-1/2}F(x)$, where $F(x)$ is Dawson's integral.[11] Also plotted in Fig. 3 is a negatively sloping straight line corresponding to the left-hand side of Eq. (90) with values appropriate to a xenon laser ($\delta=0.026$). The points at which such a straight line intersects one of the curves yields in a graphical way all possible sidebands having the same number of wavelengths between the mirrors as the fundamental cw mode. This same graphical solution technique has also been employed previously in a discussion of the longitudinal-mode splitting associated with high-gain lasers.[11] It is clear from the figure that with a moderate level of saturation two symmetrically spaced sidebands may satisfy this oscillation phase condition. A glance back at Fig. 2 shows that these sidebands are certain to have gain because they are situated at frequencies away from the minimum in the gain dip. Thus the spontaneous pulsations are a direct consequence of the perturbations in gain and dispersion associated with spectral holeburning. Besides providing this insight the graphical solutions are especially useful for understanding those cases in which the Doppler linewidth is not much greater than the homogeneous linewidth. If the Doppler width is actually less than the homogeneous width, it will be shown in Sec. II E that pulsing can not occur.

E. Homogeneous line broadening

Not all lasers have inhomogeneous linewidths greatly in excess of the homogeneous width, so the approximation which reduces Eqs. (70)–(72) to Eqs. (73)–(75) may not always apply. Thus it is worthwhile to consider the behavior of a laser in the opposite limit where the homogeneous width is much greater than the inhomogeneous width. This limit is obtained by setting the velocity v equal to its line center value of zero in Eqs. (70)–(72) except within the unsaturated population difference $N(v)$. The integrals are then trivial and these equations reduce, respectively, to

$$sI=\tfrac{1}{2}(r-1), \tag{91}$$

$$r^{-1}=[1+(\Delta\omega/\gamma)^2]^{-1}(1+2sI)^{-1}, \tag{92}$$

$$\Delta\omega t_c/r=-(\Delta\omega/2\gamma)[1+(\Delta\omega/\gamma)^2]^{-1}(1+2sI)^{-1}. \tag{93}$$

Equation (91) is the standard formula for the intensity in a homogeneously broadened laser. As in the previous stability discussions, this result is to be substituted into Eq. (93) to obtain the sideband frequency offset $\Delta\omega$. The offset $\Delta\omega$ is then inserted into Eq. (92) to test whether the sideband has net gain.

Substituting Eq. (91) into (93) and using the previous definitions, one obtains

$$U^2=-1-\delta^{-1}. \tag{94}$$

Since U^2 is always negative, the frequency offset is apparently imaginary. Thus a homogeneously broadened laser can never support sidebands of the saturating mode, and the mode is always stable with respect to coherent pulsations. This same conclusion follows from numerical and graphical solutions for mixed broadening situations in which the homogeneous linewidth is dominant.

III. ONE-DIRECTIONAL RING LASERS

Many lasers are built in a one-directional ring resonator configuration, and for completeness we consider briefly the stability characteristics of such resonators. The starting point for these calculations is again the density-matrix equations (1)–(4) but now the rapid time and space variations are factored out using the traveling-wave forms

$$E(z,t)=\tfrac{1}{2}E'(t)\exp(ikz-i\omega t)+\text{c.c.}, \tag{95}$$

$$\rho_{ab}(v,z,t)=P'(v,t)\exp(ikz-i\omega t)/2\mu. \tag{96}$$

With these substitutions Eqs. (1)–(4) reduce to

$$\frac{\partial P'(v,t)}{\partial t}=i(\omega-\omega_0-kv)P'(v,t)-\gamma P'(v,t)-\frac{i\mu^2}{\hbar}E'(t)D(v,t), \tag{97}$$

$$\frac{\partial D(v,t)}{\partial t}=\lambda_a(v)-\lambda_b(v)-\frac{\gamma_a+\gamma_b}{2}D(v,t)-\frac{\gamma_a-\gamma_b}{2}M(v,t)+\frac{1}{2\hbar}[iE'(t)P'^*(v,t)+\text{c.c.}], \tag{98}$$

$$\frac{\partial M(v,t)}{\partial t}=\lambda_a(v)+\lambda_b(v)-\frac{\gamma_a+\gamma_b}{2}M(v,t)-\frac{\gamma_a-\gamma_b}{2}D(v,t). \tag{99}$$

Together with the wave equation, Eqs. (97)–(99) characterize the transient phenomena that can occur in a ring laser.

If the laser is capable of periodic spontaneous pulsations, a useful set of substitutions analogous to those used in the study of standing-wave oscillators include

$$E'(t)=\sum_n E_n\exp(-in\Delta\omega t), \tag{100}$$

$$P'(v,t)=\sum_n P_n(v)\exp(-in\Delta\omega t), \tag{101}$$

$$D(v,t)=\sum_n D_n(v)\exp(-in\Delta\omega t), \tag{102}$$

$$M(v,t)=\sum_n M_n(v)\exp(-in\Delta\omega t)\,. \tag{103}$$

With these substitutions the nth frequency sidebands of Eqs. (97)–(99) must satisfy the equations

$$0=i(\omega+n\Delta\omega-\omega_0-kv)P_n(v)-\gamma P_n(v)$$
$$-\frac{i\mu^2}{\hbar}\sum_j E_{n-j}D_j(v)\,, \tag{104}$$

$$0=[\lambda_a(v)-\lambda_b(v)]\delta_{n0}+in\Delta\omega D_n(v)-\frac{\gamma_a+\gamma_b}{2}D_n(v)$$
$$-\frac{\gamma_a-\gamma_b}{2}M_n(v)+\frac{i}{2\hbar}\sum_j[E_{j+n}P_j^*(v)-E_{j-n}P_j(v)]\,, \tag{105}$$

$$0=[\lambda_a(v)+\lambda_b(v)]\delta_{n0}+in\Delta\omega M_n(v)$$
$$-\frac{\gamma_a+\gamma_b}{2}M_n(v)-\frac{\gamma_a-\gamma_b}{2}D_n(v)\,. \tag{106}$$

In a similar manner the wave equation reduces to

$$\left(\frac{i}{2t_c}+(\omega+n\Delta\omega-\Omega)\right)E_n=-\frac{\omega_0 l}{2\epsilon_0 L}\int_{-\infty}^{\infty}P_n(v)\,dv\,, \tag{107}$$

where we have integrated over the length of the cavity L with an active medium of length l.

To test the stability of the laser oscillation, we assume as before that the zero-order frequency component is dominant. Then Eqs. (104)–(107) for this component are

$$0=(\omega-\omega_0-kv)C_0(v)-\gamma S_0(v)-\frac{\mu^2}{\hbar}E_0D_0(v)\,, \tag{108}$$

$$0=-(\omega-\omega_0-kv)S_0(v)-\gamma C_0(v)\,, \tag{109}$$

$$0=\lambda_a(v)-\lambda_b(v)-\frac{\gamma_a+\gamma_b}{2}D_0(v)-\frac{\gamma_a-\gamma_b}{2}M_0(v)+\frac{E_0S_0(v)}{\hbar}\,, \tag{110}$$

$$0=\lambda_a(v)+\lambda_b(v)-\frac{\gamma_a+\gamma_b}{2}M_0(v)-\frac{\gamma_a-\gamma_b}{2}D_0(v)\,, \tag{111}$$

$$\frac{E_0}{2t_c}=-\frac{\omega_0 l}{2\epsilon_0 L}\int_{-\infty}^{\infty}S_0(v)\,dv\,, \tag{112}$$

$$(\omega-\Omega)E_0=-\frac{\omega_0 l}{2\epsilon_0 L}\int_{-\infty}^{\infty}C_0(v)\,dv\,, \tag{113}$$

where E_0 is real and $P_0(v)=C_0(v)+iS_0(v)$. Equations (108)–(111) may be solved for $S_0(v)$ and $D_0(v)$, and the results are

$$S_0(v)=-\frac{\mu^2E_0D_0(v)/\gamma\hbar}{1+[(\omega-\omega_0-kv)/\gamma]^2}\,, \tag{114}$$

$$D_0(v)=\frac{\gamma_a+\gamma_b}{2\hbar\gamma_a\gamma_b}E_0S_0(v)+N(v)\,. \tag{115}$$

Combining these equations, the polarization components may be related explicitly to the unsaturated population inversion and the intensity according to

$$S_0(v)=-\frac{\mu^2E_0N(v)/\gamma\hbar}{1+[(\omega-\omega_0-kv)/\gamma]^2+sI}\,, \tag{116}$$

$$C_0(v)=\left(\frac{\omega-\omega_0-kv}{\gamma}\right)\frac{\mu^2E_0N(v)/\gamma\hbar}{1+[(\omega-\omega_0-kv)/\gamma]^2+sI}\,, \tag{117}$$

where the normalized intensity is now defined by

$$sI=\frac{\mu^2E_0^2}{2\hbar^2}\frac{\gamma_a+\gamma_b}{\gamma\gamma_a\gamma_b}\,. \tag{118}$$

Equations (112) and (116) may be combined to yield the unsaturated intensity gain coefficient

$$g=\frac{\mu^2\omega_0}{c\epsilon_0\gamma\hbar}\int_{-\infty}^{\infty}\frac{N(v)dv}{1+[(\omega-\omega_0-kv)/\gamma]^2}\,, \tag{119}$$

which is the same as Eq. (49). In terms of this gain coefficient Eq. (112) can be written

$$\frac{1}{t_c}=\frac{gcl}{L}\int_{-\infty}^{\infty}\frac{N(v)\,dv}{1+[(\omega-\omega_0-kv)/\gamma]^2+sI}\times\left(\int_{-\infty}^{\infty}\frac{N(v)dv}{1+[(\omega-\omega_0-kv)/\gamma]^2}\right)^{-1}\,. \tag{120}$$

Similarly, Eqs. (113), (117), and (119) may be combined to obtain

$$\omega-\Omega=-\frac{gcl}{2L}\int_{-\infty}^{\infty}\left(\frac{\omega-\omega_0-kv}{\gamma}\right)\frac{N(v)\,dv}{1+[(\omega-\omega_0-kv)/\gamma]^2+sI}\times\left(\int_{-\infty}^{\infty}\frac{N(v)dv}{1+[(\omega-\omega_0-kv)/\gamma]^2}\right)^{-1}\,. \tag{121}$$

These equations govern the fundamental saturating mode.

Next it is necessary to determine whether the first-order sideband can satisfy the oscillation conditions. From Eq. (104) the infinitesimal first-order polarization must satisfy

$$0=i(\omega+\Delta\omega-\omega_0-kv)P_1(v)-\gamma P_1(v)-\frac{i\mu^2}{\hbar}E_1D_0(v)\,. \tag{122}$$

If E_1 is assumed to be real, this can be separated into the two equations

$$0=(\omega+\Delta\omega-\omega_0-kv)C_1(v)-\gamma S_1(v)-\frac{\mu^2}{\hbar}E_1D_0(v)\,, \tag{123}$$

$$0=-(\omega+\Delta\omega-\omega_0-kv)S_1(v)-\gamma C_1(v)\,. \tag{124}$$

These combine to yield

$$S_1(v)=\frac{-\mu^2E_1D_0(v)/\gamma\hbar}{1+[(\omega+\Delta\omega-\omega_0-kv)/\gamma]^2}\,. \tag{125}$$

This equation resembles Eq. (114), and one obtains

$$S_1(v)=\frac{1+[(\omega-\omega_0-kv)/\gamma]^2}{1+[(\omega+\Delta\omega-\omega_0-kv)/\gamma]^2}\frac{E_1}{E_0}S_0(v)\,. \tag{126}$$

Similarly, the first sideband of C_1 is related to the fundamental component by

$$C_1(v)=\frac{\omega+\Delta\omega-\omega_0-kv}{\omega-\omega_0-kv}\,\frac{1+[(\omega-\omega_0-kv)/\gamma]^2}{1+[(\omega+\Delta\omega-\omega_0-kv)/\gamma]^2}\times\frac{E_1C_0(v)}{E_0}\,. \tag{127}$$

Using Eq. (107) the oscillation conditions for the sideband may now be written

$$\frac{E_1}{2t_c}=-\frac{\omega_0 l}{2\epsilon_0 L}\int_{-\infty}^{\infty} S_1(v)\,dv\,, \qquad (128)$$

$$(\omega+\Delta\omega-\Omega)E_1=-\frac{\omega_0 l}{2\epsilon_0 L}\int_{-\infty}^{\infty} C_1(v)\,dv\,. \qquad (129)$$

With Eqs. (116), (117), (119), (125), and (127), these conditions become

$$\frac{1}{t_c}=\frac{gcl}{L}\int_{-\infty}^{\infty}\frac{1+[(\omega-\omega_0-kv)/\gamma]^2}{1+[(\omega+\Delta\omega-\omega_0-kv)/\gamma]^2}\,\frac{N(v)\,dv}{1+[(\omega-\omega_0-kv)/\gamma]^2+sI}\Bigg/\int_{-\infty}^{\infty}\frac{N(v)dv}{1+[(\omega-\omega_0-kv)/\gamma]^2}\,, \qquad (130)$$

$$\omega+\Delta\omega-\Omega=-\frac{gcl}{2L}\int_{-\infty}^{\infty}\frac{\omega+\Delta\omega-\omega_0-kv}{\gamma}\,\frac{1+[(\omega-\omega_0-kv)/\gamma]^2}{1+[(\omega+\Delta\omega-\omega_0-kv)/\gamma]^2}\,\frac{N(v)\,dv}{1+[(\omega-\omega_0-kv)/\gamma]^2+sI}$$
$$\times\left(\int_{-\infty}^{\infty}\frac{N(v)dv}{1+[(\omega-\omega_0-kv)/\gamma]^2}\right)^{-1}. \qquad (131)$$

In principle the stability analysis of ring lasers is now complete. Equations (120) and (121) are a coupled set which yield the intensity and frequency of the dominant frequency component. Equations (130) and (131) show whether in the presence of this component a sideband at frequency offset $\Delta\omega$ can simultaneously satisfy the oscillation phase condition and exhibit net gain.

In practice the equations just listed are tedious to solve, but some simplifications are usually possible. If the laser is tuned to line center, Eq. (121) is satisfied trivially provided that $N(v)$ is an even function of v. Then Eqs. (120), (130), and (131) reduce to

$$\frac{1}{r}=\int_{-\infty}^{\infty}\frac{N(v)\,dv}{1+(kv/\gamma)^2+sI}\Bigg/\int_{-\infty}^{\infty}\frac{N(v)\,dv}{1+(kv/\gamma)^2}\,, \qquad (132)$$

$$\frac{1}{r}=\int_{-\infty}^{\infty}\frac{1+(kv/\gamma)^2}{1+[(\Delta\omega-kv)/\gamma]^2}\,\frac{N(v)\,dv}{1+(kv/\gamma)^2+sI}\times\left(\int_{-\infty}^{\infty}\frac{N(v)\,dv}{1+(kv/\gamma)^2}\right)^{-1}, \qquad (133)$$

$$\frac{\Delta\omega t_c}{r}=-\int_{-\infty}^{\infty}\frac{\Delta\omega-kv}{2\gamma}\,\frac{1+(kv/\gamma)^2}{1+[(\Delta\omega-kv)/\gamma]^2}\,\frac{N(v)\,dv}{1+(kv/\gamma)^2+sI}\times\left(\int_{-\infty}^{\infty}\frac{N(v)\,dv}{1+(kv/\gamma)^2}\right)^{-1}. \qquad (134)$$

If the Doppler width is much greater than the homogeneous linewidth, $N(v)$ may be replaced by its line center value. In this limit the integrals in Eq. (132) can be performed analytically, and the result is

$$sI=r^2-1\,. \qquad (135)$$

This is the same form as the standing-wave result in Eq. (73) except for the factor of 2. Equations (133) and (134) can now be written

$$\frac{1}{r}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1+V^2}{1+(V-U)^2}\,\frac{dV}{r^2+V^2}\,, \qquad (136)$$

$$\frac{\gamma U t_c}{r}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(V-U)(1+V^2)}{1+(V-U)^2}\,\frac{dV}{r^2+V^2}\,. \qquad (137)$$

These equations may be shown to be identical to Eqs. (74) and (75). Thus the stability criteria for one-directional ring lasers are identical in form to the stability criteria for standing-wave lasers, and all of the previous considerations are still valid.

IV. CONCLUSIONS

A rigorous solution of the time-dependent density-matrix equations and Maxwell's equations shows that some lasers which one might expect would produce a cw output are in fact unstable and produce a periodic train of short pulses. In this work we have derived the conditions that a laser must satisfy in order to exhibit the pulsation instability. The resulting stability criteria in their most general form consist of a set of four coupled integral equations. For most purposes, however, it is sufficient to simply examine Fig. 1 to determine whether there is any possibility of instability. Basically, the laser medium must be inhomogeneously broadened and the product of the homogeneous linewidth and the cavity lifetime must be less than unity. The xenon laser at 3.51 μm has been our archetypical unstable system, but the unstable regime is broad so that similar behavior should be found with many other laser types.

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