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Elastic electron-hydrogen scattering in a modified approach to the Glauber approximation

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Recently a modified Glauber theory was proposed by one of the authors (S.R.). In this theory a finite value of the mean excitation energy of the atom in its intermediate states is retained in each order (except the first) of the Glauber series, in a consistent way. In the present paper this theory is examined by analyzing the elastic electron-hydrogen scattering at 50, 100, and 200 eV. The excitation energy is determined by the total cross section, and turns out to be essentially independent of the energy of the incident particle. The exchange effect has been taken into account by making use of the Glauber exchange formulas of Forster and Williamson. The agreement of the theoretical results with the experimental data is very good. The proof of the unitarity of the modified Glauber amplitude is also given.

I. INTRODUCTION

The study of particle-atom scattering of intermediate energies by the Glauber impact-parameter method[1] and other related methods has been with us for some time. Although the Glauber amplitude is of a relatively simple nature, the Glauber theory takes into account the complex multiple-scattering processes of the charged particle in the atom. It was first introduced into the realm of electron-atom scattering by Franco,[2] and independently by the present authors.[3]

The Glauber amplitude suffers from a number of shortcomings both for elastic and inelastic collisions. For elastic scattering the main problem of the Glauber amplitude is the sharp peak and logarithmic divergence at small angles. The divergence appears in the second-order term of the amplitude. This failure can be traced back[3-5] to the neglect of the excitation energies of the atom in its intermediate states in the derivation of the Glauber amplitude.

Several methods have been proposed to remedy this shortcoming. In one method[6] (the eikonal-Born-series method) the Glauber series is truncated after the third-order term and the second-order Glauber term is replaced by the corresponding Born term, which is calculated at some finite excitation energy. Another method, closely related, in which the second-order Glauber term is replaced by the corresponding Born term but the rest of the Glauber series is retained, has recently been proposed by Gien.[7]

Recently, a different approach has been put forward by one of us.[8] Earlier attempts to remedy the divergence of $\mathrm{Im} f^{(2)}$ were *ad hoc* corrections. In contrast, the derivation of the new amplitude, valid for both elastic and inelastic collisions, is a rigorous approach in which the value of the mean excitation energy of the atom in its intermediate states is kept finite in each order term (except the first) of the scattering amplitude. Only the leading term in inverse powers of the momentum k of the scattered particle is retained in each order. Next, the usual eikonization process is performed, and taking the axis of quantization along the direction of the initial beam it is possible to sum over all orders. The amplitude obtained resembles the conventional Glauber amplitude in character but most of its shortcomings are absent. In particular, for elastic scattering the scattering amplitude is finite in the forward direction. We shall refer to the new amplitude as the modified Glauber amplitude. The Glauber theory has other shortcomings: for instance, we get zero excitation cross sections for states for which $(L_f - M_f) - (L_i - M_i)$ is an odd number, L_f, M_f, and L_i, M_i being the angular momentum quantum numbers of the final and initial states, respectively. This problem also is removed by our modified theory. One shortcoming of the Glauber theory which is *not* solved by the modified theory, is that the cross section does not distinguish between scattering of positive and negative particles. More details concerning this point are given in Sec. II.

A different approach—that of introducing a finite excitation energy into the theory, strictly in the framework of the Glauber theory—has also been communicated[9] recently by the authors.

The purpose of the present paper is to examine the modified Glauber amplitude by analyzing the experimental data for elastic scattering of electrons by hydrogen in its ground state at 50, 100, and 200 eV. In Sec. II a brief account concerning the background of the theory as well as the expressions of the scattering amplitude and the exchange amplitude are given. The mean excitation energy is determined via the optical theorem. In Sec. III the comparison with experimental data is represented and discussed.

II. THEORETICAL BACKGROUND

As discussed above, the modified Glauber scattering amplitude under consideration here is similar in many respects to the conventional Glauber amplitude. Like the latter, it is an impact-parameter representation valid for energies much above ionization threshold. However, it differs from the Glauber theory in one important respect: All terms in the amplitude of order higher than 1 are evaluated by assuming a *finite* value for the mean excitation energy E_{exc} of the atom in its intermediate states. This fact does away, with many shortcomings from which the conventional Glauber theory suffers (as shown in Ref. 8). Taking the axis of quantization z along the direction of the initial beam, it was shown in the same reference that a close form of the scattering amplitude can be obtained by summing over all orders. This is true for elastic as well as inelastic processes.

The amplitude for scattering of charged particles by atoms from the ground state σ_i to the state σ_n is given by

$$f_{ni}^{(MG)}(\vec{k}_n\vec{k}_i) = -i^{M_n+1}\sqrt{k_n k_i}\, e^{-iM_n\varphi_s} \times \int_0^\infty b\, db J_{M_n}(bQ)\langle\sigma_n e^{-iM_n\varphi}\,|i\Lambda_{k_n-k_i} + \alpha_{k_n-k_i}(e^{i\Lambda_0} - i\Lambda_0 - 1)|\sigma_i\rangle\,, \quad (1)$$

where $\vec{Q}=\vec{k}_i-\vec{k}_n$ is the momentum transfer and b is the impact parameter. φ_s and φ are the azimuthal angles of $\vec{Q}$ and $\vec{b}$, respectively. The quantity $\alpha_{k_n-k_i}$ is a measure of the deviation of our amplitude from the Glauber amplitude. It is defined by

$$\alpha_{k_n-k_i} = \Lambda_{k_n-\bar{k}}\Lambda_{\bar{k}-k_i}/\Lambda_0^2\,, \quad (2)$$

and the modified Glauber phase function $\Lambda_{\Delta k}$ is defined by

$$\Lambda_{\Delta k} = -\frac{\mu}{\hbar^2\bar{k}}\int_{-\infty}^{\infty} V(\vec{b}+\hat{k}_i z;\vec{\zeta})e^{-i\Delta k z}dz\,, \quad (3)$$

where μ is the reduced mass of the system. $\Lambda_{\Delta k}$ reduces to the conventional Glauber phase function Λ_0 when $\Delta k=0$. $\bar{k}$ is the mean momentum of the scattered particle, which is related to the mean excitation energy E_{exc} of the atom in the intermediate state by

$$k_i^2 - \bar{k}^2 = 2\mu E_{exc}\,. \quad (4)$$

When $k_i = k_n = \bar{k}$, then $\alpha_{k_n-k_i}=1$; in this case our amplitude reduces to the conventional Glauber amplitude.

For the interaction

$$V = -e^2/r + e^2/|\vec{r}-\vec{\zeta}|\,, \quad (5)$$

the modified phase function is given by

$$\Lambda_{\Delta k} = \frac{2e^2\mu}{\hbar^2\bar{k}}\,[K_0(b\Delta k) - e^{i\zeta_\parallel\Delta k}K_0(\tau\Delta k)]\,, \quad (6)$$

where $\zeta_\parallel$ and $\zeta_\perp$ are the components of the radius vector of the bound electron along the initial beam and perpendicular to it, respectively, and $\tau = |\vec{b}-\zeta_\perp|$. The functions K_0 are the modified Bessel functions of the third kind. In the limit $\Delta k=0$, $\Lambda_{\Delta k}$ reduces to the Glauber phase

$$\Lambda_0 = (2e^2\mu/\hbar^2\bar{k})\ln(\tau/b)\,. \quad (7)$$

In the original version[8] $Q_\perp$, the perpendicular component of Q, appears in the expression of the amplitude instead of Q. The modified Glauber theory (like the conventional Glauber theory) is a small-angle theory, therefore, it is quite alright to use Q instead of $Q_\perp$ because $Q_\perp \sim Q$ for small angles. It turns out, however, that our amplitude, Eq. (1), reproduces the experimental data very well over essentially the whole angular range provided Q is kept in Eq. (1), and not $Q_\perp$. Although there is no real theoretical justification for this substitution, it is common practice among authors in this field. Also, in Eq. (1), for reasons of symmetry, we have replaced the unsatisfactorily defined momentum k of the original version by the factor $\sqrt{k_n k_i}$. Our amplitude of Eq. (1) satisfies the unitarity theorem. The proof is given in the Appendix.

Making use of the integral

$$\int_{-\infty}^{\infty} e^{-(2/a_0)(x^2+y^2)^{1/2}}e^{ixt}dx = \frac{4y}{a_0(4/a_0^2+t^2)^{1/2}}K_1[y(4/a_0^2+t^2)^{1/2}]\,, \quad (8)$$

the integration over $\zeta_\parallel$ in Eq. (1) is easily performed. We thus remain with the three-dimensional integral for the scattering amplitude:

$$f_{ii}^{(MG)} = -\frac{4ik}{\pi a_0^3}\int_0^\infty b\,db\,J_0(bQ) \times\int_0^\pi d\varphi\int_0^\infty \zeta_\perp^2 d\zeta_\perp\left[i\Lambda_0 K_1\left(\frac{2\zeta_\perp}{a_0}\right) + \beta(e^{i\Lambda_0} - i\Lambda_0 - 1)\right]\,, \quad (9a)$$

where

$$\beta = \left(\frac{2}{ka_0\Lambda_0}\right)^2\left[[K_0^2(b\Delta k) + K_0^2(\tau\Delta k)]K_1\left(\frac{2\zeta_\perp}{a_0}\right) - 4\frac{K_0(b\Delta k)K_0(\tau\Delta k)}{a_0(4/a_0^2+\Delta k^2)^{1/2}} \times K_1[\zeta_\perp(4/a_0^2+\Delta k^2)^{1/2}]\right]\,. \quad (9b)$$

The mean momentum $\bar{k}$ and with it the mean excitation energy E_{exc} of the atom in its intermediate states is an undetermined quantity in the theory which leads to the amplitude, Eq. (1). To determine its value use has been made of the optical theorem

$$\mathrm{Im} f_{ii}^{(MG)}(\theta=0)=(k/4\pi)\sigma_{tot}\,. \qquad (10)$$

In order to estimate the total cross section σ_{tot} we have taken the Bethe-Born (three-term) expression of Kim and Inokuti[10]

$$\sigma_{tot}=\pi a_0^2\left(\frac{4}{(ka_0)^2}\ln(ka_0)^2+\frac{9.676}{(ka_0)^2}-\frac{8}{(ka_0)^4}\right) \qquad (11)$$

which describes[11] σ_{tot} very well in the energy region under consideration in spite of the fact that it is calculated in lowest-order perturbation theory. To put it differently, $\mathrm{Im} f_B^{(n)}(0)$, $n>2$ contributes very little to σ_{tot}. More specifically, $f_B^{(3)}$ and $f_B^{(4)}$ contribute to the term k^{-4}, $f_B^{(5)}$ and $f_B^{(6)}$ to k^{-6}, etc. In other words, the Bethe-Born formula is dominantly determined by the lowest-order amplitude $f^{(2)}$. In short, Eq. (11) is well suited for the numerical evaluation of Δk. The contribution of the exchange effect has been included neither in the expression of σ_{tot} nor in $f_{ii}^{(MG)}$ of Eq. (10). The results for Δk and the corresponding values of E_{exc} as function of the momentum k are listed in Table I. The surprising result is that the mean excitation energy E_{exc} is essentially independent of energy. Not only does it not increase with energy, as we expected intuitively, if actually *decreases* slightly with increasing energy.

We are now in a position to evaluate the scattering amplitude of Eq. (1) numerically. It turns out that the real part of the modified Glauber amplitude is not very different from the real part of the conventional Glauber amplitude. On the other hand, the corresponding imaginary parts are significantly different from each other. The main difference between the two amplitudes is at small angles: whereas our amplitude is finite when $\theta\to 0$, the Glauber amplitude becomes infinite. This, of course, was to be expected. However there is also a non-negligible difference between the two amplitudes for the rest of the angular range. For $\theta>20°$ ($E=50$ eV) the imaginary part of our amplitude is always slightly higher than the imaginary part of the Glauber amplitude, it reaches 16% in the neighborhood of 30°. In Fig. 1 the imaginary parts of the two amplitudes for incident energy of 50 eV are plotted versus scattering angles. The imaginary part of the amplitude suggested by Gien[7] is also shown. It is systematically lower than our amplitude; at intermediate angles the difference is up to 45%, pointing to the strong dependence of $\mathrm{Im} f^{(n)}$, $n>2$ on Δk.

As already pointed out in the Introduction, the absolute values of the conventional and modified Glauber amplitudes do not depend on the sign of the charge of the scattered particle. In Ref. 8 it was shown that this is attributable to the neglect of what is there called the "$j_l n_l$ terms." These terms should be incorporated into each order term of the amplitude. This is a rather complicated problem, but if the mathematical difficulties involved are solved, we shall be left with an amplitude in which the phase function includes higher-order terms, thus giving rise to cross sections which distinguish between positive and negative

TABLE I. Excitation energy E_{exc} and Δk as function of incident energy E.

E (eV)	$\Delta k(a_0^{-1})$	E_{exc} (Ry)
50	0.2800	0.993
100	0.1845	0.965
200	0.1270	0.956
400	0.0890	0.956

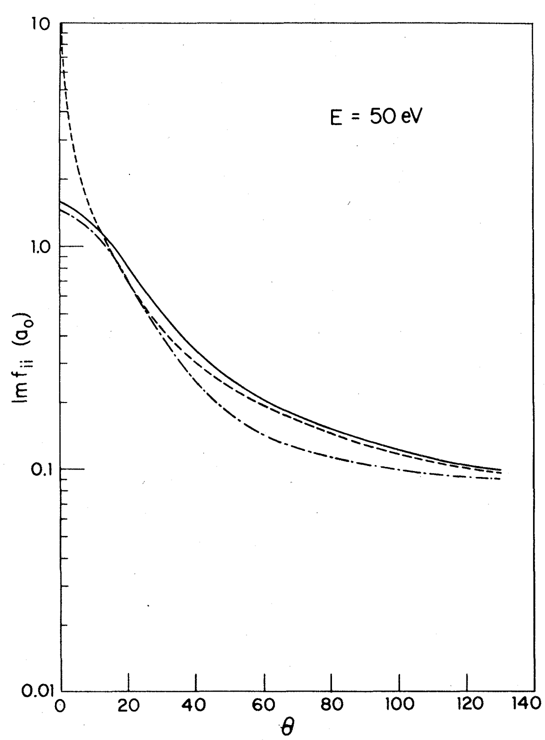

FIG. 1. Imaginary part of scattering amplitude at 50 eV. Solid line, modified Glauber amplitude; dashed line, conventional Glauber amplitude; dash-dotted line, Gien's amplitude.

particles. In the meantime, we shall have to be content with the correction of the second-order term of the amplitude only. It is easy to see that this correction is simply the real part of the second-order Born amplitude $f_{ii}^{(2)}(\theta)$. We have thus to add to our modified Glauber amplitude $f_{ii}^{(MG)}$ the term

$$\operatorname{Re} f_{ii}^{(2)}=\frac{2}{(\pi a_0)^2}\operatorname{Re}\sum_n\int\frac{[F_{in}(\vec{k}_f-\vec{q})-\delta_{in}][F_{ni}(\vec{q}-\vec{k}_i)-\delta_{ni}]}{(\vec{k}_f-\vec{q})^2(q^2-k_n^2-i\epsilon)(\vec{q}-\vec{k}_i)^2}\,d\vec{q}\,, \tag{12}$$

where F_{ik} is the form-factor of the atom. In order to evaluate the sum over the intermediate states, the mean momentum $\bar{k}$ of the scattered particle is substituted instead of the actual intermediate momenta k_n. We then obtain

$$\operatorname{Re} f_{ii}^{(2)}=\frac{2}{(\pi a_0)^2}\operatorname{Re}\int\frac{F_{ii}(\vec{Q})-F_{ii}(\vec{k}_f-\vec{q})-F_{ii}(\vec{q}-\vec{k}_i)+1}{(\vec{k}_f-\vec{q})^2(q^2-\bar{k}^2-i\epsilon)(\vec{q}-\vec{k}_i)^2}\,d\vec{q}\,, \tag{12$'$}$$

which has been calculated by the Feynman technique in the usual way. We have $\bar{k}=k-\Delta k$, where Δk was taken to be equal to the previously found values given in Table I. The reason for chosing the same $\bar{k}$ in $f^{(\mathrm{MG})}$ and $\operatorname{Re} f^{(2)}$ is as follows: In Ref. 8 it is shown that the value of $\bar{k}$ is defined by a certain sum over the intermediate states [Eqs. (5.7) and (5.9)]. Now it turns out that if one performs the same kind of analysis for $\operatorname{Re} f^{(2)}$ one finds that $\bar{k}$ is defined by *exactly* the same expression (this material is still unpublished). It is worth pointing out that $\operatorname{Re} f_{ii}^{(2)}$ depends significantly on the value of Δk. For instance, for 50-eV electrons in the forward direction we get $\operatorname{Re} f_{ii}^{(2)}(\Delta k=0.28)=1.858$ and $\operatorname{Re} f_{ii}^{(2)}(\Delta k=0)=0.387$. This strong dependence on Δk is due to the fact that $\operatorname{Re} f_{ii}^{(2)}$ at $\Delta k=0$ has a discontinuity equal to π/k.

The direct amplitude is thus given by

$$f_D=f_{ii}^{(\mathrm{MG})}+\operatorname{Re} f_{ii}^{(2)}. \tag{13}$$

As to the exchange amplitude we have used the "post" eikonal exchange amplitude proposed recently by Forster and Williamson.[12] A similar expression has been derived independently by Madan.[13] It is given by the two-dimensional integral

$$\begin{aligned} f_E=&-\frac{2^{4-i/ka_0}}{a_0^4}\,\frac{\Gamma(1-i/ka_0)}{\Gamma(-i/ka_0)}\\ &\times\int_0^\infty d\lambda\,\lambda^{-i/ka_0-1}\int_0^1\frac{d\chi}{\chi}\\ &\times\left[\frac{1}{a_0}\left(\frac{d}{d\mu^2}\right)^2\mathfrak{F}_{(+)}(1,0,0,0,0)\right.\\ &\left.-\frac{1}{\chi}\left(\frac{d}{d\mu^2}\right)^2\mathfrak{F}_{(+)}(1,0,0,1,0)\right]_{\mu=1/a_0}\,, \end{aligned} \tag{14a}$$

where

$$\begin{aligned}\mathfrak{F}_{(+)}(1,0,0,s,0)=&\lambda^s(1-\chi)^s(\beta_{(+)}^2+Q_{(+)}^2)^{i/ka_0-1}\\ &\times(\beta_{(+)}-iQ_{(+)z})^{-i/ka_0}\,,\end{aligned} \tag{14b}$$

$$\beta_{(+)}=\Lambda_{(+)}+1/a_0\,, \tag{14c}$$

$$\begin{aligned}\Lambda_{(+)}^2=&\chi^2(1-\chi)^2+\mu^2\chi+k^2\chi(1-\chi)\\ &-2i\lambda\chi(1-\chi)k\cos\theta\,,\end{aligned} \tag{14d}$$

$$\vec{Q}_{(+)}=\vec{k}_i-\chi\vec{k}_f-i\lambda(1-\chi)\hat{z}\,. \tag{14e}$$

The z direction was taken along $\vec{k}_i$ in compliance with the direct amplitude where the same direction was chosen.

The differential cross section is calculated according to

$$\frac{d\sigma}{d\Omega}=\frac{3}{4}|f_D-f_E|^2+\frac{1}{4}|f_D+f_E|^2. \tag{15}$$

III. RESULTS AND DISCUSSION

The differential cross section for elastic e-H scattering was calculated numerically for 50, 100, and 200 eV and scattering angles from 0.001° to 140° according to Eq. (15). The direct amplitude is given by Eqs. (9), (12), and (13) with Δk given by Table I. The exchange amplitude is given by Eqs. (14).

In Table II the differential cross sections calculated with and without the exchange effect, as well as the experimental data by Williams,[14] are presented. In Figs. 2–4, the differential cross sections including exchange are plotted, along with the experimental data of Ref. 13. For comparison, the first Born approximation cross section and the conventional Glauber cross section, both without exchange, are also plotted. The latter was calculated according to the analytic expression of Thomas and Gerjuoy.[15] At 50 eV the agreement of the modified Glauber cross section with the experimental data is extremely good. For 100 and 200 eV the agreement is also very satisfactory but less good than for 50 eV. In order to put this on a more quantitative basis, use has been made of the well-known χ^2 test. Let X_i be the theoretical expectation of the cross section, and x_i and σ_i the experimen-

TABLE II. Differential cross sections of elastic e-H scattering in $a_0^2\ \mathrm{sr}^{-1}$ units for 50, 100 and 200 eV. The numbers in parentheses are the standard deviations in the last significant digits.

E Angles (deg)	50 eV Without exchange	50 eV With exchange	50 eV Expt. data (Williams)	100 eV Without exchange	100 eV With exchange	100 eV Expt. data (Williams)	200 eV Without exchange	200 eV With exchange	200 eV Expt. data (Williams)
10^{-3}	10.75	12.05	···	7.419	8.044	···	5.15	5.4	···
10	5.03	5.83	5.04(51)	2.123	2.436	···	1.02	1.132	···
15	3.21	3.82	3.18(37)	1.236	1.464	···	0.607	0.679	···
20	2.07	2.51	2.17(23)	0.786	0.946	1.10(10)	0.374	0.416	0.419(40)
30	0.95	1.17	1.12(12)	0.363	0.430	0.509(49)	0.149	0.161	0.172(17)
40	0.50	0.598	0.551(59)	0.186	0.211	0.288(27)	0.065 0	0.068 5	0.070 6(68)
50	0.299	0.333	0.308(27)	0.103	0.111	0.132(12)	0.032	0.033 0	0.031 4(32)
60	0.190	0.200	0.205(19)	0.061	0.063	0.072(7)	0.017 4	0.017 8	0.018 7(19)
70	0.129	0.130	0.146(14)	0.038	0.039	0.049(5)	0.010 4	0.010 5	0.012 5(14)
80	0.0909	0.0897	0.0993(121)	0.025 5	0.025 6	0.0295(30)	0.006 8	0.006 8	0.008 6(9)
90	0.0667	0.0658	0.0716(82)	0.017 9	0.017 9	0.0209(20)	0.004 7	0.004 7	0.005 8(6)
100	0.0505	0.0504	0.0558(66)	0.013 3	0.013 3	0.0155(15)	0.003 45	0.003 46	0.004 12(41)
110	0.0400	0.0407	0.0421(43)	0.010 3	0.010 3	0.0115(12)	0.002 68	0.002 68	0.003 23(31)
120	0.0326	0.0339	0.0349(33)	0.008 3	0.008 4	0.0092(9)	0.002 14	0.002 15	0.002 72(35)
130	0.0273	0.0291	0.0288(30)	0.007 0	0.007 1	0.0078(7)	0.001 79	0.001 80	0.001 99(25)
140	0.0238	0.0259	0.0273(26)	0.005 88	0.006 03	0.0065(7)	0.001 61	0.001 62	0.001 78(26)

tal value of the cross section and the corresponding standard deviation for the same energy and scattering angle, then

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{X_i - x_i}{\sigma_i}\right)^2 , \tag{16}$$

where n is the number of measurements for given energy. Obviously, the smaller χ^2, the better the fit. In the literature[16] the values of the function $P_{n-1}(>\chi^2)$ are given, which expresses the proba-

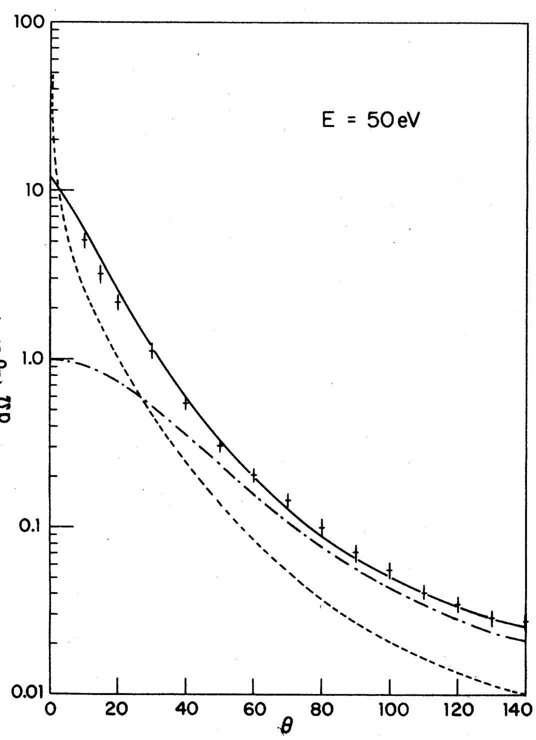

FIG. 2. Differential cross sections of e-H elastic scattering at 50 eV. Solid line, modified Glauber with Glauber exchange; dashed line, conventional Glauber without exchange; dash-dotted line, first Born without exchange. Data by Williams.

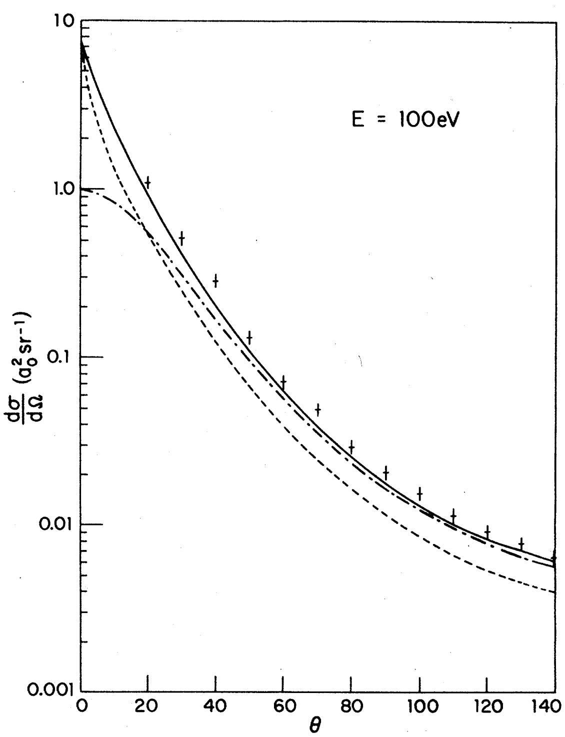

FIG. 3. Same as Fig. 2 at 100 eV.

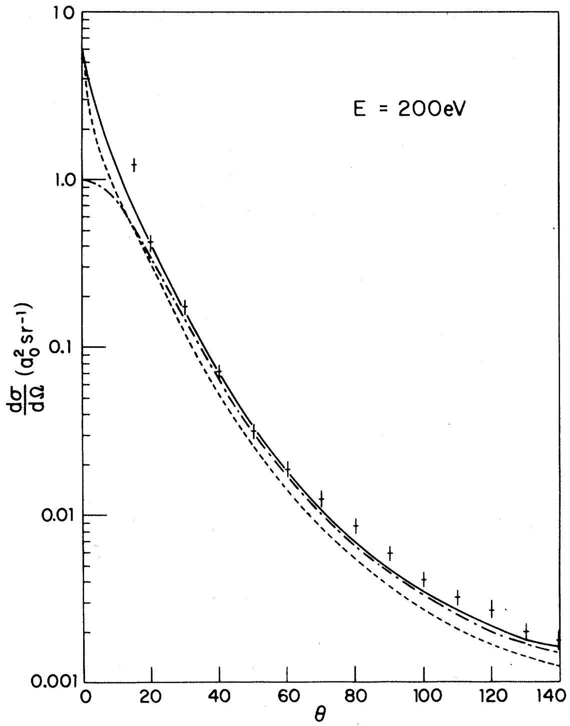

FIG. 4. Same as Fig. 2 at 200 eV.

bility of obtaining any value worse than χ^2. In Table III are listed the values of χ^2 and the corresponding probability function P_{n-1} for different energies based on the modified Glauber theory. Similarly, the values calculated according to the eikonal-Born-series amplitude[6] and Gien's amplitude,[7] both including exchange have also been given whenever the numerical values were available in the literature. All these calculations are based on the experimental data of Williams.[14] As can be seen the agreement with the experimental data of 50 eV is really very good. At 200 eV, and even more so at 100 eV, the results of the modified Glauber theory as well as the eikonal-Born-series theory are less satisfactory, though acceptable. But even at 50 eV there is still room for improvement. Whether the fault is with the theoretical prediction or with the experimental data is difficult to say. Probably both are in need of amendment. A possible amendment of the modified Glauber theory has already been pointed out in Sec. II, namely, inclusion of higher-order terms of the phase function in the expression of the amplitude. As to the measurements at 100 and 200 eV, more recent data by Van Wingerden *et al.*[17] do not improve the fit. On the contrary, their data of the differential cross section are systematically higher than those by Williams, thus increasing the gap between theory and experiment even further.

For 50 eV, we also checked the sensitivity of the results to changes of Δk. We have found the following: (i) Increasing Δk by 10% to 0.308 gives $\chi^2 = 19.94$, thus making the agreement worse considerably. (ii) Decreasing Δk by 10% to 0.252 gives $\chi^2 = 12.66$, thus improving the results only slightly. In other words, the value of Δk found via the optical theorem seems to yield results which cannot be considerably improved upon by changing the value of Δk arbitrarily. This is an indication of the internal consistency of the theory.

APPENDIX

We here prove the unitarity of the modified Glauber amplitude of Eq. (1). Making use of the integral representation of the Bessel function [see Ref. 8, Eq. (2.21)] the amplitude becomes

$$f_{ni}^{(MG)}(\vec{k}_n,\vec{k}_i) = -i\,\frac{\sqrt{k_n k_i}}{2\pi}\, e^{-iM_n\varphi_s}\int d^{(2)}b\, e^{-i(\vec{k}_n-\vec{k}_i)\cdot\vec{b}-iM_n\varphi}\langle\sigma_n e^{-iM_n\varphi}|i\Lambda_{k_n-k_i}+\alpha_{k_n-k_i}(e^{i\Lambda_0}-i\Lambda_0-1)|\sigma_i\rangle . \qquad \text{(A1)}$$

TABLE III. χ^2 values and probability function $P_{n-1}(>\chi^2)$.

E (eV)	Number of points n	χ^2 (Present work)	$P_{n-1}(>\chi^2)$ (Present work)	χ^2 (Others)	$P_{n-1}(>\chi^2)$ (Others)
50	15	13.02	0.53	20.15 [a]	0.13
100	12 [b]	31.33	0.0011	34.33 [c]	0.0004
200	10 [d]	16.13	0.065	13.80 [c]	0.13

[a] Reference 7.
[b] Without the point at 140°.
[c] Reference 6.
[d] Without the points at 120°, 130°, 140°.

Similarly,

$$(f_{fn}^{(\mathrm{MG})}(k_f,k_n))^\dagger = i\,\frac{\sqrt{k_f k_n}}{2\pi}\,e^{-i(M_f-M_n)\varphi_s}\int d^{(2)}b\,e^{-i(\vec{k}_f-\vec{k}_n)\cdot\vec{b}-i(M_f-M_n)\varphi}\langle\sigma_n e^{-iM_n\varphi}|-i\Lambda_{k_f-k_n} + \alpha_{k_f-k_n}(e^{-i\Lambda_0}+i\Lambda_0-1)|\sigma_f e^{-iM_f\varphi}\rangle, \tag{A2}$$

since $\Lambda_{k_1-k_2}$ and $\alpha_{k_1-k_2}$ are Hermitian. Now the only dependence of $\Lambda_{k_1-k_2}$ on φ_ξ and φ, the azimuthal angles of $\vec{\xi}$ and $\vec{b}$, respectively, is through τ, where

$$\tau^2 = b^2+\xi_\perp^2-2b\xi_\perp\cos(\varphi_\xi-\varphi)\,. \tag{A3}$$

Thus, the integral over φ_ξ is of the form

$$\int_0^{2\pi} e^{-i(M_n-M_f)(\varphi_\xi-\varphi)}g(\cos(\varphi_\xi-\varphi))d\varphi_\xi\,. \tag{A4}$$

This integral does not depend on the sign of (M_n-M_f), and therefore the two states in the matrix element of (A1) are interchangeable.

Hence we have

$$\frac{k_n}{\sqrt{k_f k_i}}\int (f_{fn}^{(\mathrm{MG})}(\vec{k}_f,\vec{k}_n))^\dagger f_{ni}^{(\mathrm{MG})}(\vec{k}_n,\vec{k}_i)\,d\Omega_n$$
$$=\frac{k_n^2}{(2\pi)^2}\,e^{-iM_f\varphi_s}\int\!\!\int\!\!\int d^{(2)}b\,d^{(2)}b'\,d\Omega_n\,e^{-i(\vec{k}_f-\vec{k}_n)\cdot\vec{b}-i(\vec{k}_n-\vec{k}_i)\cdot\vec{b}'}e^{-i(M_f-M_n)\varphi-iM_n\varphi'}\langle\sigma_f e^{-iM_f\varphi}|-i\Lambda_{k_f-k_n} + \alpha_{k_f-k_n}(e^{-i\Lambda_0}+i\Lambda_0-1)|\sigma_n e^{-iM_n\varphi}\rangle\langle\sigma_n e^{-iM_n\varphi'}|i\Lambda_{k_n-k_i}+\alpha_{k_n-k_i}(e^{i\Lambda_0}-i\Lambda_0-1)|\sigma_i\rangle\,. \tag{A5}$$

Now the present theory is a high-energy, small-angle theory; we can therefore, according to Glauber,[18] as long as k_i and k_f lie close together in direction, replace the integration over the sphere by an integration over the plane in k_n space which is tangent to the sphere at $\vec{k}_n=\vec{k}_i$. In other words, we may put $d\Omega_n=d^{(2)}k_n/k_n{}^2$. With this in mind, the angular integration becomes a two-dimensional δ function of $\vec{b}-\vec{b}'$. We therefore get

$$\frac{k_n}{\sqrt{k_f k_i}}\int (f_{fn}^{(\mathrm{MG})}(\vec{k}_f,\vec{k}_n))^\dagger f_{ni}^{(\mathrm{MG})}(\vec{k}_n,\vec{k}_i)d\Omega_n$$
$$= e^{-iM_f\varphi_s}\int d^{(2)}b\,e^{-i(\vec{k}_f-\vec{k}_i)\cdot\vec{b}-iM_f\varphi}\langle\sigma_f e^{-iM_f\varphi}|-i\Lambda_{k_f-k_n}+\alpha_{k_f-k_n}(e^{-i\Lambda_0}+i\Lambda_0-1)|\sigma_n e^{-iM_n\varphi}\rangle \times\langle\sigma_n e^{-iM_n\varphi}|i\Lambda_{k_n-k_i}+\alpha_{k_n-k_i}(e^{i\Lambda_0}-i\Lambda_0-1)|\sigma_i\rangle\,. \tag{A6}$$

Next, by introducing the mean momentum $\bar{k}$ the sum over all intermediate states σ_n can be performed [see Ref. 8, Eqs. (5.7) and (5.9)]. We obtain

$$\frac{1}{\sqrt{k_f k_i}}\sum_{\sigma_n} k_n\int (f_{fn}^{(\mathrm{MG})}(\vec{k}_f,\vec{k}_n))^\dagger f_{ni}^{(\mathrm{MG})}(\vec{k}_n,\vec{k}_i)d\Omega_n$$
$$= e^{-iM_f\varphi_s}\int d^{(2)}b\,e^{-i(\vec{k}_f-\vec{k}_i)\cdot\vec{b}-iM_f\varphi}\langle\sigma_f e^{-iM_f\varphi}|[-i\Lambda_{k_f-\bar{k}}+\alpha_{k_f-\bar{k}}(e^{-i\Lambda_0}+i\Lambda_0-1)][i\Lambda_{\bar{k}-k_i}+\alpha_{\bar{k}-k_i}(e^{i\Lambda_0}-i\Lambda_0-1)]|\sigma_i\rangle. \tag{A7}$$

By the definition of $\alpha_{k_1-k_2}$, Eq. (2), we have

$$\Lambda_{k_f-\bar{k}}\Lambda_{\bar{k}-k_i}=\alpha_{k_f-k_i}\Lambda_0^2\,,\quad \Lambda_{k_f-\bar{k}}\alpha_{\bar{k}-k_i}=\alpha_{k_f-\bar{k}}\Lambda_{\bar{k}-k_i}=\alpha_{k_f-k_i}\Lambda_0\,,\quad \alpha_{k_f-\bar{k}}\alpha_{\bar{k}-k_i}=\alpha_{k_f-k_i}\,.$$

Thus the r.h.s. of Eq. (A7) becomes

$$e^{-iM_f\varphi_s}\int d^{(2)}b\,e^{-i(\vec{k}_f-\vec{k}_i)\cdot\vec{b}-iM_f\varphi}\langle\sigma_f e^{-iM_f\varphi}|2\alpha_{k_f-k_i}(1-\cos\Lambda_0)|\sigma_i\rangle\,. \tag{A8}$$

On the other hand, by Eqs. (A1) and (A2) we have

$$f_{fi}^{(\mathrm{MG})}(\vec{k}_f,\vec{k}_i)-(f_{fi}^{(\mathrm{MG})}(\vec{k}_f,\vec{k}_i))^\dagger = i\,\frac{\sqrt{k_f k_i}}{2\pi}\,e^{-iM_f\varphi_s}\int d^{(2)}b\,e^{-i(\vec{k}_f-\vec{k}_i)\cdot\vec{b}-iM_f\varphi}\langle\sigma_f e^{-iM_f\varphi}|2\alpha_{k_f-k_i}(1-\cos\Lambda_0)|\sigma_i\rangle\,. \tag{A9}$$

In conclusion, we have verified that the modified Glauber amplitude indeed satisfies the unitarity theorem

$$\frac{1}{2i}\left[f_{fi}^{(\mathrm{MG})}(\vec{k}_f,\vec{k}_i)-(f_{fi}^{(\mathrm{MG})}(\vec{k}_f,\vec{k}_i))^\dagger\right]=\frac{1}{4\pi}\sum_n k_n\int(f_{fn}^{(\mathrm{MG})}(\vec{k}_f,\vec{k}_n))^\dagger f_{ni}^{(\mathrm{MG})}(\vec{k}_n,\vec{k}_i)d\Omega_n\,. \tag{A10}$$

Finally, we show that the limitation imposed on the angles and relative directions of $\vec{k}_i$ and $\vec{k}_f$ is not necessary for the verification of unitarity. Making use of the connection between the Bessel function and spherical harmonics, and replacing the integral over the impact parameter b by a summation over angular momentum l [Ref. 8, Eqs. (3.17) and (3.18)], we obtain, for the modified Glauber amplitude,

$$f_{ni}^{(\mathrm{MG})}(\vec{k}_n,\vec{k}_i)=-i^{M_n+1}\left(\frac{2\pi}{k_nk_i}\right)^{1/2}\sum_l\sqrt{l+\tfrac{1}{2}}\,Y_l^{-M_n}(\Omega_{ni})\langle\sigma_ne^{-iM_n\varphi}|i\Lambda_{k_n-k_i}+\alpha_{k_n-k_i}(e^{i\Lambda_0}-i\Lambda_0-1)|\sigma_i\rangle\,. \tag{A11}$$

Similarly we have

$$(f_{fn}^{(\mathrm{MG})}(\vec{k}_f,\vec{k}_n))^\dagger=i^{M_f-M_n+1}\left(\frac{2\pi}{k_fk_n}\right)^{1/2}\sum_{l'}\sqrt{l'+\tfrac{1}{2}}\,Y_{l'}^{-(M_f-M_n)}(\Omega_{fn})\langle\sigma_fe^{-iM_f\varphi'}|-i\Lambda_{k_f-k_n}+\alpha_{k_f-k_n}(e^{-i\Lambda_0}+i\Lambda_0-1)|\sigma_ne^{-iM_n\varphi'}\rangle \tag{A12}$$

if $\vec{k}_n$ were in the z direction. However $\vec{k}_n$ is not in the z direction. We, therefore, perform a rotation of the coordinate system so that $\vec{k}_i$ will be along the new z axis. Hence

$$Y_{l'}^{-(M_f-M_n)}(\Omega_{fn})=\sum_{M'=-l'}^{l'}D_{M',M_n-M_f}^{(l')}(\varphi_{ni},\theta_{ni},0)Y_{l'}^{M'}(\Omega_{fi}) \tag{A13}$$

where $D_{MM'}^{(l)}$ is the rotation matrix. As

$$Y_l^{-M_n}(\Omega_{ni})=[(l+\tfrac{1}{2})/2\pi]^{1/2}D_{-M_n,0}^{(l)*}(\varphi_{ni},\theta_{ni},0)\,, \tag{A14}$$

we obtain

$$\frac{k_n}{\sqrt{k_fk_i}}\int(f_{fn}^{(\mathrm{MG})}(\vec{k}_f,\vec{k}_n))^\dagger f_{ni}^{(\mathrm{MG})}(\vec{k}_n\vec{k}_i)\,d\Omega_{ni}$$
$$=i^{M_f}\frac{\sqrt{2\pi}}{k_fk_i}\sum_{l,l',M'}[(l+\tfrac{1}{2})^2(l'+\tfrac{1}{2})]^{1/2}Y_{l'}^{M'}(\Omega_{fi})\int D_{M',M_n-M_f}^{(l')}D_{-M_n,0}^{(l)*}d\Omega_{ni}\langle\sigma_fe^{-iM_f\varphi'}|-i\Lambda_{k_f-k_n}+\alpha_{k_f-k_n}(e^{-i\Lambda_0}+i\Lambda_0-1)|\sigma_ne^{-iM_n\varphi'}\rangle\langle\sigma_ne^{-iM_n\varphi}|i\Lambda_{k_n-k_i}+\alpha_{k_n-k_i}(e^{i\Lambda_0}-i\Lambda_0-1)|\sigma_i\rangle\,. \tag{A15}$$

After making use of the orthogonality relation of the rotation matrix, (A15) becomes the analogous expression of (A6). The remaining part of the proof follows the same lines as above.

[1]R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin and L. G. Dunham (Interscience, New York, 1959), Vol. 1, pp. 315–414.
[2]V. Franco, Phys. Rev. Lett. 20, 709 (1968).
[3]A. Birman and S. Rosendorff, Nuovo Cimento B 63, 89 (1969).
[4]B. L. Moiseiwitsch and A. Williams, Proc. R. Soc. A 250, 337 (1959).
[5]C. J. Joachain and M. H. Mittleman, Phys. Rev. A 4, 1492 (1971).
[6]F. W. Byron and C. J. Joachain, Phys. Rev. A 8, 1267 (1973).
[7]T. T. Gien, J. Phys. B 9, 3203 (1976); Phys. Rev. A 16, 123 (1977).
[8]S. Rosendorff, Proc. R. Soc. A 353, 11 (1977).
[9]A. Birman and S. Rosendorff, J. Phys. B 9, L189 (1976).
[10]Y. K. Kim and M. Inokuti, Phys. Rev. A 3, 665 (1971).
[11]F. J. de Heer, M. R. C. McDowell, and R. W. Wagenaar, J. Phys. B 10, 1945 (1977).
[12]G. Forster and W. Williamson, Jr., Phys. Rev. A 13, 2023 (1976).
[13]R. N. Madan, Phys. Rev. A 12, 2631 (1975).
[14]J. F. Williams, J. Phys. B 8, 2191 (1975).
[15]B. K. Thomas and E. Gerjuoy, J. Math. Phys. 12, 1567 (1971).
[16]See, e.g., N. C. Barford, *Experimental Measurements: Precision, Error and Truth* (Addison-Wesley, London, 1967), p. 114.
[17]B. van Wingerden, E. Weigold, F. J. de Heer, and K. J. Nygaard, J. Phys. B 10, 1345 (1977).
[18]See Ref. 1, pp. 347 and 370.