Soliton production and solutions to perturbed Korteweg-deVries equations

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The author discusses the production of a secondary soliton when the wave equation describing the soliton undergoes a small perturbation. Discussion is limited to the Korteweg-deVries equation, where it is found that, for an appropriate sign of the perturbation, soliton production always occurs. The methods of inverse scattering are used where this production can be clearly demonstrated. Previous authors have ignored the production of secondary solitons and consequently have arrived at erroneous conclusions regarding the conservation laws. Two examples of perturbations are discussed in some detail.

This paper is a result of an attempt to understand in a simple way some of the features of soliton production. I have in mind physical systems represented by equations that are perturbations of equations that are exactly soluble by inverse scattering methods. There are several reasons why one might be interested in describing soliton reaction. One application of such an understanding is to the statistical mechanics of a system of particles. When a soliton is produced a continuum degree of freedom is changed into a discrete one, and it is important to incorporate correctly the counting of degrees of freedom. In conventional quantum-statistical mechanics this is treated by the introduction of Levinson's theorem directly into the partition function.[1] In my work here I will use a corollary to the theorem, namely that the scattering length changes sign through infinity as another bound state is created. The one-dimensional version of that result states that one piece of the scattering data goes through zero if a new bound state (soliton) is created. In this paper I do not actually look at a statistical system, but rather show the importance of soliton production and discuss the simple signature representing a new soliton, which could then be used in a partition function.

A second application is obviously just to understand in a simple way what can be expected in the way of soliton production. This has not really been addressed before other than to say that some excitation may have a soliton content. I show how for weak perturbations one can make a more quantitative statement. I first appreciated the importance of soliton production when I attempted to apply the recently developed formalisms for perturbation theory[2-5] to a numerical analysis of the perturbed Korteweg-deVries (KdV)[6] equation. I could not give a correct description if I ignored soliton production, as previous authors have done.[2,7]

The methods of Refs. 3 and 7 cannot at present treat a change in the number of discrete degrees of freedom. The formalism of Refs. 2 and 5, however, are sufficiently general, and I will use their methods.

In this paper I point out that the creation of a new soliton has a unique signature and that soliton production is very important for perturbations to single-soliton solutions of the KdV equation,[5] in that one cannot understand correctly the changes in the low-order conservation equations without accounting for soliton production. The new soliton may be important for some aspects of a problem, say the momentum, but has no effect on the energy. I shall make these statements quantitative for short times, but for long times I have only a qualitative understanding. It will become apparent that for an appropriate sign of the perturbation, a new soliton is always produced. Consequently, it is necessary to understand this production in order to understand the long-time behavior of the solution. Also, the new soliton represents the creation of long-wavelength slowly varying excitations which may cause difficulties in the multiple-time-scale methods of Ref. 2. Previous papers[2,7] have not correctly accounted for soliton production, so their predicted long-time behavior of the new excitations is not correct.

I shall consider the KdV equation with a perturbing term,

$$q_t(x,t)+6qq_x+q_{xxx}=\epsilon F(q,t)\,, \tag{1}$$

where $\epsilon \ll 1$ and

$$q(x,0)=2\eta_1^2\,\mathrm{sech}^2\eta_1(x-x_0)\,. \tag{2}$$

I shall treat the cases

$$F(q,t)=\Gamma(t)q\,, \tag{3a}$$

$$F(q,t)=q_{xx}\,, \tag{3b}$$

although the general form for F is unimportant. My work also applies to the equation

$$q_t+6\alpha(t)qq_x+\beta(t)q_{xxx}=0\,, \tag{4}$$

which has been studied by a number of authors[7] and can be transformed easily into Eq. (1) with F given by Eq. (3a). I shall demonstrate that for the correct sign of ϵ a secondary soliton is generated. My results will be obtained within the framework of inverse scattering theory,[2,5,8-10] as that formalism gives a clear meaning to soliton production. The solitons correspond to discrete degrees of freedom and the waves to continuum degrees of freedom in the associated inverse problem. The initial condition of Eq. (2) corresponds to one discrete degree of freedom excited. At a later time there will be, for the appropriate sign of ϵ, another discrete state as well as continuum states, even for infinitesimal ϵ. This is not the case for other equations (nonlinear Schrödinger, Sine-Gordon, etc.), which are soluble by the methods of Zakharov and Shabat.[9,10]

In this paper I use the perturbation scheme of Kaup and Newell,[2] as their formalism is well suited to understanding soliton production. In order to make this paper self-contained, I repeat here the necessary formulas.[2] The inverse scattering problem inovlves the Schrödinger equation

$$v_{xx}(x,t)+[\xi^2+q(x,t)]\,v(x,t)=0\,, \tag{5}$$

with the independent solutions ϕ, ψ

$$\phi\to\begin{cases} e^{-i\xi x}\,, & x\to-\infty\,, \quad (6a)\\ a(\xi,t)e^{-i\xi x}+b(\xi,t)e^{i\xi x}, & x\to+\infty\,, \quad (6b)\end{cases}$$

$$\psi\to\begin{cases} e^{i\xi x}, & x\to+\infty\,, \quad (6c)\\ a(\xi,t)e^{i\xi x}-b(-\xi,t)e^{-i\xi x}, & x\to-\infty\,. \quad (6d)\end{cases}$$

In the following I suppress the time t. The coefficients a, b satisfy

$$a(-\xi)=a^*(\xi^*)\,, \tag{7a}$$

$$b(-\xi)=b^*(\xi^*)\,, \tag{7b}$$

$$|a|^2-|b|^2=1, \quad \xi \text{ real}\,; \tag{7c}$$

$a(\xi)$ is analytic in the upper half-ξ plane, and, in the neighborhood of $\xi=0$, a and b have the expansions[11]

$$\xi a(\xi)=a_0+\xi a_1+\xi^2 a_2+\cdots\,, \tag{8a}$$

$$\xi b(\xi)=-a_0+\xi b_1+\xi^2 b_2+\cdots\,. \tag{8b}$$

This analytic behavior of $a(\xi)$ is in disagreement with the suggestion of Ref. 2 that $a(\xi)$ will develop an essential singularity at $\xi=0$. According to Ref. 11, Eq. (8) is true if[12]

$$\int_{-\infty}^{\infty}|q(x,t)|(1+|x|)dx<\infty\,.$$

We are starting with an initial solution, Eq. (2), that falls off exponentially at large x. As q is small for large x, it is sufficient to consider the linearized form of Eq. (1), which can be solved by Fourier-transform methods, and it is easily verified that the exponential behavior is valid at later times. Thus Eq. (8) remains valid.

I wish to emphasize the importance of Eqs. (7) and (8). The discrete spectrum is determined by the zeros of $a(\xi)$ in the upper half-ξ plane. If a_0 is not zero, it is clear that $a(\xi)$ and $b(\xi)$ have simple poles at $\xi=0$. For the Schrödinger equation a_0 is in general different from zero. However, if the potential is reflectionless or vanishes identically, $a_0=0$ and $a_1=\pm1$. Almost any arbitrary attractive perturbation about either a reflectionless potential or no potential will lead to a small a_0 and a zero of $a(\xi)$ at $\xi\simeq -a_0/a_1$ (ξ is small and pure imaginary). I will show below that this new zero emerges from $\xi=0$, and as time passes it moves continuously up (or down) the imaginary ξ axis. For small time the zero moves approximately linearly in t. This zero, if in the upper half-ξ plane, represents a secondary soliton that will eventually emerge from the original soliton. If the perturbation is left on for a sufficiently long time, more solitons may be produced, but I have nothing to say about them. Note that we are able to make these qualitative statements as a result of simple analyticity arguments, and no detailed calculations are needed. Of course to make quantitative statements we must do some calculations. Many of the other inverse scattering problems (e.g., Zakharov-Shabat) have a plus sign in Eq. (7c) and hence a_0 vanishes identically. In those equations small perturbations about single-soliton solutions do not necessarily lead to new solitons.

Using the formalism of Kaup and Newell,[2] one can compute a_0 in perturbation theory; this then gives directly the size of any new solitons. The initial condition of a single soliton correspond to

$$\begin{aligned} &a(\xi)=(\xi-i\eta_1)/(\xi+i\eta_1), \quad a_0=0, \quad a_1=-1\,,\\ &b(\xi)=0\,,\\ &\psi(x,\xi)=[e^{i\xi x}/(\xi+i\eta_1)][\xi+i\eta_1\tanh\eta_1(x-x_0)]\,,\\ &\phi(x,\xi)=a(\xi)\psi(x,-\xi)\,. \end{aligned} \tag{9}$$

The time development of $a(\xi,t)$ and $b(\xi,t)$ is given by the following equations[2]:

$$a_t(\xi,t)=\frac{\epsilon}{2i\xi}\int_{-\infty}^{\infty}F(q,t)\phi(x,\xi,t)\psi(x,\xi,t)dx\,, \tag{10}$$

$$\begin{aligned} b_t(\xi,t)=&\,8i\xi^3 b(\xi)\\ &-\frac{\epsilon}{2i\lambda}\int_{-\infty}^{\infty}F(q,t)\phi(x,\xi,t)\psi(x,-\xi,t)dx\,, \end{aligned} \tag{11}$$

$$q(x,t)=\frac{2}{i\pi}\int_{-\infty}^{\infty}\xi\,\frac{b(\xi,t)}{a(\xi,t)}\,\psi^2(x,\xi,t)d\xi+q_{\mathrm{sol}}\,. \tag{12}$$

We shall not need the explicit form of q_{sol}.[2] The first two of the infinite number of conservation laws for the KdV equation have a known time dependence if $F=q\Gamma(t)$. For that particular case the time dependence is given by

$$I_1(t)=\int_{-\infty}^{\infty} q(x,t)dx=I_1(0)\exp\left(\epsilon\int_0^t \Gamma(t')dt'\right) \tag{13}$$

and

$$I_2(t)=\int_{-\infty}^{\infty} q^2(x,t)dx=I_2(0)\exp\left(2\epsilon\int_0^t \Gamma(t')dt'\right). \tag{14}$$

The soliton parameter in Eq. (2) is defined by

$$a(i\eta_1,t)=0. \tag{15}$$

The functions I_1 and I_2 can be written in terms of the scattering data by

$$I_1=4\sum_k \eta_k+\frac{1}{\pi}\int_{-\infty}^{\infty}\log\left(1-\frac{|b(\xi)|^2}{|a(\xi)|^2}\right)d\xi \tag{16}$$

and

$$I_2=\tfrac{16}{3}\sum_k \eta_k^3-\frac{4}{\pi}\int_{-\infty}^{\infty}\xi^2\log\left(1-\frac{|b(\xi)|^2}{|a(\xi)|^2}\right)d\xi. \tag{17}$$

We now want to understand how the various contributions to I_1 and I_2 change as a function of time. For simplicity, we take $\Gamma=1$. Then, using the form of ϕ and ψ for single solitons,[2] we can integrate Eq. (10) to obtain

$$a_0=\tfrac{2}{3}i\eta_1\epsilon t. \tag{18}$$

For small ϵt the perturbation is dominated by small ξ, and from Eqs. (8a) and (18) we see that a second zero of $a(\xi)$ appears at $\xi=i\eta_2$,

$$\eta_2=\tfrac{2}{3}\eta_1\epsilon t. \tag{19}$$

We now are in a position to examine the quantities I_1 and I_2. It is clear from Eq. (16) that the continuum contribution to I_1 is always negative, which is contrary to the estimate of Kaup and Newell. They used Eqs. (12) and (13) and obtained a result proportional to the sign of ϵ. The error comes from the small-ϵ behavior of q as determined by Eq. (12). In fact terms with amplitude ϵ^2, can, when integrated, give a contribution of order ϵ to I_1. For small ϵt, by using just a_0 as given by Eq. (18), and with $a_1=-1$, it is possible to evaluate the continuum and discrete contributions to I_1.

Clearly from Eq. (17) the continuum is at most $O(\epsilon^2t^2)$ and η_2^3 is $O(\epsilon^3t^3)$; so, to $O(\epsilon^2t^2)$, I_2 is satisfied entirely by η_1, which gives immediately

$$\eta_1(t)=\eta_1(0)\exp\left(+\tfrac{2}{3}\epsilon\int_0^t \Gamma(t')dt'\right). \tag{20}$$

This is in agreement with Refs. 2 and 7. The problem has been to understand how Eqs. (13) and (16) are satisfied.[2,7]

From Eqs. (16) and (18)–(20) it is easy to find the following contributions to δI_1 (the superscript "con" denotes the continuum contribution):

$$\delta I_1^{\text{con}}=-\tfrac{4}{3}\eta_1 t|\epsilon|, \tag{21}$$

$$\delta I_1(\eta_1)=\tfrac{8}{3}\eta_1\epsilon t, \tag{22}$$

$$\delta I_1(\eta_2)=\begin{cases}\tfrac{8}{3}\eta_1\epsilon t, & \epsilon>0,\\ 0, & \epsilon<0\end{cases}. \tag{23}$$

The sum of these contributions should give Eq. (13) to $O(\epsilon)$ if our work is correct. From Eqs. (13) and (20) we have

$$\delta I_1^{\text{tot}}=4\eta_1\epsilon t. \tag{24}$$

If we add Eqs. (21)–(23) we independently obtain the same total contribution to δI_1. Thus it is clear that the secondary soliton has twice as big a contribution to I_1 as the continuum. For later times more terms in the expansions of Eq. (8) are needed and the integrals must be done numerically. The continuum appears to gain on the secondary soliton for later times. As previous estimates[2,7] ignored the new soliton their estimates cannot be correct.

If $a_0(t)$ is calculated from Eq. (10) or (18), we see that if $\Gamma>0$, it will increase indefinitely to $O(\epsilon)$. This increase must, of course, stop, when higher-order corrections are kept. Without doing the second-order calculation, it is possible to estimate the value of a_0 obtained when growth stops. Assuming that the dominant contributions are from $\xi\simeq 0$, we will see a_0 in the combination a_0/ξ on the right-hand side of (10). This presumably leads to a term of the form

$$\frac{\epsilon}{2i}a_0\int_{-\infty}^{\infty}F(q,t)\left(\psi(x,\xi,t)\frac{d\phi(x,\xi,t)}{d\xi}\right)_{\xi=0}dx. \tag{25}$$

The integral is easily done, and, if we assume that the coefficient of the term is one, we obtain a correction to Eq. (10)

$$\frac{da_0}{dt}\simeq\tfrac{2}{3}i\eta_1\epsilon(1+ia_0x_0), \tag{26}$$

where x_0 is the distance the original soliton has traveled during the interaction. This estimate of the size of the secondary soliton is in qualitative agreement with the numerical integrations of Eq. (2) by Wingate.[13] Since $x_0\propto t$ and $a_0\propto t$, we expect that the second-order corrections come in at least by

$$\epsilon t^2\sim 1.$$

Once the secondary soliton separates from the original soliton and any continuum radiation, its size $\eta_2(t)$ will be given by an equation similar to Eq. (20).

A final point which needs some further study is the extension of these results to times longer than $O(1)$. The implicit assumption[2] is that the multiple-time-scale or averaging methods can be used. This assumes the unperturbed motions are "fast." However, if $\xi=0$ dominates, this is no longer the case as the phases are proportional to $\exp(8i\xi^3t)$ and $a(\xi)$ can change more quickly than that. Note that this is true even when no solitons are produced. We expect some small long-wavelength excitations due to the dominance of $\xi=0$.

We now turn to a different perturbation to the KdV equation that has been discussed recently[15,16]

$$q_t+6qq_x+q_{xxx}=\epsilon q_{xx}\ . \tag{27}$$

This equation is referred to as the KdV-Burgers equation. The perturbation provides damping for $\epsilon>0$. If the initial condition is a soliton as given by Eq. (9), the dissipation slows down the initial soliton and generates a new soliton. For the opposite sign of ϵ, no new soliton is produced. The analysis proceeds as before except that now only I_1 is known exactly. It is easy to see directly from Eq. (27) that

$$\frac{dI_1}{dt}(t)=0\ . \tag{28}$$

From Eq. (16) we see that the net soliton component must increase. We now proceed to demonstrate how that happens. Again all the corrections to I_2 except the change in the original soliton are $O(\epsilon^2)$. We find

$$\frac{dI_2}{dt}=-2\int_{-\infty}^{\infty}\left(\frac{dq}{dx}\right)^2dx\simeq-\tfrac{128}{15}\,\epsilon\eta^3\ . \tag{29}$$

From Eq. (17) we have

$$\frac{d\eta_1}{dt}=-\tfrac{8}{15}\epsilon\eta_1\,,\quad \eta_1\simeq-\tfrac{8}{15}\epsilon\eta_1t+\eta_1(0)\ . \tag{30}$$

This gives the rate of decay of the soliton. Clearly to satisfy Eq. (28) a new soliton must be generated. Substituting the appropriate F in Eq. (10), we easily find a_0:

$$a_0\simeq\tfrac{16}{15}i\epsilon\eta_1t\ , \tag{31}$$

and then

$$\eta_2\simeq\tfrac{16}{15}\epsilon\eta_1t\ . \tag{32}$$

From Eq. (16), we calculate the continuum contribution to I_1,

$$\delta I_1^{\text{con}}=-\tfrac{32}{15}\epsilon\eta_1t\ . \tag{33}$$

Combining the contributions and using the relationship between I_1 and given by Eq. (16), we get, as expected

$$\delta I_1(\eta_1)+\delta I_2(\eta_2)+\delta I_1^{\text{con}}=0\ . \tag{34}$$

It is clear from the above discussion that any attempt to discuss the production of a shelf behind the soliton,[15] should take into account the new soliton component. In particular, Eqs. (13) and (16) show that the secondary soliton creates a positive shelf, while the continuum creates a predominantly negative shelf. Note the curious feature that for the perturbation of Eq. (3a), if the original soliton increased in size a new soliton was generated, whereas for Eq. (3b) the opposite effect is noted.

In this note I have explained some qualitative predictions of soliton production in the KdV equation. I find that up to a sign, if a single soliton is perturbed, it will generate a secondary soliton which is at least as important a contribution to I_1 as the continuum. Other equations, nonlinear Schrödinger, Sine-Gordon, etc., do not have this feature of instant soliton production, but the dominance of the perturbation by small ξ should still be the main feature when there is soliton production in those equations. I am currently making detailed quantitative calculations and comparisons of perturbation results with exact results over long time scales. I will also present some numerical examples of soliton production.[14]

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