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Theory of free-electron lasers

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A general analysis is presented of free-electron lasers in which a static periodic magnetic pump field is scattered from a relativistic electron beam. The steady-state formulation of the problem is fully relativistic and contains beam thermal effects. Growth rates associated with the radiation field, efficiencies, and saturated-field amplitudes are derived for various modes of operation. Effects of space charge on the scattering process are included and shown to play a dominant role in certain situations. Scaling laws for the growth rates and efficiencies at a fixed radiation frequency as a function of the magnetic-pump amplitude are obtained. The shear in beam axial velocity due to self-fields is discussed, and various methods of reducing it are suggested. Finally, a detailed illustration of a far-infrared ($\lambda = 2\ \mu$m) two-stage free-electron laser using a 3-MeV electron beam and a 2-cm-wavelength magnetic pump field is presented.

I. INTRODUCTION

The class of free-electron lasers (FEL's) in which a pump field is scattered from a relativistic electron beam is of great interest as a potential high-power, tunable source of coherent radiation, particularly in the infrared, visible, and ultraviolet spectral regions. The concept involves the stimulated backscatter of a pump wave from a relativistic electron beam. The pump wave may be either an electromagnetic wave or a static periodic electric or magnetic field. For a static periodic pump wave, the backscattered radiation frequency from a relativistic electron beam is

$$\omega \approx (1+\beta_z)\gamma_z^2 \beta_z c k_0 \approx 2\gamma_z^2 c k_0 ,$$

where $c\beta_z = v_z$ is the axial-drifting beam velocity, $\gamma_z = (1-\beta_z^2)^{-1/2}$, $k_0 = 2\pi/l$, and l is the period of the pump wave. The modulated source currents for the coherent scattered radiation are generated by axial bunching of the electron beam at the radiation wavelength through coupling of the scattered waves and the pump field. The mechanism responsible for this axial bunching is the ponderomotive force acting on the electrons in the combined fields of the pump and radiation waves.

Analysis and design considerations pertaining to the single-particle scattering process have been carried out, both classically[1-14] and quantum mechanically.[15-17] When the electron beam is sufficiently intense, collective effects become important and indeed may dominate the process. Linear analysis of the FEL has been performed, including collective effects,[4,8,12,14,18-24] and scattering efficiencies have been derived for various FEL scattering regimes.[3,5,12,14,22]

Free-electron-laser experiments with pulsed intense relativistic electron beams have been conducted at a number of laboratories.[25-30] Submillimeter radiation at MW power levels were generated with electron beams of energies up to a few MeV and currents in the multi-kA range. In these experiments collective effects play a dominant role in the scattering mechanism because of the high beam currents.

In another class of FEL experiments at Stanford University,[31,32] relatively low-current, high-energy beams were employed ($I_0 \sim 2$A, 24 MeV $< E_0 <$ 43 MeV). In these tenuous-beam experiments collective effects are negligible, and single-particle scattering physics applies. Operating in the oscillator mode,[32] peak powers of ~7 kW at 3.4 μ were generated with an efficiency of ~0.01%.

In this paper we present a general analysis of the FEL process utilizing a right-handed circularly polarized, spatially periodic magnetic pump. A schematic of the FEL configuration is shown in Fig. 1. The analysis is fully relativistic and performed explicitly in the laboratory frame. Our formulation shows that, depending on the beam and pump parameters, several distinct interaction processes can be distinguished. Our results are applicable to both the tenuous- and intense-beam-type experiments. Growth rates (or gains) together with saturation efficiencies are derived for the various FEL regimes. A condition for the neglect of collective effects for the low-gain FEL process is derived. Scaling laws for the growth rates and efficiencies at a fixed output frequency as a function of the pump amplitude are given. The detrimental effect of axial-velocity shear on the beam due to self-fields is discussed, and various methods of reducing this shear are suggested. In addition, an illustration of a far-

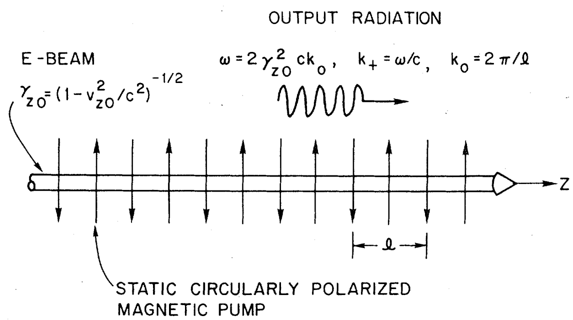

FIG. 1. Schematic of the free-electron-laser model. The adiabatic buildup of the pump field is not shown, and occurs to the left of the figures where the unmodulated beam enters.

infrared two-stage FEL using a 3-MeV electron beam is presented. Here the output radiation wavelength is decreased approximately by the factor $8\gamma_z^4$ compared to the pump wavelength, instead of the factor $2\gamma_z^2$ for a single-stage FEL.

II. DERIVATION OF GENERAL FEL DISPERSION RELATION

The pump is chosen to be a right-handed circularly polarized magnetic field given for $z \geq 0$ by

$$\vec{B}_0 = B_0[\hat{e}_x \cos(k_0 z) + \hat{e}_y \sin(k_0 z)], \tag{1}$$

where B_0 is constant and $k_0 = 2\pi/l$ (see Fig. 1). The representation of the pump field in Eq. (1) is a good approximation near the axis of an appropriate coil winding. The vector potential associated with $\vec{B}_0$ is $\vec{A}_0 = -\vec{B}_0/k_0$. For particles in the field given by Eq. (1) the canonical momenta in the x and y directions, as well as the total momentum, are constants of the motion and are given, respectively, by

$$\alpha(p_x, z) = p_x - (|e|/c)A_{0x}(z), \tag{2a}$$

$$\beta(p_y, z) = p_y - (|e|/c)A_{0y}(z), \tag{2b}$$

$$u(\vec{p}) = |\vec{p}|, \tag{2c}$$

where $\vec{p}$ is the momentum.

We assume that the interaction between the relativistic electron beam and the pump field has reached the temporal steady state so that the radiation fields are proportional to $\exp(-i\omega t)$, where ω is the frequency of radiation. The radiation and space-charge fields are given by

$$\vec{E}(z,t) = \frac{1}{2}\left(-\frac{\partial\tilde{\phi}(z)}{\partial z}\hat{e}_z + \frac{i\omega}{c}\tilde{A}(z)(\hat{e}_x + i\hat{e}_y)\right)e^{-i\omega t} + \text{c.c.}, \tag{3a}$$

$$\vec{B}(z,t) = -\frac{i}{2}\frac{\partial\tilde{A}(z)}{\partial z}(\hat{e}_x + i\hat{e}_y)e^{-i\omega t} + \text{c.c.}, \tag{3b}$$

where the associated potentials are

$$\phi(z,t) = \tfrac{1}{2}\tilde{\phi}(z)\exp(-i\omega t) + \text{c.c.}$$

and

$$\vec{A}(z,t) = \tfrac{1}{2}\tilde{A}(z)(\hat{e}_x + i\hat{e}_y)\exp(-i\omega t) + \text{c.c.}$$

Using the relativistic Vlasov equation we expand the electron distribution function to first order in the scattered fields $\vec{E}$ and $\vec{B}$ about its equilibrium. It proves very convenient first to transform the independent variables $(\vec{p}, z, t)$ of the distribution function to the new independent variables (α, β, u, z, t). The electron distribution is then written as

$$g(\alpha,\beta,u,z,t) = g^{(0)}(\alpha,\beta,u) + g^{(1)}(\alpha,\beta,u,z,t), \tag{4}$$

where $g^{(0)}(\alpha,\beta,u)$ is the equilibrium distribution and is an arbitrary function of the constants of the motion, and $g^{(1)}(\alpha,\beta,u,z,t)$ is the perturbed part of the distribution which is proportional to either ϕ or $\vec{A}$. It is straightforward to show that the perturbed part of the Vlasov equation for $g^{(1)}$ is

$$\frac{\partial g^{(1)}}{\partial t} + \frac{p_z}{\gamma m_0}\frac{\partial g^{(1)}}{\partial z} = \tilde{H}e^{-i\omega t} + \text{c.c.}, \tag{5}$$

where

$$\tilde{H} = \tilde{H}(\alpha,\beta,u,z) = \frac{|e|}{2c}\left\{\left[\left(i\omega - v_z\frac{\partial}{\partial z}\right)\tilde{A}\right]\left(\frac{\partial}{\partial\alpha} + i\frac{\partial}{\partial\beta}\right) + \left(i\omega p_-\tilde{A} - cp_z\frac{\partial\tilde{\phi}}{\partial z}\right)\frac{1}{u}\frac{\partial}{\partial u}\right\}g^{(0)}, \tag{6}$$

and the dependent variables are

$$v_z = v_z(\alpha,\beta,u,z) = p_z(\alpha,\beta,u,z)/\gamma(u)m_0, \tag{7a}$$

$$p_- = p_-(\alpha,\beta,z) = p_x + ip_y = \alpha + i\beta - (|e|B_0/ck_0)e^{ik_0 z}, \tag{7b}$$

$$p_z = p_z(\alpha,\beta,u,z) = (u^2 - p_x^2 - p_y^2)^{1/2}, \tag{7c}$$

$$p_x = p_x(\alpha, z) = \alpha + (|e|/c)A_{0x}(z), \tag{7d}$$

$$p_y = p_y(\beta, z) = \beta + (|e|/c)A_{0y}(z), \tag{7e}$$

$$\gamma = \gamma(u) = (1 + u^2/m_0^2c^2)^{1/2}. \tag{7f}$$

The general solution of Eq. (5) is

$$g^{(1)} = \tilde{g}^{(1)}e^{-i\omega t} + \text{c.c.},$$

where

$$\tilde{g}^{(1)} = \tilde{g}^{(1)}(\alpha,\beta,u,z) = \int_{-\infty}^{z} dz' M(\alpha,\beta,u,z,z')\tilde{H}(\alpha,\beta,u,z'), \tag{8}$$

$$M(\alpha,\beta,u,z,z')=\frac{e^{i\omega\tau(\alpha,\beta,u,z,z')}}{v_z(\alpha,\beta,u,z'')},$$

and

$$\tau(\alpha,\beta,u,z,z')=\int_{z'}^{z}\frac{dz''}{v_z(\alpha,\beta,u,z'')}.$$

The beam has been taken to be unperturbed at $z=-\infty$, i.e., $\tilde{g}^{(1)}(\alpha,\beta,u,-\infty)=0$.

Rearranging Eq. (8), the Fourier transform of the perturbed part of the distribution function becomes

$$\tilde{g}^{(1)}(\alpha,\beta,u,z)=\left[\tilde{G}_+(\alpha,\beta,u,z)\left(\frac{\partial}{\partial\alpha}+i\frac{\partial}{\partial\beta}\right)+\tilde{G}_z(\alpha,\beta,u,z)\frac{\partial}{\partial u}\right]g^{(0)}(\alpha,\beta,u),\tag{9}$$

where

$$\tilde{G}_+=\frac{|e|}{2c}\int_{-\infty}^{z}dz'M(\alpha,\beta,u,z,z')\times\left(i\omega-v_z(\alpha,\beta,u,z')\frac{\partial}{\partial z'}\right)\tilde{A}(z')=-\frac{|e|}{2c}\tilde{A}(z),\tag{10a}$$

$$\tilde{G}_z=\frac{|e|}{2c}\frac{1}{u}\int_{-\infty}^{z}dz'M(\alpha,\beta,u,z,z')\times\left(i\omega p_-(\alpha,\beta,z')\tilde{A}(z')-cp_z(\alpha,\beta,u,z')\frac{\partial\tilde{\phi}(z')}{\partial z}\right).\tag{10b}$$

In Eq. (10a) we have integrated by parts, using the fact that the radiation field vanishes at $z=-\infty$. The expression for $\tilde{g}^{(1)}$ in Eq. (9) determines the first-order perturbation of the electron distribution function due to scattered fields with arbitrary axial spatial dependence, is correct to all orders in the pump field amplitude, and contains thermal effects in g_0.

The perturbed current density which drives the scattered fields is given by

$$J(z,t)=[\tilde{J}_+(z)\hat{e}_++\tilde{J}_z(z)\hat{e}_z]e^{-i\omega t}+\text{c.c.},\tag{11}$$

where

$$\left.\begin{matrix}J_+(z)\\J_z(z)\end{matrix}\right\}=\frac{-|e|}{m_0}\int_0^\infty du\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}d\beta\,\frac{u\tilde{g}^{(1)}(\alpha,\beta,u,z)}{\gamma(u)p_z(\alpha,\beta,u,z)}\left\{\begin{matrix}p_+(\alpha,\beta,z)\\p_z(\alpha,\beta,u,z)\end{matrix}\right.,\tag{12}$$

with

$$p_+(\alpha,\beta,z)=p_x-ip_y=\alpha-i\beta-(|e|B_0|ck_0)\exp(-ik_0z),$$

and

$$\hat{e}_+\equiv\tfrac{1}{2}(\hat{e}_x+i\hat{e}_y).$$

To evaluate the current density in Eq. (12), we take

$$g^{(0)}(\alpha,\beta,u)=n_0\delta(\alpha)\delta(\beta)g_0(u),\tag{13}$$

where n_0 is the unperturbed beam density, assumed to be uniform in space, and $g_0(u)$ is arbitrary but subject to the normalization condition $\int_0^\infty dug_0(u)u/u_z=1$. The delta functions for α and β arise from assuming that the equilibrium transverse momentum is due solely to the pump field, i.e., that transverse thermal effects can be neglected. Using the distribution function (13), we find, after some lengthy algebra, that the Fourier coefficients of the current densities given in Eq. (12) are

$$\tilde{J}_+(z)=-\frac{\omega_b^2}{4\pi}\int_0^\infty du\left[\frac{u}{\gamma u_zc}(1-\tfrac{1}{2}\beta_\perp^2)\tilde{A}(z)+\tfrac{1}{2}m_0\beta_\perp e^{i\psi(z)}\int_0^z dz'\left(\frac{\partial\tilde{\phi}(z')}{\partial z'}e^{-i\omega z'/v_z}+\frac{i\omega\beta_\perp}{c}\tilde{A}(z')e^{-i\psi(z')}\right)\frac{\partial}{\partial u}\right]g_0,\tag{14a}$$

$$\tilde{J}_z(z)=\frac{\omega_b^2}{8\pi}m_0\int_0^\infty du\,e^{i\omega z/v_z}\int_0^z dz'\left(\frac{\partial\tilde{\phi}(z')}{\partial z'}e^{-i\omega z'/v_z}+\frac{i\omega\beta_\perp}{c}A(z')e^{-i\psi(z')}\right)\frac{\partial g_0}{\partial u},\tag{14b}$$

where

$$\omega_b = (4\pi n_0 |e|^2/m_0)^{1/2} ,$$
$$\beta_\perp = \beta_\perp(u) = \Omega_0/\gamma(u) v_z(u) k_0 ,$$
$$\Omega_0 = |e| B_0/m_0 c ,$$
$$\psi(z) = (\omega/v_z(u) - k_0) z ,$$
$$u_z = u_z(u) = p_z(\alpha = 0, \beta = 0, u, z) = (u^2 - m_0^2 \Omega_0^2/k_0^2)^{1/2} ,$$

and

$$v_z = u_z/\gamma(u) m_0 .$$

The limits of integration over z' are from 0 to z and not from $-\infty$ to z, because the amplitude of the various fields, i.e., A_0, $\tilde{\phi}$, and $\tilde{A}$, are assumed to build up from zero at $z = -\infty$ to their initial values at $z = 0$ in a distance which is small compared to $(k_+ + k_0 - \omega/v_z)^{-1}$, where k_+ is the wave number associated with $\tilde{A}(z)$. The limits of integration over z' can therefore be changed from $(-\infty, z)$ to $(0, z)$ without loss of accuracy. Because the characteristic length $(k + k_0 - \omega/v_z)^{-1}$ is much longer than the wavelength of the pump field, this situation is necessarily satisfied in any experimental configuration. The driving current density in Eqs. (14) contain: (i) the ponderomotive potential manifested in the term $\beta_\perp \tilde{A}(z)$; (ii) collective effects from the scalar potential; (iii) arbitrary axial variation of the excited fields ϕ and $\vec{A}$; (iv) ballistic terms propagating from the boundary at $z = 0$ associated with the lower limit on the z' integral; and (v) arbitrary thermal nature of the beam manifested in $g_0(u)$.

The analysis is closed by taking the perturbed current density of Eqs. (14) to be the source current in the wave equations for $\tilde{A}$ and ϕ:

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\right) \tilde{A}(z) = \frac{-4\pi}{c} \tilde{J}_+(z) , \tag{15a}$$

$$\frac{\partial \tilde{\phi}(z)}{\partial z} = \frac{4\pi i}{\omega} \tilde{J}_z(z) . \tag{15b}$$

A number of different scattering regimes can be distinguished using the general form for the driving currents expressed in Eqs. (14). We shall discuss in detail those regimes which appear to be important for the development of efficient, high-power FEL.

III. LOW-GAIN LIMIT

The first case we consider is the low-gain or short-cavity regime, where collective effects do not play a dominant role and the electromagnetic wave is nearly of constant amplitude. By low-gain limit we mean that the total integrated gain of the radiation field is much less than unity. This limit corresponds to the parameter regime of the experiments carried out at Stanford University[31,32] with highly relativistic (≤ 48 MeV), low-current (≤ 2.4 A) beams. Neglecting collective effects implies that $\tilde{\phi}(z) \ll \tilde{A}(z) \langle \beta_\perp \rangle$; the condition on the beam density for this inequality to be satisfied is given later. Taking the electromagnetic field to be of the form

$$\tilde{A}(z) = A(0) \exp\left(i \int_0^z k_+(z') dz'\right) , \tag{16}$$

where $|\mathrm{Im}(k_+)| \ll \mathrm{Re}(k_+)$ and $\exp\int_0^z dz' \mathrm{Im}[k_+(z')] \approx 1$. With this representation, together with (14a) and (15a), the dispersion relation takes the form

$$k_{0+}^2 - \omega^2/c^2 + 2k_{0+}\delta k(z) = \frac{-\omega_b^2}{c^2} \int_0^\infty du \left(\frac{u}{\gamma u_z}(1 + \tfrac{1}{2}\beta_\perp^2) - \frac{m_0}{2} \beta_\perp^2 \omega v_z \frac{1 - \exp\{i[\omega/v_z - (k_{0+} + k_0)]z\}}{\omega - v_z(k_{0+} + k_0)} \frac{\partial}{\partial u}\right) g_0(u) , \tag{17}$$

where $k_+(z) = k_{0+} + \delta k(z)$, $|\delta k| \ll k_{0+}$, and $k_{0+} \simeq 2\gamma_z^2 c k_0$ is real and constant. Solving for the imaginary part of $\delta k(z)$ we obtain

$$\mathrm{Im}[\delta k(z)] = -\frac{\omega_b^2/c^2}{2k_{0+}} \int_0^\infty du \left(\frac{m_0}{2} \beta_\perp^2 \omega \frac{\sin[\omega/v_z - (k_{0+} + k_0)]z}{\omega/v_z - (k_0 + k_0)}\right) \frac{\partial g_0}{\partial u} . \tag{18}$$

The total integrated gain in the wave amplitude over the interaction region of length L is defined as $G_L = -\int_0^L \mathrm{Im}[\delta k(z')] dz'$. It is straightforward to show that if the thermal energy spread of the beam, E_{th}, is such that

$$E_{\mathrm{th}}/E_0 \ll \gamma_{z0}^2 \lambda/L , \tag{19}$$

the beam can be considered to be monoenergetic in Eq. (18). In the above inequality $E_0 = (\gamma_0 - 1) m_0 c^2 \approx \gamma_0 m_0 c^2$ is the total kinetic energy of each beam particle, $\gamma_{z0} = (1 - v_{z0}^2/c^2)^{-1/2}$ and $\lambda = 2\pi/k_{0+}$ is the wavelength of the radiation field. For future use we note that $E_{\mathrm{th}} = \gamma_0 \gamma_{z0}^2 m_0 v_{z0} V_{\mathrm{th}}$, where V_{th} is the beam thermal velocity. Assuming Eq. (19) to be satisfied we can use the distribution function $g_0(u) = (u_z/u)\delta(u - u_0)$ and find that

$$G_L=\frac{\omega_b^2/c^2}{8k_{0+}}m_0\omega L^2 \times\int_0^\infty du\,\beta_\perp^2\left(\frac{\sin(k_{0+}+k_0-\omega/v_z)L/2}{(k_{0+}+k_0-\omega/v_z)L/2}\right)^2\frac{\partial g_0}{\partial u} \approx\frac{\xi^2}{8}\beta_{0\perp}^2(k_0L)^3\frac{\partial}{\partial\theta_0}\left(\frac{\sin\theta_0}{\theta_0}\right)^2\ll 1\,, \tag{20}$$

where $\xi=\omega_b/(\sqrt{\gamma_0}ck_0)$, $\gamma_0=\gamma(u_0)$, $\gamma_{z0}=\gamma_z(u_0)$, $\beta_{0\perp}=\beta_\perp(u_0)$, $\theta_0=(\omega/v_{z0}-k_{0+}-k_0)L/2$, and u_0 is the magnitude of the total particle momentum. The function $\partial(\sin\theta_0/\theta_0)^2/\partial\theta_0$ has a maximum value of 0.54 when $\theta_0=-1.3$, hence, the maximum total gain is

$$G_{L,\max}\approx(\xi/4)^2\beta_{0\perp}^2(k_0L)^3 \tag{21}$$

and is much less than unity. Using Eq. (15b) in conjunction with Eq. (14b) we find that the condition for neglecting collective effects, i.e., $\tilde\phi(z)\ll\tilde A(z)\langle\beta_\perp\rangle$, can be written as

$$\frac{\omega_b^2}{2\omega}m_0\left|\int_0^\infty du\,\frac{1-\exp\{i[\omega/v_z-(k_{0+}+k_0)]z\}}{\omega/v_z-(k_{0+}+k_0)}\frac{\partial g_0}{\partial u}\right|\ll 1\,,$$

which reduces to

$$\left(\frac{\xi L k_0}{2\gamma_{z0}}\right)^2\ll 1\,. \tag{22}$$

The L^2 dependence in Eq. (22) is due to the dependence of the perturbed density on length. The density modulations on the beam can be shown to increase as z^2 in our present limit.

It will be necessary to obtain the difference between the phase velocity of the ponderomotive wave and the axial electron velocity when deriving the saturation efficiency in Sec. V. From the definition of θ_0, this velocity difference is simply

$$v_{\rm ph}-v_{z0}=-\Delta v=\frac{\omega}{k_{0+}+k_0}-v_{z0}=\frac{\theta_0 c}{\gamma_{z0}^2Lk_0}\,, \tag{23}$$

where $k_{0+}\approx 2\gamma_{z0}^2k_0\gg k_0$ and the value of θ_0 extends from 0 to ≈ -3 for the domain of maximum positive gain.

IV. HIGH-GAIN LIMIT

In contrast to the first case we now consider the long-cavity limit, where the excited-field amplitude spatially exponentiates several times within the interaction region. Under these conditions the terms containing the boundary conditions at $z=0$ can be neglected, i.e., $\tilde\phi(0)\ll\tilde\phi(z)$ and $\tilde A(0)\ll\tilde A(z)$. We assume that conditions appropriate to a cold beam are satisfied, hence $g_0(u)=(u_z/u)\delta(u-u_0)$; we shall return to this point later. The potentials can be represented by

$$\tilde\phi(z)=\tilde\phi(0)e^{ikz}\,, \tag{24a}$$

$$\tilde A(z)=\tilde A(0)e^{ik_+z}\,, \tag{24b}$$

where $k=k_++k_0$ and k_+ is complex. Substituting the potentials represented by Eqs. (24) into Eqs. (14), in conjunction with the cold-beam assumption, and making use of the wave equations (15) we obtain the following dispersion relation for ω and k:

$$D(\omega,k-k_0)[(\omega-v_{z0}k)^2-\omega_b^2/\gamma_{z0}^2\gamma_0] =-\frac{\omega_b^2}{2\gamma_0^3}\frac{\Omega_0}{ck_0}D(\omega,k)\,, \tag{25}$$

where

$$D(\omega,k)=\omega^2-c^2k^2-\omega_b^2/\gamma_0\,,$$

$$\gamma_0=\gamma(u_0)\,,\quad v_{z0}=v_z(u_0)\,,\quad \gamma_{z0}=\gamma_z(u_0)\,.$$

The electromagnetic wave approximately satisfies the dispersion relation $D(\omega,k_+=k-k_0)\approx 0$; hence, we can replace $D(\omega,k)$ on the right-hand side of Eq. (25) by $-2kk_0c^2$. Also, since $k_+\approx\omega/c$, we approximate $D(\omega,k_+)$ by $-2c^2k_+[k_+-(\omega^2-\omega_b^2/\gamma_0)^{1/2}/c]$. The dispersion relation can now be put into the simple form

$$[k-(K+k_0)][k-(\omega/v_{z0}+\kappa)][k-(\omega/v_{z0}-\kappa)] \approx-\tfrac{1}{2}\alpha^2k_0\,, \tag{26}$$

where

$$K=(\omega^2-\omega_b^2/\gamma_0)^{1/2}/c\,,$$

$$\kappa=\omega_b/v_{z0}\gamma_{z0}\gamma_0^{1/2}\,,$$

$$\alpha^2=(\Omega_0/ck_0)^2(\omega_b^2/\gamma_0^3v_{z0}^2)=(\xi\beta_{0\perp}k_0)^2\,,$$

$$\beta_{0\perp}=\Omega_0/\gamma_0v_{z0}k_0\,,$$

and

$$\xi=\omega_b/\sqrt{\gamma_0}ck_0\,.$$

Further simplification of Eq. (26) is obtained by setting

$$k=\omega/v_{z0}+\kappa+\delta k\,, \tag{27}$$

where δk is in general complex and $|\delta k|\ll k$. Since $v_{z0}\approx c$ and $\omega\gg\omega_b/\sqrt{\gamma_0}$, we find that $K\approx\omega/c-\xi^2k_0/(4\gamma_{z0}^2)$, $\kappa\approx\xi k_0/\gamma_{z0}$, so that Eq. (26) reduces to

$$\delta k(\delta k+2\xi k_0/\gamma_{z0})(\delta k-\Delta k)=-\alpha^2k_0/2\,, \tag{28}$$

where $\Delta k=k_0-\omega/2c\gamma_{z0}^2$ is a mismatch parameter. At this point it is convenient to evaluate the difference between the phase velocity of the longitudinal wave and the initial axial beam velocity in the high-gain limit. This velocity difference is given by

$$v_{\rm ph}-v_{z0}\equiv-\Delta v=\omega/{\rm Re}(k)-v_{z0}=\frac{-[\kappa+{\rm Re}(\delta k)]c}{2\gamma_{z0}^2k_0}\,, \tag{29}$$

where we have used the expression for k in Eq. (27). The expression for $\Delta\nu$ in Eq. (29) will be used later to obtain an estimate for the saturation efficiency and maximum radiation field. We now discern two important limits of the dispersion relation (28).

A. Weak-pump limit

For a pump magnetic field amplitude such that $\xi_{0\perp} \ll \beta_{\rm crit} \equiv 4(\xi/\gamma_{z0}^3)^{1/2}$ the space-charge potential dominates the ponderomotive potential and collective effects are important.[14] That is, in this case the electron susceptibility

$$\chi = -(\omega - \nu_{z0}k)^{-2}(\omega_b^2/\gamma_{z0}^2\gamma_0) \approx -1$$

and the electrostatic wave is nearly an eigenmode of the system. This regime of scattering corresponds to setting $|\delta k| \ll 2\xi k_0/\gamma_{z0}$ in the dispersion relation (28), which becomes

$$\delta k(\delta k - \Delta k) = -\alpha^2\gamma_{z0}/4\xi\ , \tag{30}$$

with the growing root given by

$$\delta k = \tfrac{1}{2}\Delta k \pm \tfrac{1}{2}i[\alpha^2\gamma_{z0}/\xi - (\Delta k)^2]^{1/2}\,. \tag{31}$$

The condition for instability is clearly $\alpha^2\gamma_{z0}/\xi > (\Delta k)^2$ and the maximum spatial growth rate occurs when $\Delta k = 0$ and is

$$\Gamma_{\max} \equiv -{\rm Im}(\delta k)_{\max} = \tfrac{1}{2}\beta_{0\perp}(\xi\gamma_{z0})^{1/2}k_0\,. \tag{32}$$

Using Eq. (32) we see that the condition $|\delta k| \ll 2\xi k_0/\gamma_{z0}$ is equivalent to the weak-pump condition, i.e., $\beta_{0\perp} \ll \beta_{\rm crit}$. In this FEL regime we find

$$\nu_{\rm ph} - \nu_{z0} = -\Delta\nu = [(\xi k_0/\gamma_{z0} + \Delta k/2)/2\gamma_{z0}^2 k_0]\,, \tag{33}$$

where Eq. (31) has been used for ${\rm Re}(\delta k)$ and $\Delta k \leqslant (\xi\gamma_{z0})^{1/2}\beta_{0\perp}k_0$.

B. Strong-pump limit

In this regime the pump magnetic field amplitude is sufficiently strong to satisfy the inequality $\beta_{0\perp} \gg \beta_{\rm crit} \equiv 4(\xi/\gamma_{z0}^3)^{1/2}$. The ponderomotive potential, which is proportional to the pump amplitude, completely dominates the space-charge potential in the strong-pump regime[14] and $|\chi| \ll 1$. This is a single-particle scattering regime where collective effects are negligible. For $\beta_{0\perp} \gg \beta_{\rm crit}$ we neglect $2\xi k_0/\gamma_{z0}$ compared with δk in Eq. (28) with the dispersion relation becomes

$$(\delta k)^2(\delta k - \Delta k) = -\tfrac{1}{2}\alpha^2 k_0\,. \tag{34}$$

The maximum spatial linear growth rate occurs for exact frequency matching, i.e., $\Delta k = 0$, and is given by

$$\Gamma_{\max} \equiv -{\rm Im}(\delta k)_{\max} = (\sqrt{3}/2^{4/3})(\xi\beta_{0\perp})^{2/3}k_0\,, \tag{35}$$

while at this frequency ${\rm Re}(\delta k) = (\xi\beta_{0\perp})^{2/3}k_0/2^{4/3}$. The real part of δk, which is a function of Δk, has the maximum value ${\rm Re}(\delta k)_{\max} = (\xi\beta_{0\perp})^{2/3}k_0$ when $\Delta k = \frac{3}{2}(\xi\beta_{0\perp})^{2/3}k_0$. The velocity difference in Eq. (29) also attains its maximum value when ${\rm Re}(\delta k) = {\rm Re}(\delta k)_{\max}$ which is the point where the growth rate vanishes. As we shall see in Sec. V the energy extraction is proportional to $\Delta\nu$, and hence the maximum efficiency is attained close to the point of vanishing growth rate.

V. SATURATION LEVELS

To obtain estimates for the saturation levels in the different FEL regimes we resort to arguments based on electron trapping.[14] In the cold-beam limit, we assume that saturation occurs when the beam electrons become trapped in the total longitudinal wave, i.e., space charge plus ponderomotive potential. The difference between the longitudinal-wave phase velocity and the axial electron velocity is initially $\nu_{\rm ph} - \nu_{z0} = -\Delta\nu$, where the difference $\Delta\nu$ is greater than zero for instability and depends on the particular FEL regime as well the frequency mismatch parameter Δk [see Eq. (32)]. Assuming all the particles to be deeply trapped, we may estimate that at saturation $\nu_{\rm ph} - \nu_{z,\rm sat} \approx \Delta\nu$, where $\nu_{z,\rm sat}$ is the average axial electron velocity at saturation and $\nu_{\rm ph}$ is assumed to remain fixed. The maximum decrease in the axial beam velocity is ~$2\Delta\nu$ corresponding to a change of particle kinetic energy by an amount

$$\Delta E_{\rm KE} \approx -\frac{\partial\gamma}{\partial\nu_z}\bigg|_{\nu_z = 2\nu_{z0}}\Delta\nu m_0c^2 = -2\gamma_0\gamma_{z0}^2 m_0\nu_{z0}\Delta\nu\,. \tag{36}$$

The energy-conversion efficiency is, therefore,

$$\eta = -\Delta E_{\rm KE}/(\gamma_0 - 1)m_0c^2 \approx 2\gamma_{z0}^2\Delta\nu/c\,. \tag{37}$$

Similar arguments have been used to obtain good estimates for efficiency in two-steam interaction processes.[33] The vector potential at saturation, $z = z_{\rm sat}$, can be found by applying the conservation law for total energy flux. The result is

$$|\tilde{A}(z = z_{\rm sat})| = \left[|\tilde{A}(z=0)|^2 + \left(\frac{\xi\gamma_0}{2\gamma_{z0}^2}\frac{m_0c^2}{|e|}\right)^2\eta\right]^{1/2}. \tag{38}$$

In the low-gain FEL limit described in Sec. III the efficiency, which is given by Eq. (37) together with Eq. (23), is

$$\eta = -2\theta_0/Lk_0\,, \tag{39}$$

for the highest-gain band; θ_0 ranges from 0 to -3. The maximum gain occurs when $\theta_0 = -1.3$ and is given in Eq. (21). The amplitude of the vector potential at saturation is

$$|\tilde{A}(z = z_{\rm sat} = L)| \approx |\tilde{A}(z=0)|(1 + G_L)\,. \tag{40}$$

Comparing Eq. (38) with (40) we find that the input signal needed to cause saturation at $z=L$ is

$$|\tilde{A}(0)|\approx\left(\frac{\xi\gamma_0}{2\gamma_{z0}^2}\frac{m_0c^2}{|e|}\right)\left(\frac{\eta}{2G_L}\right)^{1/2}, \tag{41}$$

with G_L given by Eq. (21).

In the high-gain, weak-pump FEL regime discussed in Sec. IV A, the efficiency is

$$\eta=(\xi/\gamma_{z0}+\Delta k/2k_0), \tag{42}$$

where we have used Eq. (33) in conjunction with (37), noting that the mismatch parameter is $\Delta k \leq (\xi\gamma_{z0})^{1/2}\beta_{0\perp}k_0$. Equation (42) is valid in the high-gain, weak-pump-parameter regime and hence, the second term is somewhat smaller than the first. The amplitude of the vector potential at saturation in this case is

$$|\tilde{A}(z=z_{\text{sat}})|\approx\frac{\xi\gamma_0}{2\gamma_{z0}^2}\frac{m_0c^2}{|e|}\eta^{1/2}, \tag{43}$$

where Eq. (38) was used together with the condition $|\tilde{A}(z=z_{\text{sat}})|\gg|\tilde{A}(0)|$.

Finally, we consider the high-gain, strong-pump case. The efficiency, using Eqs. (29) and (37), is given by

$$\eta=\frac{\text{Re}(\delta k)}{k_0}+\frac{\xi}{\gamma_{z0}}, \tag{44}$$

where $\text{Re}(\delta k)\leq(\xi\beta_{0\perp})^{2/3}k_0$, the equality holding where the spatial growth rate vanishes. When the growth rate is maximum [see Eq. (35)] the efficiency is

$$\eta=2^{-4/3}(\xi\beta_{0\perp})^{2/3}+\xi/\gamma_{z0}. \tag{45}$$

The second term in Eq. (44) is small compared to the first in the strong-pump limit. The saturated value of the vector potential is given by Eq. (43) together with (45).

VI. GROWTH RATE (GAIN) VERSUS B_0 FOR FIXED OUTPUT FREQUENCY

It is of interest to determine the scaling laws for the growth rate (or total gain) and efficiency, for a fixed output frequency, as a function of the magnetic-pump amplitude B_0. To obtain these scaling laws for a fixed-output frequency $\omega=2\gamma_{z0}^2ck_0$, i.e., fixed γ_{z0} and k_0, we note that the total gamma can be written as

$$\gamma_0=\gamma_{z0}[1+(|e|B_0/m_0c^2k_0)^2]^{1/2}. \tag{46}$$

Therefore, when B_0 approaches and becomes larger than the critical magnetic field amplitude

$$B_{\text{crit}}=(m_0c^2/|e|)k_0=(10.6/l)\text{ kG}, \tag{47}$$

(where l is in cm), the axial gamma γ_{z0} becomes significantly smaller than the total gamma γ_0. For fixed γ_{z0} and k_0, when $B_0\ll B_{\text{crit}}$, the total gamma is nearly equal to γ_{z0}; when $B_0\gg B_{\text{crit}}$ however, $\gamma_0\approx\gamma_{z0}B_0/B_{\text{crit}}$.

In the low-gain FEL regime, Eq. (21) shows that the maximum total gain is proportional to B_0^2 for $B_0\ll B_{\text{crit}}$ and falls off as B_0^{-1} for $B_0\gg B_{\text{crit}}$. The efficiency given in Eq. (39) is independent of the pump magnetic field amplitude for fixed γ_{z0} and k_0.

In the high-gain, weak-pump FEL case the maximum spatial growth rate [see Eq. (32)] is proportional to B_0 for $B_0\ll B_{\text{crit}}$ and decreases as $B_0^{-1/4}$ for $B_0\gg B_{\text{crit}}$. The efficiency, on the other hand, [see Eq. (42)] is independent of B_0 for $B_0\ll B_{\text{crit}}$ and falls off as $B_0^{-1/2}$ for $B_0\gg B_{\text{crit}}$. For the high-gain, strong-pump case Eqs. (35) and (45) show that, for $B_0\ll B_{\text{crit}}$, both the maximum growth rate and efficiency increase as $B_0^{2/3}$, whereas, for $B_0\gg B_{\text{crit}}$, both the growth rate and efficiency fall off as $B_0^{-1/3}$. These scaling laws for fixed output frequency indicate that, for all the FEL regimes which have been considered, the optimal magnetic-pump amplitude is one where $B_0\approx B_{\text{crit}}$.

VII. DISCUSSION

A. Energy shear

In the preceding formulation of FEL's, we have neglected any effects of energy shear across the beam. Such a shear arises owing to the self-electrostatic-potential drop within the beam. This leads to a radial dependence of the beam kinetic energy in the equilibrium state. The energy shear results in a shear in the axial equilibrium velocity across the beam and, therefore, is equivalent to a beam temperature. For an axially propagating beam of radius r_0 the effective beam temperature is of the order

$$\Delta E\approx|e|\Delta\phi=(\xi k_0r_0/2)^2E_0,$$

where $\Delta\phi$ is the self-potential drop across the beam from $r=0$ to $r=r_0$, and $E_0=(\gamma_0-1)m_0c^2$ is the kinetic energy of the electrons. A necessary condition for the validity of the cold-beam approximation in all the FEL regimes which have been considered is

$$\Delta E/E_0\ll\eta.$$

This inequality may be invalid at sufficiently high beam densities.

A more refined analysis taking account of the energy shear should also consider the radial gradient of the pump field, which is necessary to satisfy $\nabla\cdot\vec{B}_0=\nabla\times\vec{B}_0=0$. The radial dependence of the pump produces a shear in the equilibrium transverse velocity which will tend to compensate for the shear in axial velocity due to self-field effects. Other possible approaches which may be considered to eliminate the axial velocity shear include: (i) establishing Brillouin flow in the beam by applying an axial magnetic field, or (ii) creating the beam on a nonequipotential surface so that the applied-potential shear just cancels out the self-potential shear. In the following example self-field effects will be neglected.

B. Two-stage FEL

As an illustration of a far-infrared radiation source we consider a two-stage FEL generator. In a two-stage FEL, two consecutive and distinct scattering processes take place within a single electron beam. The output radiation from the first stage, in which the pump is a circularly polarized static magnetic field, is reflected back on the beam and used as the pump wave in the second stage. This configuration is schematically depicted in Fig. 2. The final wavelength of the output radiation, from the second stage is $\lambda \approx l/8\gamma_z^4$ instead of $l/2\gamma_z^2$, as would be the case in a single-stage device. Hence, in a two-stage FEL, far shorter output wavelengths can be realized for the same electron-beam energy. The pump field in the second stage is a circularly polarized electromagnetic wave and not a circularly polarized static magnetic field as in Eq. (1). Our results for a magnetic pump, however, also apply to a circularly polarized electromagnetic pump if the electron beam is highly relativistic. To see this we note that, in the beam frame, the two pump waves are equivalent if we set $B_{02} = 2E_1(z = z_{\text{sat}})$ and $k_{02} = 2k_{+1}$, where B_{02} and k_{02} are the magnetic field amplitude and wave number of the equivalent magnetic pump in the second stage and $E_1(z = z_{\text{sat}})$ and k_{+1} are the saturated electric field amplitude and wave number of the reflected output radiation from the first stage. The relevant parameters for this

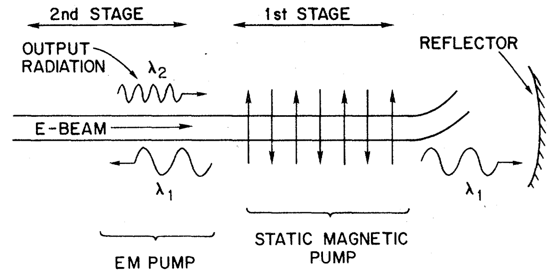

FIG. 2. Schematic of the two-stage free-electron-laser concept. The electron beam enters at left. Radiation scattered at wavelength λ_1 from the static magnetic pump of wavelength l in the first stage is reflected to act as an electromagnetic pump in the second stage. The final scattered radiation is at wavelength $\lambda_2 \sim l/8\gamma_z^4$.

example are contained in Table I. The results outlined in Table I demonstrate that in principle a rather low-energy electron beam ($E_0 = 3$ MeV) is necessary to generate far-infrared radiation using a 2-cm-wavelength magnetic pump. The radiation to beam power efficiency of 0.085% may be greatly enhanced by adiabatically varying the longitudinal wavelength of the electromagnetic pump in the second stage. Contouring the pump period for the purpose of enhancing efficiency has been suggested in Ref. 14.

Recent nonlinear calculations have shown that efficiency enhancement factors greater than 100 can be achieved by varying the wavelength of the static magnetic-pump field.[34] In the case of an electromagnetic pump the axial wavelength may be contoured by varying the waveguide wall radius. Work is now underway at the Naval Research Laboratory to evaluate this approach fully.

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TABLE I. Illustration of a far-infrared ($\lambda = 2\ \mu$) two-stage FEL. Electron beam energy: $E_0 = 3$ MeV ($\gamma_0 = 7$); current: $I_0 = 10$ kA; radius: $r_0 = 0.3$ cm.

	First stage	Second stage
Pump amplitude	$B_{01} = 5$ kG	$B_{02} = 2E_1(z = z_{\text{sat}}) = 15.7$ kG
Pump wavelength	$l_{01} = 2$ cm	$l_{02} = \lambda_1/2 = 0.019$ cm
Longitudinal gamma	$\gamma_{z1} = 5.1$	$\gamma_{z2} = 7$
Beam-strength parameter	$\xi_1 = 0.61$	$\xi_2 = \xi_1/(4\gamma_{z1}^2) = 5.9 \times 10^{-3}$
Transverse velocity	$\beta_{0\perp} = 0.135$	$\beta_{0\perp,2} = 0.4 \times 10^{-2}$
Critical velocity	$\beta_{\text{crit},1} = 0.27$	$\beta_{\text{crit},2} = 1.65 \times 10^{-2}$
Output wavelength	$\lambda_1 = 0.038$ cm	$\lambda_2 = 2.0\ \mu$
Spatial growth rate	$\Gamma_{\max,1} = 0.37$ cm^{-1}	$\Gamma_{\max,2} = 0.13$ cm^{-1}
Efficiency	$\eta_1 = 12\%$	$\eta_2 = 0.085\%$
Output power	$P_{01} = 3.6$ GW	$P_{02} = 25$ MW

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