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Quantum theory of a two-mode laser with coupled transitions

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The quantum-statistical properties of the optical field of a two-mode laser with two coupled transitions have been studied by means of a generalization of the Scully and Lamb theory. The photon number distribution and the mode intensity distribution are obtained in the steady state for a system of homogeneously broadened atoms in resonance with the laser field. It is shown that the mode-coupling constant ξ is unity and that near threshold the results of earlier treatments are recovered. Furthermore, it is shown that certain limit measures for the relative intensity fluctuations predicted by the semiclassical Fokker-Planck treatments based on the third-order theory are valid even in the limit of arbitrarily high gain levels.

I. INTRODUCTION

It is known that in multimode lasers different modes compete against one another for contribution, partly from the same population of excited atoms in the gain medium. Their interaction with the atomic medium gives rise to a coupling between different modes and, depending on the strength of this coupling, one mode can influence the properties of the other modes to varying degrees. The simplest example of a multimode laser is a two-mode laser where the effects of mode competition show up most markedly in the statistical properties of the light in the two modes. In the bidirectional gas-ring laser, for example, one mode grows at the expense of the other when the cavity is tuned to the center of the atomic line.[1] The stronger mode tends to become coherent as gain is increased much above threshold; the weaker mode becomes completely chaotic and obeys thermal statistics. Under certain circumstances strong competition may lead to a bistable behavior which may be reflected in the relative intensity fluctuations of the weaker mode. Some of these predictions have already been observed.[2]

In a two-mode laser with two-level atomic gain medium, mode coupling arises because both modes compete for gain from the same atomic transition, viz., the one from the upper level to the lower level [Fig. 1(a)] with the possibility of emission in either of the two interacting modes. However, in a three-level gain medium [Fig. 1(b)], two modes may be supported by two different transitions. But, if these transitions share a common atomic level, mode competition will arise, for then both modes will compete for the atomic population in the upper level $|a\rangle$. In addition to this the transitions are also coupled via a Raman or quadrupole-type interaction. A direct transition between the levels $|b_1\rangle$ and $|b_2\rangle$ is not allowed (we assume this to be the case in our discussion), but a Raman-type transition involving the upper level as the intermediate state is possible. This process provides additional coupling between the two transitions and hence between the two modes. Thus in a three-level gain-medium mode, coupling occurs because the two supportive transitions share a common level and interact via a Raman-type process. This situation, in practice, may correspond to some two-mode Zeeman lasers or to a two-mode ring laser where the counter-propagating waves may be supported by the two transitions.[3] In this paper we treat a two-mode laser with the latter type of gain medium.

The theory of two-mode lasers has been developed by many authors. Most of these treatments have been semiclassical and deterministic,[4] in which a set of coupled Maxwell-Bloch equations is solved, usually perturbatively, up to third order in the field strength, for a system of two-level atoms interacting with a two-mode electromagnetic field. To discuss the statistical properties of the light field, these theories have been generalized to include the effect of spontaneous emission noise, by replacing the Maxwell-Bloch equations with Langevin-type equations which may be converted into a Fokker-Planck equation. Many such treatments have been given.[5] There have been one or two quantum-mechanical treatments as well, but there a perturbative approach has been used.[6] Systems with two coupled transitions have also been discussed using similar procedures but the emphasis has been on such different aspects as application to sub-Doppler spectroscopy and line-shape studies.[7] The statistical nature of the field in such media has been ignored.

In the following analysis we present a second quantized treatment of a two-mode laser with two coupled atomic transitions using a procedure similar to the Scully and Lamb treatment of a single-mode laser.[8] Starting from the equation of motion for the combined atom-field density matrix, a master equation for the reduced density matrix of the field is obtained. This master equation is solved in the steady state to obtain the photon distribution function. Analytic expressions for various moments of the photon numbers in the two modes are calculated. To make comparison with semiclassical theories, the master equation is converted into a "Fokker-Planck" equation using the coherent-state-diagonal representation of the density matrix.[9] The resulting Fokker-Planck equation is solved and the statistics of the light field are discussed. It is shown that for equal losses neither mode approaches a coherent state as gain is increased. The relative intensity fluctuations of both modes approach the value $\frac{1}{3}$. Intensity cross correlation has a negative value, which reflects mode competition, and it approaches the value $-\frac{1}{3}$. For unequal losses the favored mode grows and becomes coherent. The weaker mode does not lase and obeys thermal statistics. Finally, the results are generalized to the case of N-coupled transitions and simple results are derived for the moments when all the losses are equal.

II. EQUATIONS OF MOTION

We consider a system of atoms having a level structure shown in Fig. 1(b). The atoms are pumped to the upper level $|a\rangle$. An excited atom in the state $|a\rangle$ can make a transition to level $|b_1\rangle$ ($|b_2\rangle$) by emitting a photon into mode 1 (2) under the influence of the laser field of frequency ω_1 (ω_2). In addition to participating in the laser action, levels $|a\rangle, |b_1\rangle, |b_2\rangle$ can decay to a set of levels $|c\rangle, |d_1\rangle, |d_2\rangle$, respectively, via their interaction with a set of independent zero-temperature reservoirs. The radiation emitted in the last processes does not contribute to the laser radiation. The Hamiltonian $\hbar\hat{H}$ for such a system is

$$\begin{aligned}\hat{H} = &\sum_{i=1}^{2} \omega_i \hat{a}_i^\dagger \hat{a}_i + \sum_{\substack{\alpha = a, b_1, b_2 \\ c, d_1, d_2}} W_\alpha \hat{A}_\alpha^\dagger \hat{A}_\alpha + \sum_s \omega_s \hat{a}_s^\dagger \hat{a}_s \\ &+ \sum_{j=1}^{2} \sum_{\sigma_j} \omega_{\sigma_j} \hat{a}_{\sigma_j}^\dagger \hat{a}_{\sigma_j} + \sum_{i=1}^{2} (g_i \hat{a}_i^\dagger \hat{A}_{b_i}^\dagger \hat{A}_a + \text{H.c.}) \\ &+ \sum_s (g_s \hat{a}_s^\dagger \hat{A}_c^\dagger \hat{A}_a + \text{H.c.}) \\ &+ \sum_{j=1}^{2} \sum_{\sigma_j} (g_{\sigma_j} \hat{a}_{\sigma_j}^\dagger \hat{A}_{d_j}^\dagger \hat{A}_{b_j} + \text{H.c.}) \\ = &\hat{H}_0^F + \hat{H}_0^A + \hat{H}_0^s + \sum_j \hat{H}_0^{\sigma_2} + \hat{V} + \sum_s \hat{V}^s + \sum_j \sum_{\sigma_j} \hat{V}^{\sigma_j} .\end{aligned} \tag{1}$$

Here ω_1 (ω_2) is the frequency of mode 1 (2) and $\hat{a}_1$ ($\hat{a}_2$), $\hat{a}_1^\dagger$ ($a_2^\dagger$) are the corresponding annihilation and creation operators; $\hbar W_\alpha$ is the energy eigenvalue associated with the atomic level α and $\hat{A}_\alpha^\dagger$ represent the annihilation and creation operators for the atomic levels; $\hbar\hat{H}_0^s, \hbar\hat{H}_0^{\sigma_1}, \hbar\hat{H}_0^{\sigma_2}$, are the free Hamiltonians of the reservoirs interacting with levels $|a\rangle, |b_1\rangle, |b_2\rangle$, respectively. The reservoir variables carry subscripts s, σ_1, and σ_2; $\hbar\hat{H}_0^F$ and $\hbar\hat{H}_0^A$ describe free-field and atomic Hamiltonians. The atom-field interaction is described by $\hat{V}$. Similarly, the reservoir–atomic-level interactions are represented by $\hat{V}^s$, $\hat{V}^{\sigma_1}$, and $\hat{V}^{\sigma_2}$.

$$g_{1(2)} = (e x_{a b_1 (b_2)} / \hbar) \mathcal{E}_{1(2)} ,$$

with

$$\mathcal{E}_i(\vec{r}) = (\hbar\omega_i / 2\epsilon_0 V)^{1/2} U_i(\vec{r}) ,$$

where $e x_{a b_i}$ is the dipole matrix element connecting state $|a\rangle$ with $|b_i\rangle$. $U_i(\vec{r})$ describes the spatial dependence of the field, which we shall ignore. In general, g is a complex quantity but we shall take it to be real because the final expressions for the quantities of interest involve g_i^2, which in the case of complex g_i can be replaced by $|g_i|^2$. Similar expressions exist for g_s, g_{σ_1}, and g_{σ_2}.

The equation of motion for the total density matrix in the interaction representation is

$$\frac{d\hat{\rho}}{dt} = -i[\hat{V}^I(t), \hat{\rho}] - i\Big(\sum_s (\hat{V}^I)^s + \sum_j \sum_{\sigma_j} (\hat{V}^I)^{\sigma_j}, \hat{\rho}\Big), \tag{2}$$

where $\hat{O}^I$ is the operator $\hat{O}$ in the interaction picture.

We are interested in the following states of the system:

$$\begin{aligned}&(a, n_1, n_2, O_s) \equiv \alpha; \quad (c, n_1, n_2, 1_s) \equiv \gamma , \\ &(b_1, n_1+1, n_2, O_{\sigma_1}) \equiv \beta_1; \\ &(d_1, n_1+1, n_2, 1_{\sigma_1}) \equiv \delta_1; \\ &(b_2, n_1, n_2+1, O_{\sigma_2}) \equiv \beta_2; \\ &(d_2, n_1, n_2+1, 1_{\sigma_2}) \equiv \delta_2.\end{aligned} \tag{3}$$

The quantities in the definition of states refer to the atomic state, the number of photons in mode 1, the number of photons in mode 2 and the number of photons in the reservoir state, respectively. In what follows a prime over the Greek letters signifies a prime over the field quantities, e.g., $\alpha' \equiv (a, n_1', n_2', O_s)$. Using Eqs. (2) and (3) we find the following equations of motion for the relevant matrix elements of the density operator:

$$\dot{\rho}_{\alpha\alpha'} = -i[\hat{V}^I, \hat{\rho}]_{\alpha\alpha'} - i\sum_s (V_{\alpha\gamma}^s \rho_{\gamma\alpha'} - \rho_{\alpha\gamma'} V_{\gamma'\alpha'}^s), \tag{4a}$$

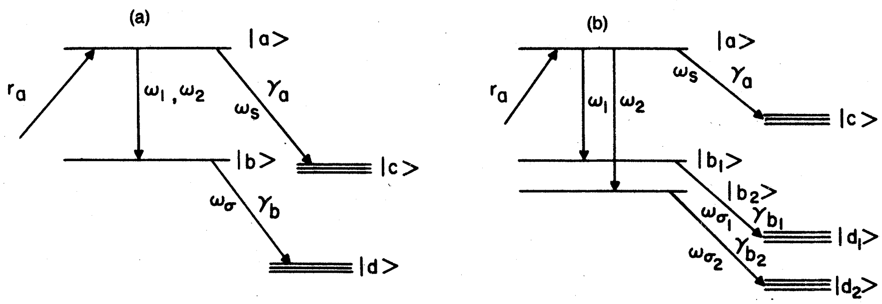

FIG. 1. (a) Two-mode competition for the same atomic transition; (b) two-mode competition for two different transitions sharing a common level.

$$\dot\rho_{\alpha\beta_1'} = -i[\hat V^I, \hat\rho]_{\alpha\beta_1'} - i\sum_{s,\sigma_1}(V^s_{\alpha\gamma}\rho_{\gamma\beta_1'} - \rho_{\alpha\delta_1'}V^{\sigma_1}_{\delta_1'\beta_1'}), \tag{4b}$$

$$\dot\rho_{\alpha\beta_2'} = -i[\hat V^I, \hat\rho]_{\alpha\beta_2'} - i\sum_{s,\sigma_2}(V^s_{\alpha\gamma}\rho_{\gamma\beta_2'} - \rho_{\alpha\delta_2'}V^{\sigma_2}_{\delta_2'\beta_2'}), \tag{4c}$$

$$\dot\rho_{\beta_1\alpha'} = -i[\hat V^I, \hat\rho]_{\beta_1\alpha'} - i\sum_{s,\sigma_1}(V^{\sigma_1}_{\beta_1\delta_1}\rho_{\delta_1\alpha'} - \rho_{\beta_1\gamma'}V^s_{\gamma'\alpha'}), \tag{4d}$$

$$\dot\rho_{\beta_1\beta_1'} = -i[\hat V^I, \hat\rho]_{\beta_1\beta_1'} - i\sum_{\sigma_1}(V^{\sigma_1}_{\beta_1\delta_1}\rho_{\delta_1\beta_1'} - \rho_{\beta_1\delta_1'}V^{\sigma_1}_{\delta_1'\beta_1'}), \tag{4e}$$

$$\dot\rho_{\beta_1\beta_2'} = -i[\hat V^I, \hat\rho]_{\beta_1\beta_2'} - i\sum_{\sigma_1,\sigma_2}(V^{\sigma_1}_{\beta_1\delta_1}\rho_{\delta_1\beta_2'} - \rho_{\beta_1\delta_2'}V^{\sigma_2}_{\delta_2'\beta_2'}), \tag{4f}$$

$$\dot\rho_{\beta_2\alpha'} = -i[\hat V^I, \hat\rho]_{\beta_2\alpha'} - i\sum_{s,\sigma_2}(V^{\sigma_2}_{\beta_2\delta_2}\rho_{\delta_2\alpha'} - \rho_{\beta_2\gamma'}V^s_{\gamma'\alpha'}), \tag{4g}$$

$$\dot\rho_{\beta_2\beta_1'} = -i[\hat V^I, \hat\rho]_{\beta_2\beta_1'} - i\sum_{\sigma_1,\sigma_2}(V^{\sigma_2}_{\beta_2\delta_2}\rho_{\delta_2\beta_1'} - \rho_{\beta_2\delta_1'}V^{\sigma_1}_{\delta_1'\beta_1'}), \tag{4h}$$

$$\dot\rho_{\beta_2\beta_2'} = -i[\hat V^I, \hat\rho]_{\beta_2\beta_2'} - i\sum_{\sigma_2}(V^{\sigma_2}_{\beta_2\delta_2}\rho_{\delta_2\beta_2'} - \rho_{\beta_2\delta_2'}V^{\sigma_2}_{\delta_2'\beta_2'}), \tag{4i}$$

$$\dot\rho_{\gamma\gamma'} = -i\sum_s(V^s_{\gamma\alpha}\rho_{\alpha\gamma'} - \rho_{\gamma\alpha'}V_{\alpha'\gamma'}), \tag{4j}$$

$$\dot\rho_{\delta_1\delta_1'} = -i\sum_{\sigma_1}(V^{\sigma_1}_{\delta_1\beta_1}\rho_{\beta_1\delta_1'} - \rho_{\delta_1\beta_1'}V^{\sigma_1}_{\beta_1'\delta_1'}), \tag{4k}$$

$$\dot\rho_{\delta_2\delta_2'} = -i\sum_{\sigma_2}(V^{\sigma_2}_{\delta_2\beta_2}\rho_{\beta_2\delta_2'} - \rho_{\delta_2\beta_2'}V^{\sigma_2}_{\beta_2'\delta_2'}), \tag{4l}$$

where we have dropped the superscipt I in the terms describing the atom-reservoir interaction for typographical convenience. Similar equations can be written for $\rho_{\alpha\alpha'}$, $\rho_{\delta_i\beta_i'}$, $\rho_{\delta_i\alpha'}$, $\rho_{\gamma\beta_i'}$. As noted earlier the radiation emitted to reservoirs is unobserved; we therefore trace Eqs. (4a)–(4l) over reservoir variables. In doing so we use the assumptions and approximations of the Wigner-Weisskopf theory and follow the procedure of Scully and Lamb.[8] This procedure yields the effective decay rates for the atomic levels. Then using the following notation for the atom-field states:

$$\begin{aligned} &1 \equiv |a, n_1, n_2\rangle, \quad 2 \equiv |b, n_1+1, n_2\rangle, \\ &3 \equiv |b_2, n_1, n_2+1\rangle, \quad 4 \equiv |c, n_1, n_2\rangle, \\ &5 \equiv |d_1, n_1+1, n_2\rangle, \quad 6 \equiv |d_2, n_1, n_2+1\rangle, \end{aligned} \tag{5}$$

where, as before, a prime over the state symbols represents a prime over the field variables. We obtain

$$\dot\rho_{11'} = -i(V^I_{12}\rho_{21'} + V^I_{13}\rho_{31'} - \rho_{12'}V^I_{2'1'} - \rho_{13'}V^I_{3'1'}) - \gamma_a\rho_{11'}, \tag{6a}$$

$$\dot\rho_{12'} = -i(V^I_{12}\rho_{22'} + V^I_{13}\rho_{32'} - \rho_{11'}V^I_{1'2'}) - \gamma_{ab_1}\rho_{12'}, \tag{6b}$$

$$\dot\rho_{13'} = -i(V^I_{12}\rho_{23'} + V^I_{13}\rho_{33'} - \rho_{11'}V^I_{1'3'}) - \gamma_{ab_2}\rho_{13'}, \tag{6c}$$

$$\dot\rho_{21'} = -i(V^I_{21}\rho_{11'} - \rho_{22'}V^I_{2'1'} - \rho_{23'}V^I_{3'1'}) - \gamma_{ab_1}\rho_{21'}, \tag{6d}$$

$$\dot\rho_{22'} = -i(V^I_{21}\rho_{12'} - \rho_{21'}V^I_{1'2'}) - \gamma_{b_1}\rho_{22'}, \tag{6e}$$

$$\dot\rho_{23'} = -i(V^I_{21}\rho_{13'} - \rho_{21'}V^I_{1'3'}) - \gamma_{b_1b_2}\rho_{23'}, \tag{6f}$$

$$\dot\rho_{31'} = -i(V^I_{31}\rho_{11'} - \rho_{32'}V^I_{2'1'} - \rho_{33'}V^I_{3'1'}) - \gamma_{ab_2}\rho_{31'}, \tag{6g}$$

$$\dot{\rho}_{32'} = -i(V^I_{31}\rho_{12'} - \rho_{31'}V^I_{1'2'}) - \gamma_{b_1b_2}\rho_{32'}, \quad (6h)$$

$$\dot{\rho}_{33'} = -i(V^I_{31}\rho_{13'} - \rho_{31'}V^I_{1'3'}) - \gamma_{b_2}\rho_{33'}, \quad (6i)$$

$$\dot{\rho}_{44'} = \gamma_a \rho_{11'}, \quad (6j)$$

$$\dot{\rho}_{55'} = \gamma_{b_1}\rho_{22'}, \quad (6k)$$

$$\dot{\rho}_{66'} = \gamma_{b_2}\rho_{33'}, \quad (6l)$$

where the decay constants γ_a, γ_{b_1}, γ_{b_2} are

$$\gamma_a = 2\pi\Omega(\omega_{ac})|V^s_{a0;c1}|^2, \quad (7a)$$

$$\gamma_{b_i} = 2\pi\Omega(\omega_{b_id_i})|V^{\sigma_i}_{b_i0;d_i1}|^2, \quad i=1,2; \quad (7b)$$

$\Omega(\omega_{ij})$ is the density of the final states at the frequency $\omega_{ij} = W_i - W_j$ in the transition $i \to j$, and

$$\gamma_{ab_i} = \tfrac{1}{2}(\gamma_a + \gamma_{b_i}), \quad i=1,2, \quad \gamma_{b_1b_2} = \tfrac{1}{2}(\gamma_{b_1} + \gamma_{b_2}). \quad (8)$$

We now replace $\hat{\rho}$ in Eqs. (6a)–(6l) by $e^{-i\hat{H}_0 t}\hat{\rho}\, e^{i\hat{H}_0 t}$, where $\hat{H}_0 = \hat{H}^F_0 + \hat{H}^A_0$, and integrate the resulting equations from time t_0 (the time at which the atom is introduced in the excited state) to t_0+T. The time T is chosen to be large compared to all atomic decay times involved but small compared to the time for growth or decay of the radiation field. If we introduce the following symbols for the integrals

$$\sigma_{jj'}(t_0+T) = \int_{t_0}^{t_0+T}\rho_{jj'}(t')dt', \quad j,j'=1,2,\ldots,6, \quad (9)$$

then the integrated equations can be written as

$$-\rho_{11'}(t_0) = -i(V_{12}\sigma_{21'} + V_{13}\sigma_{31'} - \sigma_{12'}V_{2'1'} - \sigma_{13'}V_{3'1'}) - \gamma_a\sigma_{11}, \quad (10a)$$

$$0 = -i(V_{12}\sigma_{22'} + V_{13}\sigma_{32'} - \sigma_{11'}V_{1'2'}) - i(\Delta_1 - i\gamma_{ab_1})\sigma_{12'}, \quad (10b)$$

$$0 = -i(V_{12}\sigma_{23'} + V_{13}\sigma_{33'} - \sigma_{11'}V_{1'3'}) - i(\Delta_2 - i\gamma_{ab_2})\sigma_{13'}, \quad (10c)$$

$$0 = -i(V_{21}\sigma_{11'} - \sigma_{22'}V_{2'1'} - \sigma_{23'}V_{3'1'}) + i(\Delta_1 + i\gamma_{ab_1})\sigma_{21'}, \quad (10d)$$

$$0 = -i(V_{21}\sigma_{12'} - \sigma_{21'}V_{1'2'}) - \gamma_{b_1}\sigma_{22'}, \quad (10e)$$

$$0 = -i(V_{21}\sigma_{13'} - \sigma_{21'}V_{1'3'}) - i(\Delta_1 + \Delta_2 - i\gamma_{b_1b_2})\sigma_{23'}, \quad (10f)$$

$$0 = -i(V_{31}\sigma_{11'} - \sigma_{32'}V_{2'1'} - \sigma_{33'}V_{3'1'}) + i(\Delta_2 + i\gamma_{ab_2})\sigma_{31'}, \quad (10g)$$

$$0 = -i(V_{31}\sigma_{12'} - \sigma_{31'}V_{1'2'}) + i(\Delta_1 + \Delta_2 + i\gamma_{b_1b_2})\sigma_{32'}, \quad (10h)$$

$$0 = -i(V_{31}\sigma_{13'} - \sigma_{31'}V_{1'3'}) - \gamma_{b_2}\sigma_{33'}, \quad (10i)$$

$$\rho_{44'}(t_0+T) = \gamma_a\sigma_{11'}, \quad (10j)$$

$$\rho_{55'}(t_0+T) = \gamma_{b_1}\sigma_{22'}, \quad (10k)$$

$$\rho_{66'}(t_0+T) = \gamma_{b_2}\sigma_{33'}. \quad (10l)$$

Here $\hat{V}$ is the field interaction given by Eq. (1); Δ_1 and Δ_2 are the detuning of the mode frequencies from the atomic transition frequencies given by

$$\Delta_1 = [(W_a - W_{b_1}) - \omega_1], \quad \Delta_2[(W_a - W_{b_2}) - \omega_2]. \quad (11)$$

In arriving at Eqs. (10a)–(10l) we have used the fact that T is much larger than all atomic lifetimes involved so that at time t_0+T all the matrix elements of the density operator, diagonal in the atomic states $|a\rangle$, $|b_1\rangle$, $|b_2\rangle$, will be negligibly small. Further, since at time t_0 the atoms are introduced in the upper state, all the elements except $\rho_{11'}(t_0)$ will be zero.

The reduced density matrix $\rho_{n_1n_2;n_1'n_2'}$ of the radiation field is obtained by tracing over the atomic states. This gives

$$\rho_{n_1n_2;n_1'n_2'}(t_0+T) = \rho_{11'}(t_0+T) + \rho_{22'}(t_0+T) + \rho_{33'}(t_0+T) + \rho_{44'}(t_0+T) + \rho_{55'}(t_0+T) + \rho_{66'}(t_0+T).$$

The matrix elements $\rho_{11'}(t_0+T)$, $\rho_{22'}(t_0+T)$ will be negligibly small; then using Eqs. (10j)–(10l) we obtain

$$\rho_{n_1n_2;n_1'n_2'}(t_0+T) = \gamma_a\sigma_{11'} + \gamma_{b_1}\sigma_{22'}(n_1 \to n_1 - 1,\ n_1' \to n_1' - 1) + \gamma_{b_2}\sigma_{33'}(n_2 \to n_2 - 1,\ n_2' \to n_2' - 1). \quad (12)$$

The initial condition gives

$$\rho_{n_1n_2;n_1'n_2'}(t_0) = \rho_{11'}(t_0) = \rho_{an_1n_2;an_1'n_2'}(t_0). \quad (13)$$

Our problem now reduces to solving Eqs. (10a)–(10i). It is possible to solve these equations under general conditions but the expressions are lengthy and complicated. Furthermore, not much is to be gained in terms of the understanding of the statistical properties of the optical field by keeping the most general expressions. We shall make certain simplifying assumptions. Both modes will be considered to be in resonance with corresponding atomic transition frequencies, i.e., $\Delta_1 = \Delta_2 = 0$. Mode frequencies ω_1 and ω_2 may be unequal and may correspond to two different longitudinal modes of the cavity. For equal frequencies two modes will be considered as having two different polarizations. These conditions are not difficult to meet in practice. We shall also assume that the two coupling strengths g_1 and g_2 are equal and will be denoted by a common symbol g. These assumptions hold for many systems of practical interest[3] and in Sec. III we shall comment on the case $g_1 \neq g_2$. In many cases it may be a good approximation to take $\gamma_{b_1} = \gamma_{b_2} = \gamma_b$. With these simplifications it is straightforward to solve Eqs. (10a)–(10i) for the diagonal elements ($n_1 = n_1'$, $n_2 = n_2'$) and the result is

$$\gamma_a \sigma_{11} = \frac{1 + (2g^2/\gamma_b \gamma_{ab})(n_1 + n_2 + 2)}{1 + (4g^2/\gamma_a \gamma_b)(n_1 + n_2 + 2)}\, p(n_1, n_2)\ , \quad (14a)$$

$$\gamma_b \sigma_{22} = \frac{(2g^2/\gamma_a \gamma_{ab})(n_1 + 1)}{1 + (4g^2/\gamma_a \gamma_b)(n_1 + n_2 + 2)}\, p(n_1, n_2)\ , \quad (14b)$$

$$\gamma_b \sigma_{33} = \frac{(2g^2/\gamma_a \gamma_{ab})(n_2 + 1)}{1 + (4g^2/\gamma_a \gamma_b)(n_1 + n_2 + 2)}\, p(n_1, n_2)\ , \quad (14c)$$

where $p(n_1, n_2) = \rho_{n_1 n_2; n_1 n_2}$ is the probability of finding n_1 photons in mode 1 and n_2 photons in mode 2.

If r_a is the rate at which atoms are introduced into the cavity in the excited state and each atom has the same line broadening (homogeneously broadened gain medium), the rate of change in the diagonal elements of the coarse-grained density matrix of the field is found from Eqs. (12) and (13) to be

$$\left(\frac{d}{dt} p(n_1, n_2)\right)_{\text{gain}} = r_a[\gamma_a \sigma_{11} + \gamma_b \sigma_{22}(n_1 \to n_1 - 1) + \gamma_b \sigma_{33}(n_2 \to n_2 - 1) - p(n_1, n_2)]\ . \quad (15)$$

Similarly the equation for the losses is found to be

$$\left(\frac{d}{dt} p(n_1, n_2)\right)_{\text{loss}} = -C_1 n_1 p(n_1, n_2) - C_2 n_2 p(n_1, n_2) + C_1(n_1 + 1) p(n_1 + 1, n_2) + C_2(n_2 + 1) p(n_1, n_2 + 1)\ . \quad (16)$$

The loss parameters C_i are related to the cavity Q by

$$C_i = \nu / Q_i \quad i = 1, 2\ . \quad (17)$$

We have assumed losses to be linear in photon numbers and independent for the two modes. Finally, defining the A and B coefficients by

$$A = r_a 2g^2/\gamma_a \gamma_{ab}, \quad B = r_a 8(g^2/\gamma_a \gamma_b)(g^2/\gamma_a \gamma_{ab})\ , \quad (18)$$

it follows from Eqs. (16), (17), and (19) that $p(n_1, n_2)$ satisfies the following equation

$$\frac{d}{dt} p(n_1, n_2) = -\frac{A(n_1 + 1)}{1 + (B/A)(n_1 + n_2 + 2)} p(n_1, n_2) - \frac{A(n_2 + 1)}{1 + (B/A)(n_1 + n_2 + 2)} p(n_1, n_2) + C_1(n_1 + 1) p(n_1 + 1, n_2) + C_2(n_2 + 1) p(n_1, n_2 + 1)$$
$$+ \frac{A n_1}{1 + (B/A)(n_1 + n_2 + 1)} p(n_1 - 1, n_2) + \frac{A n_2}{1 + (B/A)(n_1 + n_2 + 1)} p(n_1, n_2 - 1) - C_1 n_1 p(n_1, n_2) - C_2 n_2 p(n_1, n_2)\ . \quad (19)$$

This is the desired master equation that we set out to derive. Various terms on the right-hand side of this equation can be interpreted as outflow and inflow of probabilities in two dimensions (see Fig. 2).

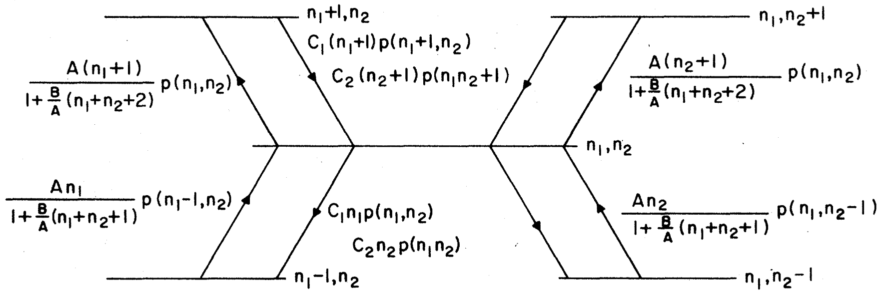

FIG. 2. Probability flow diagram for a two-mode laser.

III. STEADY-STATE SOLUTION AND PHOTON STATISTICS

In the steady state of the system $p(n_1, n_2)$ is independent of time and obeys the equation

$$-\frac{A(n_1+1)}{1+(B/A)(n_1+n_2+2)}p(n_1,n_2)-\frac{A(n_2+1)}{1+(B/A)(n_1+n_2+2)}p(n_1,n_2)+C_1(n_1+1)p(n_1+1,n_2)+C_2(n_2+1)p(n_1,n_2+1)$$
$$+\frac{An_1}{1+(B/A)(n_1+n_2+1)}p(n_1-1,n_2)+\frac{An_2}{1+(B/A)(n_1+n_2+1)}p(n_1,n_2-1)-C_1n_1p(n_1,n_2)-C_2n_2p(n_1,n_2)=0. \quad (20)$$

In view of our interpretation of Eq. (19) it follows that Eq. (20) represents the steady state of the two-dimensional probability flow. It can be shown that Eq. (20) is equivalent to the following equations:

$$C_1n_1p(n_1,n_2)=\frac{An_1}{1+(B/A)(n_1+n_2+1)}p(n_1-1,n_2), \quad (21a)$$

$$C_2n_2p(n_1,n_2)=\frac{An_2}{1+(B/A)(n_1+n_2+1)}p(n_1,n_2-1), \quad (21b)$$

which have a common solution

$$p_s(n_1,n_2)=Z^{-1}\left(\frac{A^2}{BC_1}\right)^{n_1}\left(\frac{A^2}{BC_2}\right)^{n_2}\Big/\Gamma\left(\frac{A}{B}+n_1+n_2+2\right), \quad (22)$$

where

$$Z=\sum_{n_1,n_2}\left(\frac{A^2}{BC_1}\right)^{n_1}\left(\frac{A^2}{BC_2}\right)^{n_2}\frac{\Gamma(n_1+1)\Gamma(n_2+1)}{n_1!n_2!\Gamma(A/B+n_1+n_2+2)}$$
$$=\Phi_2\left(1,1;\frac{A}{B}+1;\frac{A^2}{BC_1},\frac{A^2}{BC_2}\right) \quad (23)$$

is the normalization constant and Φ_2 is the degenerate hypergeometric series in two variables A^2/BC_1 and A^2/BC_2.[10a] With expression (22), factorial moments are easier to calculate and we give the results for the first few factorial moments

$$\langle n_1\rangle=\left(\frac{A^2}{BC_1}\right)\frac{\Phi_2(2,1;A/B+2;A^2/BC_1,A^2/BC_2)}{\Phi_2(1,1;A/B+1;A^2/BC_1,A^2/BC_2)}, \quad (24a)$$

$$\langle n_2\rangle=\left(\frac{A^2}{BC_2}\right)\frac{\Phi_2(1,2;A/B+2;A^2/BC_1,A^2/BC_2)}{\Phi_2(1,1;A/B+1;A^2/BC_1,A^2/BC_2)}, \quad (24b)$$

$$\langle n_1(n_1-1)\rangle=\left(\frac{A^2}{BC_1}\right)^2\frac{\Phi_2(3,1;A/B+3;A^2/BC_1,A^2/BC_2)}{\Phi_2(1,1;A/B+1;A^2/BC_1,A^2/BC_2)}, \quad (24c)$$

$$\langle n_2(n_2-1)\rangle=\left(\frac{A^2}{BC_2}\right)^2\frac{\Phi_2(1,3;A/B+3;A^2/BC_1,A^2/BC_2)}{\Phi_2(1,1;A/B+1;A^2/BC_1,A^2/BC_2)}, \quad (24d)$$

$$\langle n_1n_2\rangle=\frac{A^2}{BC_1}\frac{A^2}{BC_2}\frac{\Phi_2(2,2;A/B+3;A^2/BC_1,A^2/BC_2)}{\Phi_2(1,1;A/B+1;A^2/BC_1,A^2/BC_2)}. \quad (24e)$$

In general,

$$\langle n_1(n_1-1)\ldots(n_1-\mu+1)n_2(n_2-1)\ldots(n_2-\nu+1)\rangle$$
$$=\left(\frac{A^2}{BC_1}\right)^{\mu}\left(\frac{A^2}{BC_2}\right)^{\nu}\times\frac{\Phi_2(\mu+1,\nu+1;A/B+\mu+\nu+1;A^2/BC_1,A^2/BC)}{\Phi_2(1,1;A/B+1;A^2/BC_1,A^2/BC_2)}. \quad (24f)$$

The expressions for the moments (24a)–(24f) do not tell us much since Φ_2 is not a tabulated function and, when expressed as a series, it converges rather slowly. However, a qualitative understanding of the statistics can be gained by considering Eq. (22) directly. We consider two cases:

(i) $C_1=C_2=C$: The distribution function depends only on the sum n_1+n_2 and, above threshold, it peaks at

$$n_1+n_2=A(A-C)/BC. \quad (25)$$

When the laser is operated much above threshold,

$$\langle n_1+n_2\rangle\sim A^2/BC$$

and n_1+n_2 far exceeds A/B. Under these conditions

$$p_s(n_1,n_2)\sim\left(\frac{A^2}{BC}\right)^{n_1+n_2}e^{-A^2/BC}/(n_1+n_2)!$$
$$=(\langle n_1+n_2\rangle)^{n_1+n_2}e^{-\langle n_1+n_2\rangle}/(n_1+n_2)!, \quad (26)$$

which is a Poissonian in n_1+n_2 but not in n_1 and n_2 separately. This indicates that neither of the modes becomes coherent even at gains that are much above threshold. This is clearly a result of mode competition. That the two modes will be correlated is evident from Eq. (26). Actually we can do better than that. By considering $\langle(n_1+n_2)\times(n_1+n_2-1)\rangle$ over the distribution (26) we can establish

$$\frac{\langle\Delta n_1\Delta n_2\rangle}{\langle n_1\rangle\langle n_2\rangle}=-\frac{\langle(\Delta n_1)^2\rangle}{\langle n_1\rangle^2}+\frac{1}{\langle n_1\rangle}=-\frac{\langle(\Delta n_2)^2\rangle}{\langle n_2\rangle^2}+\frac{1}{\langle n_2\rangle}. \quad (27)$$

It can be shown that $\langle(\Delta n_1)^2\rangle/\langle n_1\rangle^2\geq 0$. The quantity $1/\langle n_1\rangle$ will be negligibly small above threshold. This leads to another result that the two modes may be anticorrelated and the limit of $\langle\Delta n_1\Delta n_2\rangle/\langle n_1\rangle\langle n_2\rangle$ is the same but opposite in sign to the limit of $\langle(\Delta n_1)^2\rangle/\langle n_1\rangle^2$ at high gains. We shall see that these conclusions are borne out by computer evaluation and in agreement with the results of Sec. V for $N=2$.

(ii) $C_1 \neq C_2$ $(C_1 - C_2 < 0)$: In this case, when the laser is above threshold, the distribution peaks at

$$n_1 = (A/BC_1)(A - C_1) \sim A^2/BC_1 \text{ as } A/C_1 \to \infty$$

and n_2 is very much smaller than n_1. Then we can neglect n_2 compared to n_1 in the denominator of Eq. (22), so that the distribution approaches a Poissonian in n_1. Two conclusions follow: (a) The stronger mode does approach a coherent state, and (b) Both modes become independent and cross correlations tend to die out much above threshold.

It is easy to see that, when both modes are below threshold, n_1 and n_2 can be neglected compared to A/B (typically $\sim 10^6$) and that the distribution $p_s(n_1, n_2)$ goes over into a product of two Bose-Einstein distributions. Near threshold it can be shown, by using Stirrling approximation for the factorials, that p_s has the same form as derived by Smirnov and Zhelnov[6] using a fourth-order quantum-mechanical treatment. As we shall see, these are also the predictions of the third-order semiclassical Fokker-Planck theories.[5,11] In Sec. IV we establish the connection with these theories, which give an excellent account of the statistical properties near threshold.

IV. FOKKER-PLANCK APPROACH

In semiclassical theories intensities serve as field variables rather than photon numbers. It is therefore appropriate that, in order to establish a connection with earlier treatments, we convert Eq. (20) into an equation which corresponds to the Fokker-Planck equation for the classical intensity distribution function. This can be done by using the coherent-state representation of the field, which allows a representation of $\rho_{n_1 n_2; n_1' n_2'}$ in terms[9] of the quasiprobability function $\Phi(\beta_1, \beta_1^*, \beta_2, \beta_2^*)$. The two are related by

$$\rho_{n_1 n_2; n_1' n_2'} = \int \Phi(\beta_1, \beta_1^*, \beta_2, \beta_2^*) e^{-|\beta_1|^2 - |\beta_2|^2} \times \frac{\beta_1^{n_1} \beta_1^{*n_1'} \beta_2^{n_2} \beta_2^{*n_2'}}{(n_1! n_2! n_1'! n_2'!)^{1/2}} d^2\beta_1 \, d^2\beta_2 . \quad (28)$$

Here β_1 and β_2 are the complex field amplitudes of the two modes in the coherent states of the field. Since our interest is confined to the steady-state distribution, we can simplify the procedure of converting Eq. (20) into an equation for Φ considerably. Noting that in the steady state Φ will be independent of the phases of the two modes, we obtain the following relation for the diagonal elements of the density matrix in the steady state:

$$p_s(n_1, n_2) = \int P(I_1, I_2) e^{-I_1 - I_2} \frac{I_1^{n_1}}{n_1!} \frac{I_2^{n_2}}{n_2!} dI_1 \, dI_2 , \quad (29)$$

where $I_i = |\beta_i|^2$ $(i = 1, 2)$ are the associated intensities and $P(I_1, I_2)$ which is simply related to Φ, corresponds to the semiclassical intensity distribution function. We also introduce an auxiliary function $\overline{W}(I_1, I_2)$ by[12]

$$W_s(n_1, n_2) = \int \overline{W}(I_1, I_2) e^{-I_1 - I_2} \frac{I_1^{n_1}}{n_1!} \frac{I_2^{n_2}}{n_2!} dI_1 \, dI_2 , \quad (30)$$

with

$$[1 + (B/A)(n_1 + n_2 + 2)] W_s(n_1, n_2) = p_s(n_1, n_2) . \quad (31)$$

Using Eqs. (28)–(31) we find that Eq. (20) is equivalent to the set of coupled differential equations

$$-(I_1 + I_2)\overline{W} + \left(I_1 \frac{\partial}{\partial I_1} + I_2 \frac{\partial}{\partial I_2}\right)\overline{W} + \left(\frac{C_1}{A} I_1 + \frac{C_2}{A} I_2\right) P = 0 , \quad (32)$$

$$\left[1 - \frac{B}{A}\left(I_1 \frac{\partial}{\partial I_1} + I_2 \frac{\partial}{\partial I_2} - I_1 - I_2\right)\right] \overline{W} = P . \quad (33)$$

These equations can be solved for $\overline{W}$ by eliminating P; the result is

$$\overline{W}(I_1, I_2) = \text{const}\left(1 - \frac{BC_1}{A^2} I_1 - \frac{BC_2}{A^2} I_2\right)^{A/B} e^{I_1 + I_2}, \quad \frac{BC_1}{A^2} I_1 + \frac{BC_2}{A^2} I_2 \leq 1,$$

$$= 0, \quad \frac{BC_1}{A^2} I_1 + \frac{BC_2}{A^2} I_2 > 1 . \quad (34)$$

The intensity distribution function is then found from Eqs. (33) and (34):

TABLE I. $C_2 - C_1 = 0.0$. Variation of $\langle I_1 \rangle$, μ_{11}, and μ_{12} against the ratio $(A - C_1)$ when both modes have equal losses. In this case $\langle I_1 \rangle = \langle I_2 \rangle$ and $\mu_{11} = \mu_{22}$. We have taken $A/B = 10^6$, $A = \mu\,\text{sec}^{-1}$.

$(A-C_1)/A$	$\langle I_1 \rangle \times 10^{-5}$	μ_{11}	μ_{12}
0.02	0.1022	0.3365	−0.3317
0.04	0.2084	0.3341	−0.3329
0.06	0.3192	0.3336	−0.3331
0.08	0.4348	0.3335	−0.3332
0.10	0.5556	0.3334	−0.3332
0.12	0.6818	0.3333	−0.3333
0.14	0.8139	0.3333	−0.3333

$$P(I_1, I_2) = N^{-1}\left(1 - \frac{BC_1}{A^2}I_1 - \frac{BC_2}{A^2}I_2\right)^{A/B-1} e^{I_1+I_2}, \quad \frac{BC_1}{A^2}I_1 + \frac{BC_2}{A^2}I_2 \leq 1,$$
$$= 0, \quad \frac{BC_1}{A^2}I_1 + \frac{BC_2}{A^2}I_2 > 1, \tag{35}$$

where N^{-1} is the normalization constant. Equation (35) is the two-mode analog of the single-mode result.[12,13] Note that if we set $I_2 = 0$, Eq. (35) reduces to the single-mode result.

Near threshold average values of I_1 and I_2 are small and the inequalities $(BC_1/A^2)I_1 \ll 1$, $(BC_2/A^2)I_2 \ll 1$ are satisfied. The factor in front of the exponential in Eq. (35) can be approximated as follows:

$$\left(1 - \frac{BC_1}{A^2}I_1 - \frac{BC_2}{A^2}I_2\right)^{A/B-1} = \exp\left[\left(\frac{A}{B} - 1\right)\ln\left(1 - \frac{BC_1}{A^2}I_1 - \frac{BC_2}{A^2}I_2\right)\right]$$
$$\simeq \exp\left[-\frac{C_1}{A}I_1 - \frac{C_2}{A}I_2 - \frac{B}{2A} \times\left(\frac{C_1^2}{A^2}I_1^2 - \frac{C_2^2}{A^2}I_2^2 - 2\frac{C_1C_2}{A^2}I_1I_2\right)\right]. \tag{36}$$

If we introduce the pump parameters

$$a_i = (A - C_i)(2/AB)^{1/2}, \quad i = 1, 2, \tag{37}$$

and the dimensionless intensities

$$I_i' = (2B/A)^{1/2} I_i, \quad i = 1, 2, \tag{38}$$

it follows from Eqs. (36)–(38) and (35) that

$$P(I_1', I_2') = Q^{-1}\exp\left(\tfrac{1}{2}a_1 I_1' + \tfrac{1}{2}a_2 I_2' - \tfrac{1}{4}I_1'^2 - \tfrac{1}{4}I_2'^2 - \tfrac{1}{2}I_1'I_2'\right), \tag{39}$$

where we have put $C_1 = A = C_2$ in the quadratic terms in the exponent of Eq. (39). This is justified because in the neighborhood of threshold, at $a_i \sim 10$, e.g., $A - C_i \sim 10^{-3} - 10^{-2}$ for $A/B \sim 10^6$ from Eq. (37). The restrictions on P in Eq. (35) can now obviously be removed, for the distribution (39) never really sees the boundary and both I_1, I_2 can be taken to run from 0 to ∞. Distribution (39) has the same form as the intensity distribution function in the third-order semiclassical theories of Fokker-Planck type.[5] In particular this is the same as the function derived by M-Tehrani and Mandel for a two-mode ring laser if we put $\xi = 1$ in their expression.[11] It is seen that our case corresponds to two neutrally coupled modes (the coupling constant ξ being 1). At this point a few comments are in order. We have considered a homogeneously broadened three-level gain medium. In principle both standing and running wave fields can be incorporated into the theory by defining g_i appropriately, but the coupling would never exceed unity. Actually the choice $g_1 = g_2 = g$ led us to the upper limit $\xi = 1$. Basically, the system that we have considered is a system of two weakly coupled modes. If $g_1 \neq g_2$, coupling would be smaller and would be even smaller still for an inhomogeneously broadened medium. It has been pointed out that in the homogeneously broadened ring laser with two-level atomic medium, ξ can have values greater than unity and this has a significant influence on the statistical properties of the radiation field.[14] To search for the reason why this does not happen in the present case, we go back to Eq. (12). We note that for two-level atoms additional terms like $\sigma_{23'}$ and $\sigma_{32'}$ would also enter in the evaluation of the elements of the reduced density matrix. Such terms are obviously absent in the present case. These additional terms are responsible for the enhanced coupling between the two modes in the former case under certain circumstances.

Since, near threshold, the distribution (35) coincides with those obtained in earlier treatments, nothing new can be said about the statistics of the two modes. High above threshold, Eq. (35) will have to be considered. Unfortunately, it is not possible to express the moments of the light intensities in terms of simple functions except in the case of equal losses (see Sec. V), although they can be expressed in terms of an infinite series of degenerate hypercometric functions of one variable which is hardly of any use. It is, however, possible to evaluate them numerically. Tables I and II show the variation of $\langle I_1\rangle$ and $\langle I_2\rangle$ against the ratio $(A - C_1)/A$. When $C_1 = C_2$, it is found that the average value of both I_1 and I_2 increases as $A(A - C_1)/2BC_1$. For unequal losses $(C_1 - C_2 < 0)$, the average intensity $\langle I_1\rangle$ of the stronger mode grows as $A(A - C_1)/BC_1$, whereas the intensity of the weaker mode approaches a constant value.

More interesting is the behavior of the relative intensity fluctuations (Tables I and II). For equal losses, both $\mu_{11} = \langle(\Delta I_1)^2\rangle/\langle I_1\rangle^2$ and $\mu_{22} = \langle(\Delta I_2)^2\rangle/\langle I_2\rangle^2$ approach the value $\frac{1}{3}$ with increasing gain. Actually these quantities are already close to this value when the laser is only 2% above threshold, which corresponds roughly to gain parameter values $a_1 = a_2 \simeq 30$. Thus fluctuations do not die away even in the high-gain limit and neither beam approaches a coherent state, which is regarded to be the characteristic of laser light much above threshold. Similarly the cross correlation $\mu_{12} = \langle\Delta I_1 \Delta I_2\rangle/\langle I_1\rangle\langle I_2\rangle$ (Tables I and II) has a negative value and

TABLE II. $C_2-C_1=0.005$. Variation of $\langle I_1\rangle$, $\langle I_2\rangle$, μ_{11}, μ_{22}, and μ_{12} against $(A-C_1/A)$ when the second mode is a little more lossy. The parameters for this table are the same as for Table I.

$(A-C_1)/A$	$\langle I_1\rangle\times10^{-5}$	$\langle I_2\rangle\times10^{-3}$	$\mu_{11}\times10^2$	μ_{22}	$\mu_{12}\times10^2$
0.02	0.2021	0.1959	0.2543	1.0004	−0.9750
0.04	0.4147	0.1919	0.0603	1.0005	−0.4655
0.06	0.6364	0.1879	0.0255	1.0006	−0.2971
0.08	0.8677	0.1839	0.0137	1.0007	−0.2133
0.10	1.1093	0.1799	0.0084	1.0009	−0.1632
0.12	1.3618	0.1759	0.0055	1.0011	−0.1300
0.14	1.6261	0.1719	0.0039	1.0013	−0.1064

approaches $-\frac{1}{3}$ with increasing gain. This negative correlation is a result of mode competition and the fact that this approaches a constant value shows competition between the two modes persists no matter how large the gain is.

For unequal losses the situation changes drastically. In this case the intensity fluctuations of the stronger mode (1) do go to zero rapidly above threshold, whereas that of the weaker mode approach unity. These conclusions are in agreement with the qualitative inferences drawn in Sec. II. It is remarkable to note that M-Tehrani and Mandel[11] predicted the same limiting values for various moments, starting from a Fokker-Planck equation based on a third-order theory. We emphasize, however, that those predictions are for a two-mode ring laser in an inhomogeneously broadened medium but, as noted before, the terms that cause increased coupling in a homogeneously broadened two-level system are absent in the Doppler limit. These terms are already absent in the present case because we have considered two nondegenerate lower levels. The conclusion that the predictions of the third-order semiclassical theories hold even in the high-intensity regime is not as surprising as it may seem, for it is the ratio of certain quantities predicted by earlier theories that continue to hold even beyond the range of the validity of these theories. There are, of course, quantitative differences between the predictions of the two types of theories, e.g., the two intensities grow faster than predicted by third-order theories and the absolute values of various quantities differ (see Sec. V). Much more interesting things happen near threshold and we refer the reader to Ref. 11 for a discussion of the statistics near threshold. Finally, we expect that in the limit of high intensities the role of fluctuations will become negligibly small and the results of the deterministic semiclassical and quantum treatments would agree. In the semiclassical theories, the relatively more interesting quantities are the mode intensities. We "derive" the semiclassical equations of motion for the two-mode intensities in order to facilitate a comparison with semiclassical perturbative treatment of Najmabadi, Sargent, and Hopf.[7] The details of the derivation and comparison are given in the Appendix.

V. N-MODE LASER

The treatment given above will now be extended to the case of N coupled modes. This problem for $N>2$ seems to us of only academic interest, for it seems unlikely that the conditions for the realization of such a system can be met. Nevertheless, it does bring out some useful features of multimode lasers. We consider a gain medium consisting of atoms having one pump level and N lower levels. It is assumed that the upper level is connected to each of the lower levels by a dipole transition and that each transition corresponds to a distinct mode of the electromagnetic field. Then under the conditions of exact resonance for all the modes and equal interaction strengths, the steady-state photon distribution will be given by a straightforward generalization of Eq. (22):

$$p_s(n_1,n_2,\ldots,n_N)=Z^{-1}\prod_{i=1}^{N}\left(\frac{A^2}{BC_i}\right)^{n_i}\Big/\Gamma\left(\frac{A}{B}+\sum_{i=1}^{N}n_i+N\right), \tag{40}$$

where Z is the normalization constant formally expressible in terms of the degenerate hypergeometric series in N variables $\{A^2/BC_i\}$. In writing Eq. (40) we assumed independent losses for the modes. It is possible to calculate factorial moments which come out in terms of degenerate hypergeometric series of higher orders. We shall give the expression for the most general case:

$$\langle n_1(n_1-1)\cdots(n_1-\mu+1)n_2(n_2-1)\cdots n_N(n_N-1)\cdots(n_N-\mu_{N+1})\rangle$$
$$=\prod_{i=1}^{N}\left(\frac{A^2}{BC_i}\right)^{\mu_i}\frac{\Phi_N(\mu_1+1,\mu_2+1,\ldots,\mu_N+1;A/B+N-1+\sum\mu_i;\{A^2/BC_i\})}{\Phi_N(1,1,\ldots,1;A/B+N-1;\{A^2/BC_i\})}. \tag{41}$$

For equal losses, these expressions simplify considerably and all the moments are expressible in terms of an ordinary hypergeometric series of the single variable A^2/BC. We shall discuss this case in terms of the intensity distribution function.

The mode-intensity distribution function corresponding to Eq. (40) is

$$P(I_1,I_2,\ldots,I_N)=Q^{-1}\left(1-\sum_{i=1}^{N}\frac{BC_i}{A^2}I_i\right)^{A/B-1}\exp\left(\sum_{i=1}^{N}I_i\right),\quad \sum_{i=1}^{N}\frac{BC_i}{A^2}I_i\le 1,$$
$$=0,\qquad \sum_{i=1}^{N}\frac{BC_i}{A^2}I_i>1, \tag{42}$$

which is the N-mode quantum version of the single-mode intensity distribution. It seems difficult to extract much information from Eq. (42) in general. But for equal losses considerable simplification occurs. If we write the intensity I_i of the ith mode in terms of the real and imaginary parts x_i, y_i of the corresponding field

$$I_i=x_i^2+y_i^2$$

and make use of the hyperspherical coordinates[10b, 15]

$$\begin{aligned}
x_1&=r\cos\theta_1,\\
y_1&=r\sin\theta_1\cos\theta_2,\\
x_2&=r\sin\theta_1\sin\theta_2\cos\theta_3,\\
&\;\;\vdots\\
x_N&=r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\cos\phi,\\
y_N&=r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\sin\phi,
\end{aligned}\tag{43}$$

with $0\le r\le\infty$, $0\le\theta_i\le\pi$ $(i=1,2N-2)$, and $0\le\phi\le 2\pi$, the steady-state distribution (42) can be written as

$$p(r)=Q^{-1}\left(1-\frac{BC}{A^2}r^2\right)^{A/B-1}\exp(r^2)\quad \frac{BC}{A^2}r^2\le 1,$$
$$=0,\qquad \frac{BC}{A^2}r^2>1, \tag{44}$$

where

$$Q=2\left(\frac{\pi A^2}{BC}\right)^N B\left(N,\frac{A}{B}\right)\Phi\left(N,\frac{A}{B}+N,\frac{A^2}{BC}\right). \tag{45}$$

Using Eq. (44) to evaluate moments of the intensities we obtain

$$\langle I_i\rangle=\frac{A^2}{BC(N+A/B)}\frac{\Phi(N+1;A/B+N+1;A^2/BC)}{\Phi(N;A/B+N;A^2/BC)}$$
$$\sim A^2/NBC \text{ as } A/C\to\infty,\quad i=1,2,\ldots,N, \tag{46}$$

$$\langle I_i^2\rangle=\frac{(A^2/BC)^2}{(N+A/B+1)(N+A/B)}\times\frac{\Phi(N+2;A/B+N+2;A^2/BC)}{\Phi(N;A/B+N;A^2/BC)}$$
$$\sim 2\frac{(A^2/BC)^2}{N(N+1)}\text{ as } A/C\to\infty,\quad i=1,2,\ldots,N, \tag{47}$$

$$\langle I_iI_j\rangle=\tfrac{1}{2}\langle I_i^2\rangle$$
$$\sim\frac{(A^2/BC)^2}{N(N+1)}\text{ as } A/C\to\infty,$$
$$i,j=1,2,\ldots,N;\ i\ne j. \tag{48}$$

Using Eqs. (46)–(48) and denoting $\Delta I_i=I_i-\langle I_i\rangle$, we obtain for large A/C

$$\frac{\langle(\Delta I_i)^2\rangle}{\langle I_i\rangle^2}\sim\frac{N-1}{N+1}, \tag{49a}$$

$$\frac{\langle(\Delta I_i)(\Delta I_j)\rangle}{\langle I_i\rangle\langle I_j\rangle}\sim-\frac{1}{N+1},\quad i\ne j, \tag{49b}$$

$$\frac{\langle(\Delta I_i)(\Delta I_j)\rangle}{[\langle(\Delta I_i)^2\rangle\langle(\Delta I_j)^2\rangle]^{1/2}}\sim-\frac{1}{N-1},\quad i\ne j. \tag{49c}$$

As noted earlier, interesting things happen near threshold. It can be shown that near threshold Eq. (44) reduces to the N-mode problem discussed by Hioe.[15] It is significant to note that in the limit of large gains absolute values of various moments given by Eqs. (46)–(48) differ from the asymptotic values for large $a=(A-C)(2/AB)^{1/2}$ given by him, but the normalized fluctuations have the same limiting values. We shall not discuss the N-mode case any further and refer the interested reader to Ref. 15.

VI. SUMMARY

We have investigated the statistics of the optical field emitted from a two-mode laser in a homogeneously broadened medium consisting of three-level atoms via a generalization of the Scully and Lamb procedure. It is found that our system corresponds to a system of neutrally coupled modes. For equal losses the relative intensity fluctuations of both modes approach the finite limit $\frac{1}{3}$, which is a reflection of the fact that competition between modes persists even at high intensities. A comparison with earlier treatments near threshold shows that these are also the predictions of some of these treatments. Our analysis then shows that some of these predictions extend up to arbitrarily high-gain levels.

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APPENDIX

Here we derive the equations of motion for the intensities of the two modes when the quantum correlations are neglected. It follows from Eq. (19) that the mean number of photons in the first mode, i.e.,

$$\langle n_1\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} n_1 p(n_1, n_2), \tag{A1}$$

satisfies the following equation

$$\frac{d\langle n_1\rangle}{dt} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left(\frac{A(n_1+1)}{1+(B/A)(n_1+n_2+2)} - C_1 n_1\right) \times P(n_1, n_2). \tag{A2}$$

If we neglect the correlations, i.e., we assume

$$\langle n_i^r\rangle = \langle n_i\rangle^r, \quad i=1,2, \tag{A3}$$

$$\langle n_1^{r_1} n_2^{r_2}\rangle = \langle n_1\rangle^{r_1}\langle n_2\rangle^{r_2}, \tag{A4}$$

then Eq. (A2) reduces to

$$\frac{d\langle n_1\rangle}{dt} = \frac{A\langle n_1\rangle}{1+(B/A)(\langle n_1\rangle+\langle n_2\rangle)} - C_1\langle n_1\rangle. \tag{A5}$$

In deriving Eq. (A5) we have assumed $\langle n_i\rangle \gg 1$. In a similar manner, we obtain

$$\frac{d\langle n_2\rangle}{dt} = \frac{A\langle n_2\rangle}{1+(B/A)(\langle n_1\rangle+\langle n_2\rangle)} - C_2\langle n_2\rangle. \tag{A6}$$

In order to show a comparison with the existing third-order perturbative semiclassical treatment, we consider the case $(B/A)\langle n_i\rangle \ll 1$. Under this situation Eqs. (A5) and (A6) can be written

$$\frac{dI_1}{dt} = (A - C_1)I_1 - B(I_1^2 - I_1 I_2), \tag{A7}$$

$$\frac{dI_2}{dt} = (A - C_2)I_2 - B(I_2^2 - I_1 I_2), \tag{A8}$$

where we have made the correspondence $I_i \leftrightarrow \langle n_i\rangle$ for $i=1,2$. It is apparent from Eqs. (A7) and (A8) that the coupling constant $\xi = 1$ which is in strict agreement with the results of the semiclassical theory of Najmabadi, Sargent, and Hopf[7] in the case of homogeneously broadened running-wave two-mode laser.

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