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Beyond the mean-field theory of dispersive optical bistability

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The problem of dispersive optical bistability has so far been treated only in the mean-field approximation. A rigorous justification of the mean-field theory can only be obtained from exact solutions of the steady-state Maxwell-Bloch equations which retain the spatial dependence of the field. In this paper exact analytic solutions are presented for these equations. The authors demonstrate that the mean-field equation connecting the input and the output fields follows naturally from these solutions in the limits $T \to 0$, $\delta_F \to 0$, and $\alpha L \to 0$ for the mirror transmission coefficient, the detuning of the field from the cavity resonance, and the linear absorption, respectively, with $\alpha L/2T$ and $\delta_F L/2cT$ remaining finite. The results are illustrated with the help of graphs showing the output versus input intensity for different values of the relevant parameters. The effect of these parameters on the phase shift of the output field is also displayed.

I. INTRODUCTION

The Maxwell-Bloch equations provide a basis for theoretical treatments of a large variety of optical phenomena. One of the interesting phenomena for which a semiclassical theory has been developed on the basis of these equations is optical bistability. The prediction of optical bistability by Szöke *et al.*[1] and McCall[2] was followed by the experiments of Gibbs, McCall, and Venkatesan,[3] who used a Fabry-Perot cavity containing sodium atoms on which a dye-laser beam tuned near resonance with both the atoms and the cavity was incident. Monitoring the input and output beams, they found clear evidence of optical bistability and hysteresis in the behavior of the output beam with respect to the input beam.

Theories explaining different aspects of optical bistability have since been developed by various authors. The semiclassical theories[2,4-9] are quite adequate for a description of the averaged field, i.e., if fluctuations are ignored. The electromagnetic field must be quantized, or versions of the semiclassical theory augmented by the addition of noise sources have to be employed if the fluctuations are to be accounted for.[10-13] We are concerned only with the semiclassical theory.

Much of the current understanding of optical bistability came as a result of the development of the so-called "mean-field" theories. The mean-field theory of Bonifacio and Lugiato[4(a)] for optical bistability in a ring cavity explained qualitatively the essential features of the phenomenon. They first considered purely absorptive bistability (the laser in resonance with the atoms and the cavity) and obtained steady-state solutions of the Maxwell-Bloch equations in the mean-field limit, i.e., ignoring propagation effects. It is important, however, to be able to justify rigorously the somewhat *ad hoc* approximations of the mean-field theory and to determine under what circumstances it is valid. This question was examined by Bonifacio and Lugiato[4(b)] when they performed an explicit integration of the steady-state Maxwell-Bloch equations, including the space dependence of the field. The exact solution was shown to reduce to the mean-field equation connecting the input and the output fields when $L \to 0$ and $T \to 0$, with $\alpha L/2T = C$. Here α is the linear absorption coefficient, L is the cavity length, and T is the transmission coefficient of the mirrors.

Dispersive effects in optical bistability have also been the subject of several papers. Here the input beam, the atoms, and the cavity may all be detuned with respect to each other. Mean-field theories have been developed for this case by several authors.[4(c),7,14,15] However, exact solutions, including propagation effects, have not been given either numerically or analytically. The purpose of this paper is to give exact analytic solutions to the equations for the field and show the regime of validity of the mean-field equations in the dispersive case.

In Sec. II we consider a ring cavity (Fig. 1) containing homogeneously broadened, two-level atoms on which a laser beam is incident. The beam may be detuned with respect to both the atomic and cavity resonance frequencies. Unlike purely absorptive bistability, where the field is taken to be real, we must account for a complex field amplitude. Proceeding from the Maxwell-Bloch equations for the slowly varying field, polarization, and inversion, we arrive at the z-(space)-dependent, steady-state equations for the real and imaginary parts of the field. These are two coupled, nonlinear, ordinary differential equations. In Sec. III we solve these equations analytically and proceed to calculate the relation between the input

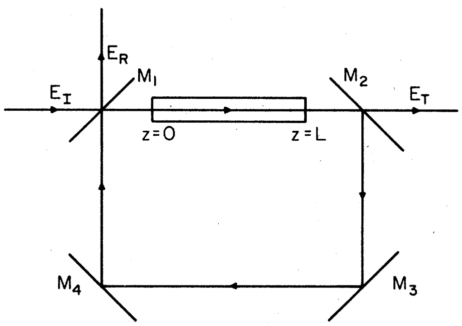

FIG. 1. Ring cavity with saturable absorber. Mirrors M_1 and M_2 have transmission coefficient T, while M_3 and M_4 are totally reflecting.

and the output intensities of the field. In Sec. IV we plot curves to show the exact and mean-field results and discuss the dependence of the exact results on various values of the parameters involved. In Sec. V we show that the exact solutions reduce to the mean-field equations in the limits $T\to 0$, $\delta_F \to 0$, and $\alpha L \to 0$, with $\delta_F L/2cT=\beta$ and $\alpha L/2T = C_1$. Here T is the transmission coefficient of mirrors M_1 and M_2 in Fig. 1, α is the linear absorption coefficient of the medium, L is the cavity length, δ_F is the detuning of the input field from the cavity resonance, and c is the speed of light *in vacuo*.

II. STEADY-STATE EQUATIONS FOR THE FIELD

The basis of our treatment is the set of Maxwell-Bloch equations for an electromagnetic field in a dispersive medium. These equations involve the polarization P, the inversion S, and the field E, where P and E are complex quantities. The constants occuring are the longitudinal and transverse atomic decay rates $\gamma_\parallel$ and $\gamma_\perp$, the detunings $\delta_A=\omega_a-\omega$ and $\delta_F=\omega_f-\omega$ of the laser frequency ω from the atomic and cavity resonance frequencies ω_a and ω_f, the dipole moment p of the atomic transition, and the number of atoms N; ϵ' is the permittivity of the medium. The equations are[16]

$$\dot{P} = -(\gamma_\perp + i\delta_A)P - (i/\hbar)|p|^2 ES\,, \tag{1}$$

$$\dot{S} = -\gamma_\parallel(\tfrac{1}{2}N+S) + (2i/\hbar)(EP^* - E^*P)\,, \tag{2}$$

$$\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{i}{c}\,\delta_F E + \frac{i\omega P}{2\epsilon' c}\,. \tag{3}$$

In the steady state,

$$P = [-i|p|^2/\hbar(\gamma_\perp + i\delta_A)]ES\,, \tag{4}$$

$$S = -\frac{N}{2}\Big/\left[1+\left(\frac{4|p|^2\gamma_\perp |E|^2}{\hbar^2(\gamma_\perp^2+\delta_A^2)\gamma_\parallel}\right)\right]\,, \tag{5}$$

and hence

$$\frac{dE}{dz} = -\frac{i}{c}\delta_F E - \frac{\omega|p|^2 N(\gamma_\perp - i\delta_A)}{4\epsilon' c\hbar(\gamma_\perp^2+\delta_A^2)}\Big/\left(1+\frac{4|p|^2\gamma_\perp|E|^2}{\hbar^2(\gamma_\perp^2+\delta_A^2)\gamma_\parallel}\right). \tag{6}$$

Let

$$\alpha = \omega|p|^2 N\gamma_\perp/4\epsilon' c\hbar(\gamma_\perp^2+\delta_A^2)\,; \tag{7}$$

α is the linear absorption coefficient. We also define

$$F = \frac{2|p|}{\hbar}\left(\frac{\gamma_\perp}{\gamma_\parallel}\right)^{1/2}\frac{E}{(\gamma_\perp^2+\delta_A^2)^{1/2}}\,. \tag{8}$$

Decomposing the rescaled field F into its real and imaginary parts F_1 and F_2, we then obtain

$$\frac{dF_1}{dz} = \frac{\delta_F F_2}{c} - \frac{\alpha F_1 + \delta_A\alpha F_2/\gamma_\perp}{1+F_1^2+F_2^2} \tag{9}$$

and

$$\frac{dF_2}{dz} = -\frac{\delta_F F_1}{c} - \frac{\alpha F_2 - \delta_A \alpha F_1/\gamma_\perp}{1+F_1^2+F_2^2}\,. \tag{10}$$

These are the steady-state equations for the field; they are coupled, nonlinear ordinary differential equations which account for the propagation of the field in the dispersive medium.

With these equations we require the boundary conditions at the ends of the cavity.[4(b)] At the far end of the cavity ($z=L$), taking the output field to be E_T, we have

$$E(L) = E_T/\sqrt{T}\,, \tag{11}$$

while at the input end ($z=0$), the condition is

$$E(0) = \sqrt{T}\,E_I + RE(L)\,, \tag{12}$$

where E_I is the incident field and $R=1-T$.

Rescaling the output and input fields via the definitions

$$X = \frac{2|p|}{\hbar}\left(\frac{\gamma_\perp}{\gamma_\parallel}\right)^{1/2}\frac{E_T}{[T(\gamma_\perp^2+\delta_A^2)]^{1/2}} \tag{13}$$

and

$$Y = \frac{2|p|}{\hbar}\left(\frac{\gamma_\perp}{\gamma_\parallel}\right)^{1/2}\frac{E_I}{[T(\gamma_\perp^2+\delta_A^2)]^{1/2}}\,, \tag{14}$$

we transform the boundary conditions to the simple equations

$$X = F(L) \tag{15}$$

and

$$Y = (1/T)[F(0) - RX]\,. \tag{16}$$

The problem now is reduced to obtaining solutions of Eqs. (9) and (10) with the boundary conditions Eqs. (15) and (16); X and Y are complex fields with $X = X_1 + iX_2$ and $Y = Y_1 + iY_2$.

III. ANALYTIC SOLUTIONS OF STEADY-STATE EQUATIONS

To obtain solutions to Eqs. (9) and (10), we first form the equation between the two physically relevant quantities

$$u(z)=F_1^2(z)+F_2^2(z) \tag{17}$$

and

$$v(z)=F_1(z)/F_2(z)\,. \tag{18}$$

Here $u(z)$ and $v(z)$ are the intensity and the cotangent of the phase at any point z in the medium. We note that from Eqs. (9) and (10)

$$\frac{du}{dz}=\frac{-2\alpha u}{(1+u)} \tag{19}$$

and

$$\frac{dv}{dz}=\left[\left(\frac{\delta_F}{c}-\frac{\delta_A}{\gamma_\perp}\right)(1+v^2)+\frac{\delta_F}{c}u(1+v^2)\right]\Big/(1+u)\,. \tag{20}$$

The equation connecting u and v is thus

$$\frac{du}{dv}=-2\alpha u\Big/\left(\frac{\delta_F}{c}-\frac{\delta_A}{\gamma_\perp}\alpha+\frac{\delta_F}{c}u\right)(1+v^2)\,. \tag{21}$$

This equation is readily integrated, and we obtain a simple relationship between the intensity and phase at any point in the medium:

$$-2\alpha\tan^{-1}v=\left(\frac{\delta_F}{c}-\frac{\delta_A\alpha}{\gamma_\perp}\right)\ln u+\frac{\delta_F}{c}u-\left(\frac{\delta_F}{c}-\frac{\delta_A\alpha}{\gamma_\perp}\right)\ln K\,, \tag{22}$$

where K is a constant of integration determined by the boundary conditions. Since $X=F(L)$,

$$K=(X_1^2+X_2^2)\exp\left[\frac{\delta_F}{c}(X_1^2+X_2^2)\Big/\left(\frac{\delta_F}{c}-\frac{\delta_A\alpha}{\gamma_\perp}\right)\right]\times\exp\left[2\alpha\tan^{-1}\left(\frac{X_1}{X_2}\right)\Big/\left(\frac{\delta_F}{c}-\frac{\delta_A\alpha}{\gamma_\perp}\right)\right]\,. \tag{23}$$

We now obtain the solutions for $u(z)$ and $v(z)$. The equation for $u(z)$, Eq. (19), gives us

$$\ln u(z)+u(z)=-2\alpha z+\ln u(0)+u(0)\,. \tag{24}$$

Substituting for $\ln u(z)$ from Eq. (22) into Eq. (24), we obtain the following relationship between $u(z)$ and $v(z)$ and z:

$$u(z)=K_1+K_2z+K_3\tan^{-1}v(z)\,, \tag{25}$$

where

$$K_1=\left[-\left(\frac{\delta_F}{c}-\frac{\delta_A\alpha}{\gamma_\perp}\right)\Big/\frac{\delta_A\alpha}{\gamma_\perp}\right][\ln u(0)+u(0)-\ln K]\,, \tag{26}$$

$$K_2=\frac{2\gamma_\perp}{\delta_A}\left(\frac{\delta_F}{c}-\frac{\delta_A\alpha}{\gamma_\perp}\right), \tag{27}$$

$$K_3=-(2\gamma_\perp/\delta_A)\,. \tag{28}$$

Equation (20) for $v(z)$ can thus be written as

$$\frac{dv}{dz}=\left(\frac{\delta_F}{c}-\frac{\delta_A\alpha}{\gamma_\perp}+\frac{\delta_F u}{c}\right)(1+v^2)/(1+u)$$
$$=\left(\frac{\delta_F}{c}-\frac{\delta_A\alpha}{\gamma_\perp}+K_1\frac{\delta_F}{c}+K_2\frac{\delta_F}{c}z+K_3\frac{\delta_F}{c}\tan^{-1}v(z)\right)\times[1+v^2(z)]/[1+K_1+K_2z+K_3\tan^{-1}v(z)]\,. \tag{29}$$

Letting

$$\tan^{-1}v(z)=q(z)\,, \tag{30}$$

we obtain

$$\frac{dq}{dz}=\left(\frac{\delta_F}{c}-\frac{\delta_A\alpha}{\gamma_\perp}+K_1\frac{\delta_F}{c}+K_2\frac{\delta_F}{c}z+K_3\frac{\delta_F}{c}q\right)\times(1+K_1+K_2z+K_3q)^{-1}\,. \tag{31}$$

We transform now to a new variable

$$s=1+K_1+K_2z+K_3q\,, \tag{32}$$

and Eq. (31) becomes

$$\frac{ds}{dz}=-(2\alpha s+2\alpha)/s\,, \tag{33}$$

with the solution

$$z=\left(-s/2\alpha-(1/2\alpha)\ln(1-s)\right)_{z=0}^{z}\,. \tag{34}$$

When we take the upper limit to be $z=L$ and use Eqs. (22), (26)–(28), (30), (32), and (34), we obtain, after some tedious algebra, the equation

$$\tan^{-1}v(L)+\frac{1}{2\alpha}\left(\frac{\delta_F}{c}-\frac{\delta_A\alpha}{\gamma_\perp}\right)\left[\ln\left(\frac{u(0)}{K}\right)+u(0)-2\alpha L\right]+\frac{\delta_A}{2\gamma_\perp}u(0)\exp\left\{-\frac{2\gamma_\perp}{\delta_A}\left[\frac{\delta_F}{c}L-\tan^{-1}v(L)-\frac{1}{2\alpha}\left(\frac{\delta_F}{c}-\frac{\delta_A\alpha}{\gamma_\perp}\right)\times\ln\left(\frac{u(0)}{K}\right)-\frac{\delta_F}{2c\alpha}u(0)\right]\right\}=0\,. \tag{35}$$

Defining the constants C_1, C_2, and β by

$$C_1=\alpha L/2T\,, \tag{36}$$

$$C_2=\alpha\delta_A L/2\gamma_\perp T\,, \tag{37}$$

and

$$\delta_F L/2cT = \beta\,, \tag{38}$$

we can show that Eq. (35) becomes

$$2C_2T = 2\beta T - \left(\frac{\beta - C_2}{2C_1}\right)\ln\left(\frac{u(0)}{K}\right) - \frac{\beta}{2C_1}u(0) - \tan^{-1}v(L)$$
$$- \frac{C_2u(0)}{2C_1}\left(\exp\left\{-\frac{2C_1}{C_2}\left[2\beta T - \tan^{-1}v(L) - \left(\frac{\beta - C_2}{2C_1}\right)\ln\left(\frac{u(0)}{K}\right) - \frac{\beta}{2C_1}u(0)\right]\right\} - 1\right).$$

The constant K can now be eliminated by using Eq. (23), and the resulting equation is

$$2C_2T = 2\beta T - \frac{\beta}{2C_1}[u(0) - u(L)] - \frac{(\beta - C_2)}{2C_1}\ln\left(\frac{u(0)}{u(L)}\right)$$
$$- \frac{C_2u(0)}{2C_1}\left(\exp\left\{-\frac{2C_1}{C_2}\left[2\beta T - \frac{\beta}{2C_1}[u(0) - u(L)] - \frac{(\beta - C_2)}{2C_1}\ln\left(\frac{u(0)}{u(L)}\right)\right]\right\} - 1\right). \tag{39}$$

Here $u(L)$ and $v(L)$ are of course given by

$$u(L) = X_1^2 + X_2^2, \quad v(L) = X_1/X_2. \tag{40}$$

Also, taking $z = 0$ in Eq. (22), we obtain

$$v(0) = \tan\left(\frac{C_2 - \beta}{2C_1}\ln\frac{u(0)}{K} - \frac{\beta}{2C_1}u(0)\right),$$

and using Eq. (23), we can rewrite this as

$$v(0) = \tan\left[\tan^{-1}v(L) - \frac{\beta}{2C_1}[u(0) - u(L)] - \frac{\beta - C_2}{2C_1}\ln\left(\frac{u(0)}{u(L)}\right)\right]. \tag{41}$$

We now use the boundary condition Eq. (16) to express the comments Y_1 and Y_2 of the output field in terms of $u(0)$, $v(0)$, $u(L)$, and $v(L)$:

$$Y_1 = \frac{1}{T}\left(\frac{[u(0)]^{1/2}v(0)}{[1 + v^2(0)]^{1/2}} - (1 - T)\,\frac{v(L)[u(L)]^{1/2}}{[1 + v^2(L)]^{1/2}}\right) \tag{42}$$

and

$$Y_2 = \frac{1}{T}\left[\frac{[u(0)]^{1/2}}{[1 + v^2(0)]^{1/2}} - (1 - T)\frac{[u(L)]^{1/2}}{[1 + v^2(L)]^{1/2}}\right]. \tag{43}$$

This last step completes our solution of the problem. Taking a given value of the output intensity $I_x \equiv X_1^2 + X_2^2 = u(L)$, we determine the intensity $u(0)$ via Eq. (39); $v(0)$ is then known from Eq. (41), where we also use the given value of $v(L)$. Equations (42) and (43) then give us the corresponding real and imaginary parts of the input field. The input intensity is simply $I_Y = Y_1^2 + Y_2^2$.

Finally we rephrase Eqs. (39), (41), (42), and (43) in a form which will be convenient for their reduction to the mean-field-theory equation. If we define

$$\gamma = \frac{\beta}{2C_1}[u(0) - u(L)] + \frac{\beta - C_2}{2C_1}\ln\left(\frac{u(0)}{u(L)}\right)$$
$$= \frac{\beta}{2C_1}[u(0) - I_X] + \frac{\beta - C_2}{2C_1}\ln\left(\frac{u(0)}{I_X}\right), \tag{44}$$

then these equations become

$$2C_2T = 2\beta T - \gamma - \frac{C_2u(0)}{2C_1}\left[\exp\left(-\frac{2C_1}{C_2}(2\beta T - \gamma)\right) - 1\right], \tag{45}$$

$$v(0) = \tan[\tan^{-1}v(L) - \gamma]$$
$$= [v(L) - \tan\gamma]/[1 + v(L)\tan\gamma]\,, \tag{46}$$

$$Y_1 = \frac{1}{T[1 + v^2(L)]^{1/2}}\{[u(0)]^{1/2}[v(L)\cos\gamma - \sin\gamma] - (1 - T)v(L)[u(L)]^{1/2}\}\,, \tag{47}$$

$$Y_2 = \frac{1}{T[1 + v^2(L)]^{1/2}}\{[u(0)]^{1/2}[v(L)\sin\gamma + \cos\gamma] - (1 - T)[u(L)]^{1/2}\}\,, \tag{48}$$

where we have substituted for $v(0)$ from Eq. (46) into Eqs. (42) and (43). It is now easy to see that

$$I_y = Y_1^2 + Y_2^2$$
$$= \frac{1}{T^2}\{u(0) + (1 - T)^2 I_x - 2(1 - T)[u(0)I_x]^{1/2}\cos\gamma\}\,. \tag{49}$$

It is therefore clear that the relationship between the input and the output intensities, Eq. (49), does not involve the phase, since both γ in Eq. (39) [which determines $u(0)$] are independent of the phase.

We now proceed to give plots of I_x vs I_y for different values of the parameters C_1, C_2, β, and T. The parameters C_2 and β are a measure of the detuning of the field from the atomic and the cavity resonance frequencies.

IV. OUTPUT VERSUS INPUT

The questions of interest which confront us now concern the variation of the output intensity with different values of the input intensity and the

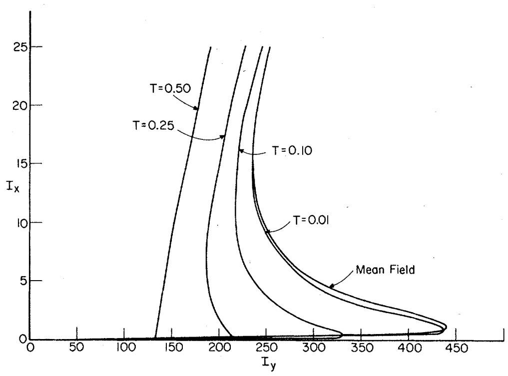

FIG. 2. Output intensity I_x vs input intensity I_y for $C_1=20$, $C_2=1$, $\beta=1$, and different values of T.

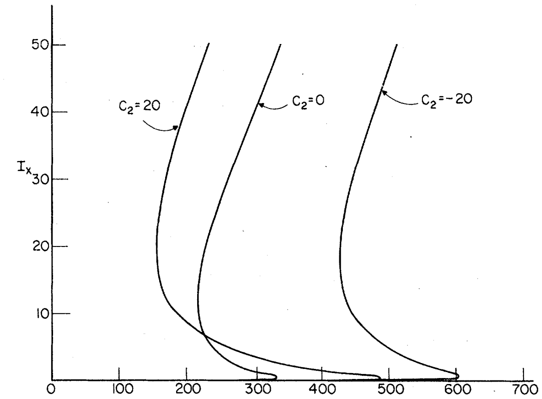

FIG. 4. I_x vs I_y for $C_1=20$, $\beta=1$, $T=0.1$, and different values of C_2.

change in the nature of these curves when we take different detunings δ_A and δ_F (C_2 and β). The structure of the equations constituting the solution, Eqs. (44)–(49), is such that we find it convenient to take a given value of I_x and find $u(0)$ from Eq. (45), and by substituting $u(0)$ in Eq. (49), we find the corresponding I_y.

In plotting our curves, we have kept C_1, which corresponds to the Bonifacio-Lugiato bistability parameter C, at the fixed value of $C_1=20$. Figure 2 shows how I_x varies with I_y for different values of T, with C_2 and β kept constant at $C_2=1$ and $\beta=1$. These values of β and C_2 correspond to $\delta_F=6$ MHz and $\delta_A=1$ MHz (for $T=0.01$, $L=1$ m, and $\gamma_\perp=2\times10^7$ sec^{-1}). The situation is similar to that in the purely absorptive case. For $T=0.0001$ the mean-field-theory curve is almost exactly reproduced. As T is increased, the deviations from the mean-field plot become more and more prominent. The degree of hysteresis decreases, and at $T=0.25$ there is comparatively little sign of hysteresis. For $T=0.5$ no hysteresis and bistability are observed.

In Fig. 3 the effect of changing β is examined for $C_1=20$, $C_2=1$, and $T=0.1$. On increasing β, i.e., the detuning between the laser and the cavity resonance, the bistable behavior becomes less pronounced. We consider both positive and negative values of β, i.e., the laser frequency is below or above the cavity resonance. A forward shift of the hysteresis loop is noticed when we take $\beta=-1$ instead of $+1$.

In Fig. 4 we have kept $C_1=20$, $\beta=1$, and $T=0.1$ constant, while changing C_2. In this case the bistability is more pronounced for larger values of C_2. We note here that actually both C_2 and C_1 depend on δ_A, C_1 through its dependence on α and C_2 through the term $\alpha\delta_A$.

Finally we may study the phase of the output field, $\theta_y=\tan^{-1}Y_2/Y_1$, with respect to the phase

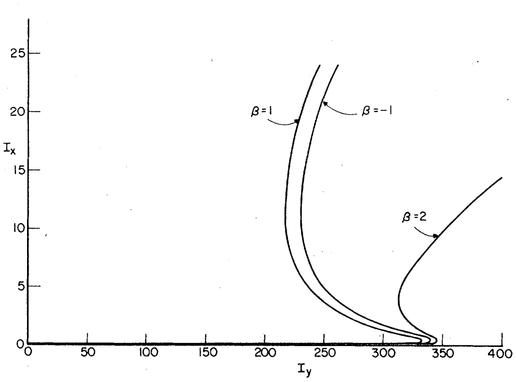

FIG. 3. I_x vs I_y for $C_1=20$, $C_2=1$, $T=0.1$, and different values of β.

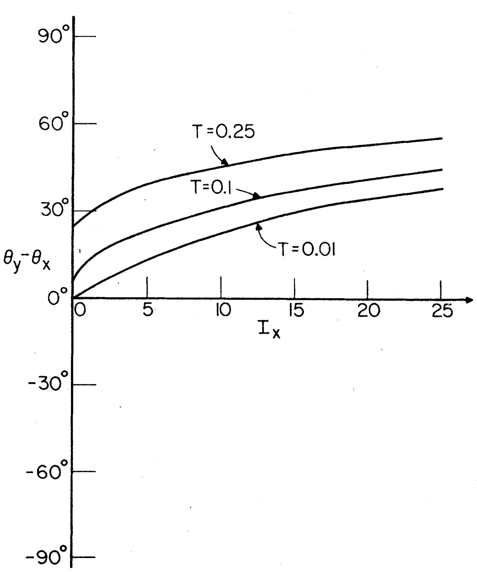

FIG. 5. Phase difference $\theta_y-\theta_x$ output intensity I_x for $C_1=20$, $C_2=1$, $\beta=1$, and different values of T.

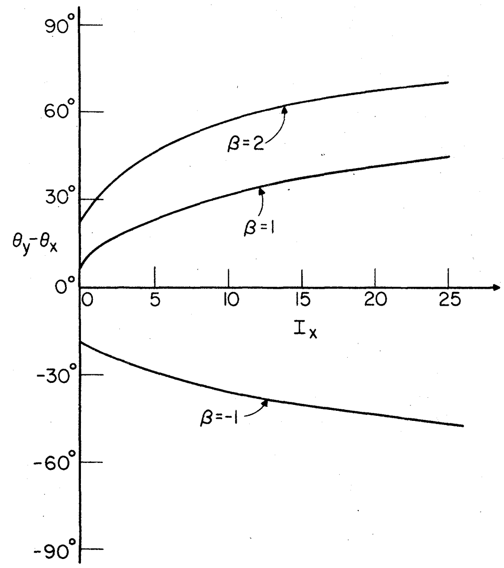

FIG. 6. $\theta_y - \theta_x$ vs I_x for $C_1=20$, $C_2=1$, $T=0.1$, and different values of β.

of the input field, $\theta_x = \tan^{-1} X_2/X_1$. It can easily be shown that for given values of the parameters C_1, C_2, β, and T, $\theta_y - \theta_x$ depends only on I_x. In fact,

$$\begin{aligned}\theta_y - \theta_x &= \tan^{-1} Y_2/Y_1 - \tan^{-1} 1/v(L)\\ &= \tan^{-1}\left[\left(\frac{v(L)Y_2}{Y_1} - 1\right)\Big/\left(\frac{Y_2}{Y_1} + v(L)\right)\right], \end{aligned} \tag{50}$$

and on substituting for Y_1 and Y_2 from Eqs. (47) and (48), we obtain

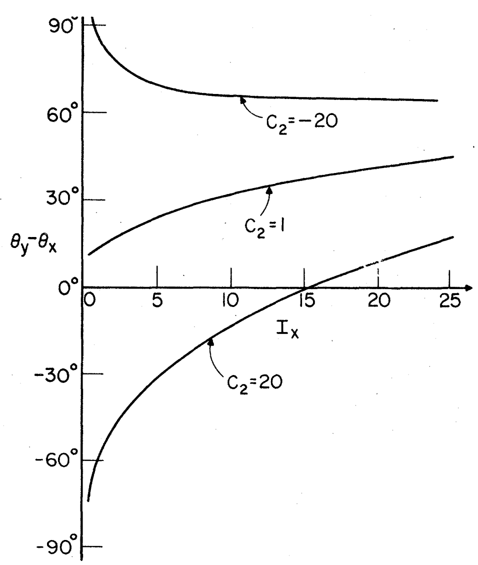

FIG. 7. $\theta_y - \theta_x$ vs I_x for $C_1=20$, $\beta=1$, $T=0.1$, and different values of C_2.

$$\theta_y - \theta_x = \tan^{-1}\left(\frac{[u(0)]^{1/2}\sin\gamma}{[u(0)]^{1/2}\cos\gamma - (1-T)[u(L)]^{1/2}}\right). \tag{51}$$

The right-hand side of (51) is independent of $v(L)$ and hence of θ_x.

We show in Figs. 5–7 how the phase shift $\theta_y - \theta_x$ changes with I_x. In Fig. 5 C_1, C_2, and β are constant while T is varied; in Fig. 6 C_1, C_2, and T are constant while β is varied; and in Fig. 7 C_1, T, and β are constant while C_2 is varied. It is clear that when the output field intensity displays bistability, i.e., makes a discontinuous change, so will the phase of the output field.

In the case $C_2 = \beta = 0$ our results for the output versus input intensity are in very good agreement with the curves of Ref. 4(b). There is of course no change in the output intensity with a change of the input field phase. The phase shift also is zero.

V. REDUCTION TO MEAN-FIELD EQUATION

In this section we undertake to reduce the results embodied in Eqs. (44)–(48) to the mean-field equation. The conditions required for this reduction are $T \to 0$, $\delta_F \to 0$, $\alpha L \to 0$, with $\alpha L/2T = C_1$ and $\delta_F L/2cT = \beta$ remaining finite. Also, the difference between the intensities $u(0)$ and $u(L)$ must be small, which we can see to be true for $T \to 0$ from the plot of $u(L) - u(0) = \epsilon$ vs I_x, shown in Fig. 8.

With these assumptions we find from Eq. (44) that, on keeping only the terms linear in ϵ,

$$\gamma \simeq \beta\epsilon/2C_1 + (\beta - C_2)\epsilon/2C_1 u(L). \tag{52}$$

Substituting this expression for γ in Eq. (45), we obtain

$$2C_2 T \simeq 2\beta T[1 + u(L)] - \left(\frac{\beta}{2C_1} + \frac{\beta - C_2}{2C_1 u(L)}\right)\epsilon[1 + u(L)].$$

It therefore follows that

$$\epsilon \simeq 4C_1 u(L)T/[1 + u(L)]. \tag{53}$$

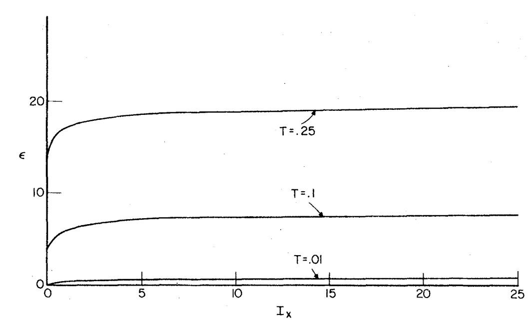

FIG. 8. ϵ vs I_x for $C_1=1$, $\beta=1$, and different values of T.

Now

$$[u(0)]^{1/2} \simeq [u(L)]^{1/2}[1+\epsilon/2u(L)]\,,$$

$$X_2 = [u(L)]^{1/2}/[1+v^2(L)]^{1/2}\,. \qquad (54)$$

Hence, by expanding $\sin\gamma$ and $\cos\gamma$ only to first order in ϵ, Eqs. (47) and (48) result in

$$Y_1 \simeq \frac{X_2}{T}\left[\left(\frac{v(L)}{2u(L)} - \frac{\beta u(L)+\beta-C_2}{2C_1 u(L)}\right)\epsilon + Tv(L)\right], \qquad (55)$$

$$Y_2 \simeq \frac{X_2}{T}\left[\left(\frac{1}{2u(L)} + \frac{v(L)[\beta+\beta u(L)-C_2]}{2C_1 u(L)}\right)\epsilon + T\right]. \qquad (56)$$

By substituting for ϵ from Eq. (53) and for $v(L)$ and $u(L)$ from Eq. (40), Eqs. (55) and (56) lead to

$$Y_1 = X_1 - 2\beta X_2 + 2(C_1X_1 + C_2X_2)/[1+(X_1^2+X_2^2)]\,, \qquad (57)$$

$$Y_2 = X_2 + 2\beta X_1 + 2(C_1X_2 - C_2X_1)/[1+(X_1^2+X_2^2)]\,. \qquad (58)$$

Combining these two equations to form the complex fields X and Y, we obtain

$$Y = X + 2C_1X/(1+|X|^2) + 2i\beta X - 2iC_2X/(1+|X|^2)\,, \qquad (59)$$

which is precisely the equation obtained from the mean-field theory.[4(b),14] It is worthwhile to mention that no constraint has been imposed on δ_A in this derivation.

We have thus been able to provide a rigorous justification of the mean-field equation (59) on the basis of our exact solutions, Eqs. (44)–(48).

VI. CONCLUSION

In conclusion, we have shown how to obtain exact solutions for the steady-state dispersive optical bistability equations for the field in a ring cavity. Graphs demonstrating the relationships between the intensity and the phase of the input and the output fields are shown. Finally the mean-field equation is derived as an approximation of the exact solutions when $T\to 0$, $\delta_F \to 0$, and $\alpha L \to 0$, with $\alpha L/2T = C_1$ and $\delta_F L/2cT = \beta$. Thus the mean-field theory of dispersive optical bistability is rigorously justified. We note that these conditions for the validity of the mean-field theory are precisely the same as those mentioned by Bonifacio and Lugiato in Ref. 4(c).

Note added in proof. The authors are grateful to Professor Bonifacio, Professor Gronchi, and Professor Lugiato for a helpful communication. We were made aware of Refs. 17 and 18. We also wish to mention that Eq. (3) is valid when δ_F is small compared to the spacing of the modes in the ring cavity.

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[1] A. Szöke, V. Daneu, J. Goldhar, and N. A. Kurnit, Appl. Phys. Lett. 15, 376 (1969).
[2] S. L. McCall, Phys. Rev. A 9, 1515 (1974).
[3] H. M. Gibbs, S. L. McCall, and T. N. C. Venkatesan, Phys. Rev. Lett. 36, 1135 (1976).
[4] (a) R. Bonifacio and L. A. Lugiato, Opt. Commun. 19, 172 (1976); (b) Lett. Nuovo Cimento 21, 505 (1978); (c) 21, 517 (1978); (d) Phys. Rev. A 18, 1129 (1978).
[5] P. Meystre, Opt. Commun. 26, 277 (1978).
[6] E. Abraham, R. K. Bullough, and S. S. Hassan, Opt. Commun. 29, 109 (1979).
[7] G. P. Agrawal and H. J. Carmichael, Phys. Rev. A 19, 2074 (1979).
[8] R. Roy and M. S. Zubairy (unpublished).
[9] R. Bonifacio, M. Gronchi, and L. A. Lugiato, Phys. Rev. A 18, 2266 (1978).
[10] C. R. Willis, Opt. Commun. 23, 151 (1977).
[11] G. S. Agrawal, L. M. Narducci, D. H. Feng, and R. Gilmore, in *Coherence and Quantum Optics IV*, edited by L. Mandel and E. Wolf (Plenum, New York, 1978), p. 281.
[12] A. Schenzle and H. Brand, Opt. Commun. 27, 485 (1978).
[13] C. M. Bowden and C. C. Sung, Phys. Rev. A 19, 2392 (1979).
[14] P. Schwendimann, J. Phys. A 12, L39 (1979).
[15] S. S. Hassan, P. D. Drummond, and D. F. Walls, Opt. Commun. 27, 480 (1978).
[16] See, for example, M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, Mass., 1974), and Ref. 4(c).
[17] R. Bonifacio, L. A. Lugiato, and M. Gronchi, in *Laser Spectroscopy IV*, edited by H. Walther and K. W. Rothe (Springer, Berlin, 1979), p. 426.
[18] R. Bonifacio, M. Gronchi, and L. A. Lugiato (unpublished).