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Fluctuations in laser theories

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The influence of fluctuations on two models relevant in solid-state laser theory is discussed. One model is the usual solid-state laser model, the other describes a laser with an unstable saturable absorber. For the usual solid-state laser it is shown that fluctuations act as singular perturbations of a bifurcation. Also derived is a simplified master equation whose stationary solution is in near-perfect agreement with the known exact stationary properties of the laser. For the laser with absorber the fluctuations are shown to act as singular perturbations for some solutions and singular destruction for other solutions. A simplified master equation is also derived for this model, and its stationary solution is compared with the known exact stationary distribution. Here too very good agreement is found whenever comparison is possible.

I. INTRODUCTION

Since the publication of the semiclassical theories of Haken and Sauermann[1] and of Lamb,[2] laser theory has enjoyed the unique status of a trailblazer in physics. This is due to the combination of two facts: (i) a laser is a real device so that any theoretical prediction may (and indeed must) be checked experimentally, and (ii) a laser may be described by a rather simple model which nevertheless exhibits all the basic properties of a highly nonlinear system. The ideas of spontaneous symmetry breaking, the emergence of macroscopic self-organized states, nonequilibrium phase transitions (to name but a few of the concepts with which laser theory deals) have spread in many other fields.[3] The recent work of Haken[4] clearly indicates how concepts developed for the simple laser model can be extended, e.g., to biology, sociology, and hydrodynamics.

For the laser action to take place, two mechanisms are required. The triggering of the laser is due to spontaneous emission, which plays the role of a fluctuating source term. Spontaneous emission is a stochastic source of radiation because it is isotropic in space and has a finite frequency spread. This initial radiation propagates in a resonant cavity which selects one mode (monomode cavity) or a set of discrete modes (multimode cavity) with well-defined energies and propagation directions. These selected modes are then amplified by stimulated emission. Stimulated emission is a coherent source of radiation because it conserves energy and momentum. In this paper we shall consider three theoretical approaches to describe monomode laser action.

The simplest theory is the semiclassical (SC) *theory.* As shown by Shirley,[5] the SC theory can be derived from a fully quantum basis by studying the moments of the atom-field distribution function. Typical moments are the mean field intensity, the mean atomic polarization and inversion. The only assumption to introduce in the moment equations is the complete neglect of spontaneous emission. It was shown independently by Paul[6] and by the author[7] that this unique assumption implies that the field is in a pure coherent state. The SC theory gives asymptotically exact results far above threshold.

The next level of description which will be considered in this paper is the semiquantum (SQ) *theory as developed by the author.* The SQ theory still deals with moments, i.e., with mean values of microscopic operators in order to retain some of the remarkable simplicity which characterizes the SC description. However, spontaneous emission is partially retained so that a more refined description is reached. In particular, a nonzero linewidth is derived implying that the field is no longer a pure coherent state. The SQ theory gives asymptotically exact results far from the transition regions, above and below threshold.

Finally we shall analyze the fully quantum (FQ) *approach to laser theory.* The FQ theory conveys, in principle, all the information about the field properties and incorporates in a quantum-mechanically consistent way the relative influences of spontaneous and stimulated emission processes. The whole art in this case is to find approximation schemes that lead to tractable equations. The two main directions followed by theoretical physicists are: the derivation of birth- and deathlike equations[8] and the derivation of Fokker-Planck-like equations.[9] We shall follow the latter direction in this paper. This paper is divided into two parts. The first deals with solid-state lasers, whereas the second deals with solid-state lasers containing an unstable saturable absorber.

In the first part we briefly recall the results of the SC theory for the solid-state laser and discuss the nature and behavior of the SC stationary intensity. We then proceed to show, via the study of the SQ description, that the fluctuations, (i.e., the spontaneous emission) induce a singular perturbation of the bifurcation which appeared in the SC theory. The theory of singular perturbation of bifurcation was recently set up by Matkowsky and Reiss,[10] and the laser theory provides an elementary example of such an effect. More precisely, it appears that the SC solutions for the intensity are the asymptotes of the SQ solutions. This is no new result but a different way to look at known results whose relevance will appear in the second part of this paper. Finally, we conclude this first part by discussing the FQ description. The whole problem is to derive a manageable equation from an original set of four coupled nonlinear differential equations. A cornerstone in laser theory is the exact stationary solution of this set of four equations which was obtained by Casagrande and Lugiato.[11] For the time-dependent description, the simplest equation is the Fokker-Planck equation with constant diffusion coefficient.[12] Unfortunately, the approximation of constant diffusion coefficient gives poor results, so that more sophisticated equations are required. The latest proposition is that of Casagrande and Lugiato,[13] which gives poor numerical results, as we shall see in this paper. However, it is possible to set up a systematic approximation scheme which leads to an equation whose stationary properties faithfully reproduce the stationary properties of the exact solution of Casagrande and Lugiato.[11] We shall call this equation the simplified master equation; it has much in common with Fokker-Planck equations but does not allow the unambiguous definition of a diffusion constant or function any longer.

In the second part we study the properties of a laser with an unstable saturable absorber. The absorber is called unstable because each of its two levels have short lifetimes. The theory of such a system was recently derived for solid-state lasers.[14] It was shown that three classes of SC solutions occur: the zero solution (I_0) which corresponds to the blackbody radiation below threshold, two solutions ($I_\pm$) which oscillate at the unperturbed frequency when there is perfect tuning, and two solutions ($\tilde{I}_\pm$) which always oscillate at different frequencies than $I_\pm$. Using the SQ description, we show that spontaneous emission has two very different effects on the five SC solutions. It induces a singular perturbation of the bifurcation created between the I_0 and the $I_\pm$ solutions; this is basically the same type of effect as the perturbation described in the first part of this paper. However, the SQ theory (as well as the FQ theory) does not produce solutions that in any way correspond to a generalization or a modification of the two SC solutions $\tilde{I}_\pm$. In this respect, spontaneous emission acts like a singular destruction of bifurcation. This effect is related to an accidental degeneracy of the SC equations which is removed by the presence of spontaneous emission. The SQ theory still preserves one aspect which appeared in the SC study, namely, the occurrence of a hysteresis cycle. This phenomenon was recently observed[15] in gas lasers. Finally, we derive a simplified (FQ) master equation which still reproduces faithfully the results of the known exact stationary solution for which a usuable expression was recently derived.[16]

Our starting point is the generalized von Neumann equation associated with the usual model of a single mode proposed by Haken and his collaborators.[9] For a normal laser the equation is

$$i\hbar\partial_t W(t) = (L_A + L_F + L_{AF} + i\Lambda_A + i\Lambda_F)W(t)\,, \qquad (1)$$

whereas, for a laser with saturable absorber, we have

$$i\hbar\partial_t W(t) = (L_A + L_{A'} + L_F + L_{AF} + L_{A'F} + i\Lambda_A + i\Lambda_{A'} + i\Lambda_F)W_{(t)}\,, \qquad (2)$$

with the definitions:

$$L_A X = H_A X - X H_A\,, \quad H_A = \sum_p \hbar\omega(p) a^\dagger(p) a(p)\,,$$

$$L_F X = \hbar\nu(\partial^*\beta^* - \partial\beta)X\,,$$

$$L_{AF} X = \hbar \sum_p \left[g(p) a^\dagger(p)\beta X + g^*(p) a(p)(\beta^* - \partial)X - g^*(p)\beta^* X a(p) - g(p)(\beta - \partial^*) X a^\dagger(p)\right],$$

$$\Lambda_F X = \hbar\kappa(\partial^*\beta^* + \partial\beta)X\,,$$

$$\Lambda_A = \frac{\hbar}{2}\sum_p \Lambda(p)\,,$$

$$\Lambda(p)X = \gamma_\uparrow(p)\{[a^\dagger(p), Xa(p)]_- + [a^\dagger(p)X, a(p)]_-\} + \gamma_\downarrow(p)\{[a(p), Xa^\dagger(p)]_- + [a(p)X, a^\dagger(p)]_-\}$$
$$- \eta(p)\left[a(p)a^\dagger(p)Xa^\dagger(p)a(p) + a^\dagger(p)a(p)Xa(p)a^\dagger(p)\right],$$

with $\partial \equiv \partial/\partial\beta$ and $\partial^* \equiv \partial/\partial\beta^*$. The operators $L_{A'}$, $L_{A'F}$, and $\Lambda_{A'}$ describe the influence of the passive atoms in the case of a saturable absorber. The structure of these operators is similar to that of the active atoms. To distinguish the two types of atoms, we describe the active atoms by the operators $a^\dagger(p)$ and $a(p)$, the passive atoms by $A^\dagger(p)$ and $A(p)$. The parameters of the passive atoms are those used for the active atoms but with a bar above them (e.g., if σ is the active-atom inversion, $\bar{\sigma}$ will be the passive-atom inversion). (As a rule we shall adhere to the notation used in Ref. 14). In Eqs. (1) and (2), κ stands for the empty cavity damping of the field whose frequency is ν, $g(p)$ measures the interaction between atoms and field and p represents in a compact way all variables needed for the quantum description of the atoms. The three atomic parameters appearing in $\Lambda(p)$ are related to the atomic inversion (σ_p), the polarization $(\gamma_\perp)$ and the atomic inversion $(\gamma_\parallel)$ relaxation times through:

$$\sigma(p) = [\gamma_\uparrow(p) - \gamma_\downarrow(p)]/[\gamma_\uparrow(p) + \gamma_\downarrow(p)] ,$$

$$\gamma_\parallel(p) = \gamma_\uparrow(p) + \gamma_\downarrow(p) ,$$

and

$$2\gamma_\perp(p) = \gamma_\uparrow(p) + \gamma_\downarrow(p) + \eta(p) .$$

II. MONOMODE SOLID-STATE LASER

A. Semiclassical theory

It is well known[5-7] that, in order to recover the traditional working equations of the SC theory, one may simply start from Eq. (1) and study the relevant moments of the operator $W(t)$ under the two factorization assumptions:

$$\langle [\alpha_1 a(p) + \alpha_2 a^\dagger(p)] X(\beta)\rangle = [\alpha_1 \langle a(p)\rangle + \alpha_2 \langle a^\dagger(p)\rangle] \langle X(\beta)\rangle , \quad (2.1)$$

$$\langle a^\dagger(p) a(p) X(\beta)\rangle = \langle a^\dagger(p) a(p)\rangle \langle X(\beta)\rangle , \quad (2.2)$$

where X is an arbitrary function of the field variable $\beta = re^{i\theta}$. Because such a description completely neglects spontaneous emission, the field linewidth vanishes identically, and therefore $\langle |\beta|^2\rangle = |\langle\beta\rangle|^2$. With this equality we may reduce the complexity of the problem to the study of $\langle\beta\rangle$ and the two additional matter equations necessary to obtain a closed set of equations. To avoid difficulties which are irrelevant for the questions we want to discuss, we introduce the oversimplified atomic model: All atoms are equivalent (Einstein solid) and homogeneously distributed. Using the SC assumptions (2.1) and (2.2), we easily derive from Eq. (1):

$$[i(\partial_t + \kappa) - \nu] \langle\beta\rangle = g^* N \langle a\rangle , \quad (2.3)$$

$$[i(\partial_t + \gamma_\perp) - \omega] \langle a\rangle = -g D(t) \langle\beta\rangle , \quad (2.4)$$

$$i(\partial_t + \gamma_\parallel) D(t) = i\sigma\gamma_\parallel + 2(g\langle\beta\rangle\langle a^\dagger\rangle - g^*\langle\beta^*\rangle\langle a\rangle) , \quad (2.5)$$

where N is the number of active atoms and $D(t) = \langle a^\dagger a\rangle - \langle a a^\dagger\rangle$ is the perturbed atomic inversion. We introduce the decompositon $\langle\beta\rangle = E(t) e^{-i\mu(t)}$. In the long time limit (i.e., when $\gamma_\perp t \gg 1$ and $\gamma_\parallel t \gg 1$), the field amplitude and phase are given by the two equations

$$\left(\frac{d}{dt} + \kappa\right) E(t) = \frac{\kappa A E(t)}{[1 + \Delta^2(t) + S E^2(t)]} , \quad (2.6)$$

$$E(t)\left(\frac{d\mu(t)}{dt} - \nu\right) = \frac{\kappa A E(t)\Delta(t)}{[1 + \Delta^2(t) + S E^2(t)]} , \quad (2.7)$$

where $A = |g|^2 N\sigma/\kappa\gamma_\perp$ is the normalized pump parameter, $S = 4|g|^2/\gamma_\parallel\gamma_\perp$ is the atomic saturation parameter, and $\Delta(t) = [\omega - \dot{\mu}(t)]\gamma_\perp^{-1}$ is the detuning function. Because we are mainly concerned with the properties of the field amplitude, we may assume perfect tuning ($\omega = \nu$) without loss of relevant properties of $E(t)$. Perfect tuning implies $\dot{\mu}(t) = \nu$. The stationary equation for the reduced intensity $I = SE^2$ is

$$I^{1/2}\left(1 - \frac{A}{1+I}\right) = 0 , \quad (2.8)$$

whose two solutions are

$$I = \begin{cases} 0, & \text{for } A \leq 1 \\ A - 1, & \text{for } A \geq 1 . \end{cases} \quad (2.9)$$

Although mathematically the solution $I = 0$ is admissible in the whole domain, it is a stable solution only if $A \leq 1$. Hence the point $A = 1$ is a bifurcation point at which the two solutions cross.

B. Semiquantum theory

In order to generalize the SC theory, we keep only the factorization (2.2). In other words we factorize products of atomic and field variables if the atomic operator is a diagonal operator. Using Eq. (1) and the SQ factorization assumption (2.2) as well as the low atomic concentration[17] in active atoms, we easily derive a closed set equations for the true intensity, which is defined as $\langle b^\dagger b\rangle = \langle |\beta|^2\rangle = n(t)$:

$$i(\partial_t + 2\kappa) n(t) = N(g^*\langle a\beta^*\rangle - g\langle a^\dagger\beta\rangle) , \quad (2.10)$$

$$[i(\partial_t + \gamma_\perp + \kappa) + \nu - \omega] g^*\langle a\beta^*\rangle = -|g|^2\{n(t)D(t) + \tfrac{1}{2}[1 + D(t)]\} , \quad (2.11)$$

$$i(\partial_t + \gamma_\parallel) D(t) = i\sigma\gamma_\parallel - 2(g^*\langle a\beta^*\rangle - g\langle a^\dagger\beta\rangle) . \quad (2.12)$$

A glance at the SC equations shows that the only difference which occurs in the set of Eqs. (2.10)–(2.12) is the appearance of an additional term

$\frac{1}{2}|g|^2[1+D(t)]$ in the second SQ equation. That this new term should be related to spontaneous emission is fairly obvious if we compare it with $|g|^2 n(t)D(t)$. The latter contribution is proportional to $|g|^2 n(t)\sigma$, whereas the former is proportional to $|g|^2(1+\sigma)$. Consequently both contributions arise from the atom-field interaction, but one is a linear function of the field intensity and can be positive or negative (corresponding to the possibility of absorption or emission), whereas the other term is independent of the field intensity and is always positive (corresponding to an emission process). Quite obviously, $|g|^2 n(t)D(t)$ will describe stimulated emission, whereas $|g|^2[1+D(t)]$ is related to spontaneous emission. The stationary solution for the reduced intensity $I=Sn$ is (assuming $\gamma_\perp \gg \kappa$):

$$I=\tfrac{1}{2}\{A-1-\tfrac{1}{2}S+[(A-1-\tfrac{1}{2}S)^2+8qS]^{1/2}\}, \quad (2.13)$$

where $q=(1+\sigma)/4\sigma_t=|g|^2 N(1+\sigma)/4\gamma_\perp\kappa$. This solution is a hyperbola whose asymptotes are precisely the two SC solutions (2.9). Furthermore the coefficient q is exactly the constant part of the diffusion coefficient one can derive in a Fokker-Planck equation (see Sec. IIC). We see that spontaneous emission (which is tantamount to random fluctuations in our problem) destroys the degeneracy of the bifurcation point and produces one continuous solution in the whole domain of variation for the pump parameter A. We can still define a transition region by the equality $A=1+\frac{1}{2}S$. As we typically have $S\simeq 10^{-4}$, this corresponds to the same transition region as in the SC theory. Three special values of the intensity can be deduced from (2.13):

$$I\begin{cases}\simeq A-1-\tfrac{1}{2}S\,, & A\gg 1+\tfrac{1}{2}S \quad (2.14)\\ =(2qS)^{1/2}\,, & A=1+\tfrac{1}{2}S \quad (2.15)\\ \simeq 2qS/(1-A)\,, & A\ll 1+\tfrac{1}{2}S\,. \quad (2.16)\end{cases}$$

Because spontaneous emission is taken into account in the SQ theory, there is a difference between $\langle|\beta|^2\rangle$ and $|\langle\beta\rangle|^2$, which indicates the existence of a finite linewidth. This is apparent if we consider the SQ field equations:

$$[i(\partial_t+\kappa)-\nu]\langle\beta\rangle=g^*N\langle a\rangle\,,$$

$$[i(\partial_t+\gamma_\perp)-\omega]\langle a\rangle=-gD(t)\langle\beta\rangle.$$

The difference with the SC scheme is that now the function $D(t)$ is given by Eq. (2.12), which is no longer a function of $|\langle\beta\rangle|^2$, but a function of the true mean photon number $\langle|\beta|^2\rangle$. This drastically modifies the mathematical structure of the equations and, in particular, deeply alters the nature of the nonlinearities. We have shown elsewhere[7] how to analyze the SQ field equations. Let us quote the main result, which is an expression for the linewidth. For the oversimplified atomic model we have:

$$\Gamma=\frac{qS}{I}\left[1-\left(1-\frac{\sigma_t}{\sigma}\right)\frac{\sigma}{\sigma+1}\right]\alpha\,, \quad (2.17)$$

where I is given by (2.13) and α is a function which varies from two well below threshold to one above threshold. Below threshold this expression is well approximated by

$$\Gamma\simeq 2qS/I\,. \quad (2.18)$$

We shall see that there is a close connection between the SQ results and the FQ results, which we now analyze.

C. Fully quantum theory

The object of the FQ theory along the line we shall follow is to study the Glauber distribution of the field defined in the interaction picture by

$$P(\beta;t)=e^{i\mathfrak{L}_F t}\,\mathrm{Tr}\,W(t)\,,$$

where $\hbar\mathfrak{L}_F=L_F$, Tr means the trace over atomic variables, and $W(t)$ is the solution of Eq. (1). In the long time limit this function satisfies a set of two coupled equations which we directly write down [see Ref. 14(a), for instance, for a detailed derivation]:

$$\left[\frac{\partial}{\partial\tau}-\left(1+\frac{1}{\sigma_t}\right)\frac{1}{x}\frac{\partial}{\partial x}x^2\right]P(x,\theta;\tau)=\frac{1}{\sigma_t}\left[\frac{S}{4}\left(\frac{1}{x}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right)-\frac{1}{x}\frac{\partial}{\partial x}x^2\right]C(x,\theta;\tau)\,, \quad (2.19)$$

$$(1+\sigma+x^2)P(x,\theta;\tau)=\left(1+x^2-\frac{Sx}{4}\frac{\partial}{\partial x}\right)C(x,\theta;\tau)\,, \quad (2.20)$$

where we have introduced

$$C(\beta;\tau)=C(x,\theta;\tau)=2e^{i\mathfrak{L}_F t}\,\mathrm{Tr}\,a^\dagger a W(t)\,,$$

and the scaled variables $\tau=\kappa t, x=S^{1/2}r$. We assumed for simplicity perfect tuning. An important result is that of Casagrande and Lugiato,[11] who showed that the exact stationary solution of (2.19) and (2.20) is

$$P(x)=\begin{cases}\mathfrak{N}e^{2x^2/S}\left(\dfrac{1+\sigma}{\sigma_T}-x^2\right)^{[2(1+\sigma_t)/S\sigma_t]-1}, & \text{if } x^2\leqslant\dfrac{1+\sigma}{\sigma_t}\\ 0, & \text{if } x^2>\dfrac{1+\sigma}{\sigma_t}\end{cases} \quad (2.21)$$

TABLE I. Mean stationary intensity vs pump parameter (1) with the solution of Eq. (2.21); (2) with the solution of Eq. (2.23); (3) with the solution of Eq. (2.24); (4) with the solution of Eq. (2.25). The parameters are $\sigma_t = 10^{-2}$, $S = 10^{-4}$.

A	(1)	(2)	(3)	(4)
0.7	1.5284×10^{-2}	1.5284×10^{-2}	1.5497×10^{-2}	1.5711×10^{-2}
0.8	2.1049×10^{-2}	2.1050×10^{-2}	2.1428×10^{-2}	2.1797×10^{-2}
0.9	3.2156×10^{-2}	3.2157×10^{-2}	3.2926×10^{-2}	3.3615×10^{-2}
1.00	5.6690×10^{-2}	5.6690×10^{-2}	5.8426×10^{-2}	5.9663×10^{-2}
1.02	6.4606×10^{-2}	6.4606×10^{-2}	6.6641×10^{-2}	6.7986×10^{-2}
1.04	7.3924×10^{-2}	7.3924×10^{-2}	7.6290×10^{-2}	7.7720×10^{-2}
1.06	8.4792×10^{-2}	8.4792×10^{-2}	8.7509×10^{-2}	8.8987×10^{-2}
1.08	9.7301×10^{-2}	9.7301×10^{-2}	1.0037×10^{-1}	1.0185×10^{-1}
1.10	1.1145×10^{-1}	1.1145×10^{-1}	1.1485×10^{-1}	1.1629×10^{-1}
1.20	2.0054×10^{-1}	2.0054×10^{-1}	2.0466×10^{-1}	2.0569×10^{-1}
1.30	3.0000×10^{-1}	3.0000×10^{-1}	3.0389×10^{-1}	3.0507×10^{-1}
1.40	4.0000×10^{-1}	4.0000×10^{-1}	4.0361×10^{-1}	4.0507×10^{-1}

where $\mathfrak{N}$ is the normalization constant. Using this result we may readily evaluate (numerically) the mean intensity $\langle I\rangle$ and the intensity fluctuation $F(I) = (\langle I^2\rangle - \langle I\rangle^2)/\langle I\rangle^2$ with the usual definition:

$$\langle I^n\rangle = \int_0^\infty dz\, z^n P(z)$$

where $z = x^2 = Sr^2$. The corresponding results are shown in the first column of Tables I and II. They are very useful because they give a reference value with which other results may be compared. For both tables we choose $\sigma_t = 10^{-2}$ and $S = 10^{-4}$.

Eqs. (2.19) and (2.20) do not provide a useful set of time-dependent equations. Indeed let us write Eq. (2.20) as $(1+\sigma+x^2)P(x,\theta;\tau) = R(x)C(x,\theta;\tau)$. If we want to have a closed equation for $P(x,\theta;\tau)$, we have to invert Eq. (2.20) and replace the function $C(x,\theta;\tau)$ in Eq. (2.19) by $R^{-1}(x)(1+\sigma+x^2)P(x,\theta;\tau)$. Except for the stationary problem, this yields a partial differential equation containing derivatives of all orders, up to infinity! Fortunately, a systematic approximation scheme can be set up which will lead to a very good approximate equation. Let $R(x)$ be the operator

$$R(x) = 1 + x^2 - \frac{Sx}{4}\frac{\partial}{\partial x} .$$

Then the inverse operator $R^{-1}(x)$ satisfies the identity

$$R^{-1}(x) = \frac{1}{1+x^2} + \frac{1}{1+x^2}\frac{Sx}{4}\frac{\partial}{\partial x}R^{-1}(x) ,$$

which may be iterated to give

$$R^{-1}(x) = \frac{1}{1+x^2}\sum_{n=0}^{\infty}\left(\frac{Sx}{4}\frac{\partial}{\partial x}\frac{1}{1+x^2}\right)^n .$$

If we now introduce

$$C(x,\theta;\tau) = R^{-1}(x)(1+\sigma+x^2)P(x,\theta;\tau)$$

into Eq. (2.19) and make use of the expansion just derived for $R^{-1}(x)$, we shall have an equation of the form

$$\partial_\tau P(x,\theta;\tau) = \Gamma(x,\theta)P(x,\theta;\tau) ,$$

where

$$\Gamma(x,\theta) = \sum_{n=0}^{\infty} S^n\Gamma(x,\theta;n) .$$

In the lowest approximation we have $\Gamma(x,\theta) \simeq \Gamma(x,\theta;0)$, and the evolution equation becomes

$$\frac{\partial}{\partial\tau}P(x,\theta;\tau) = \frac{1}{x}\frac{\partial}{\partial x}x^2\left(1-\frac{A}{1+x^2}\right)P(x,\theta;\tau) .$$

This equation can be solved by the method of characteristic equations, leading to

$$\frac{dx}{dt} = 2\kappa x\left(1-\frac{A}{1+x^2}\right),$$

which is exactly the SC [Eq. (2.6)] with perfect tuning. Therefore all corrections to $\Gamma(x,\theta;0)$ will be directly related to the influence of spontaneous

TABLE II. Same as in Table I but for the stationary fluctuation of intensity.

A	(1)	(2)	(3)	(4)
0.7	1.9229	1.9231	1.9209	1.9448
0.8	1.8713	1.8713	1.8674	1.8950
0.9	1.7677	1.7676	1.7600	1.7901
1.00	1.5705	1.5706	1.5579	1.5880
1.02	1.5190	1.5190	1.5057	1.5356
1.04	1.4649	1.4649	1.4514	1.4811
1.06	1.4099	1.4098	1.3965	1.4259
1.08	1.3555	1.3555	1.3427	1.3719
1.10	1.3037	1.3038	1.2919	1.3207
1.20	1.1228	1.1229	1.1179	1.1411
1.30	1.0561	1.0561	1.0545	1.0710
1.40	1.0315	1.0315	1.0309	1.0434

emission. The next approximation is to retain the first-order contribution in S in the evolution operator: $\Gamma(x,\theta)\simeq\Gamma(x,\theta;0)+S\Gamma(x,\theta;1)$. This yields what we shall call the simplified master equation:

$$\frac{\partial}{\partial\tau}P(x,\theta;\tau)=\left[\frac{1}{x}\frac{\partial}{\partial x}x^2\left(1-\frac{A}{1+x^2}\right)+\frac{S}{4\sigma_t}\frac{1}{x}\frac{\partial}{\partial x}\frac{x}{1+x^2}\frac{\partial}{\partial x}\frac{1+\sigma+x^2}{1+x^2}+\frac{S}{4\sigma_t}\frac{1+\sigma+x^2}{x^2(1+x^2)}\frac{\partial^2}{\partial\theta^2}\right]P(x,\theta;\tau). \tag{2.22}$$

Because we know the exact stationary solution (2.21) of the coupled Eqs. (2.19) and (2.20), an obvious test for the validity of (2.22) is to study its stationary solution and the corresponding moments in order to compare them with the results derived from the distribution (2.21). The stationary solution of (2.22) is

$$P(x)=\mathfrak{N}(1+x^2)(1+\sigma+x^2)^\alpha\exp[-x^2(x^2\sigma_t+\beta)/S], \tag{2.23}$$

where $\alpha=-1-2\sigma^2(1+\sigma_t)/S$ and $\beta=2(\sigma_t-\sigma-\sigma\sigma_t)$. The mean intensity and intensity fluctuation evaluated numerically by using Eq. (2.23) are shown in the second column of Tables I and II. The agreement between the two sets of results is quite remarkable. It indicates that the expansion in powers of S for the evolution operator is quickly convergent for the choice $S=10^{-4}$. Because reasonable values of S are equal to or smaller than 10^{-4}, we may safely use the expansion that leads to Eq. (2.22). It must be emphasized that Eq. (2.22) is not a Fokker-Planck (FP) equation in the usual sense. Indeed, in order to have a FP equation, one must have an evolution operator with the structure

$$\frac{1}{x}\frac{\partial}{\partial x}x^2E(x)+\frac{1}{x}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}F(x)+\frac{H(x)}{x^2}\frac{\partial^2}{\partial\theta^2},$$

where $E(x)$ is related only to the "systematic" force; in our problem this means that $E(x)$ must be a purely SC contribution, being therefore independent of S (apart from the obvious scaling of the radial variable r). Then one may interpret $F(x)$ and $H(x)$ as the radial and angular diffusion coefficients. Without modifying the SC drift term $1-A/(1+x^2)$, it is impossible to transform the simplified master equation into a conventional FP equation. Hence we cannot define in a clear-cut way a radial diffusion coefficient any longer. Such an attempt was made by Casagrande and Lugiato,[13] who derived a renormalized Fokker-Planck (RFP) equation by neglecting all terms which do not preserve the FP structure. In our notation their RFP is

$$\frac{\partial}{\partial\tau}P(x,\theta;\tau)=\left[\frac{1}{x}\frac{\partial}{\partial x}x^2\left(1-\frac{A}{1+x^2}\right)+\frac{S}{4\sigma_t}\frac{1+x^2+\sigma}{x^2(1+x^2)}\frac{\partial^2}{\partial\theta^2}+\frac{S}{4\sigma_t}\frac{1}{x}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\frac{1+\sigma+x^2}{(1+x^2)^2}\right]P(x,\theta;\tau). \tag{2.24}$$

Comparing this equation with (2.22), we see that a term of order S is missing in the drift coefficient. Although one may argue *a priori* that this term is small compared to the SC drift coefficient, it nevertheless gives a significant deviation from the exact results when the stationary intensity and intensity fluctuation are computed. The third column in Tables I and II display the results obtained by using the stationary solution of (2.24). The real problem in the derivation of a master equation is the relative influence of the nonlinearities in the diffusion coefficients. To have at least some idea of this influence, let us also consider the most drastic approximation, in which the diffusion coefficients are replaced by constants. This is usually assumed to be a good approximation at low intensities. The corresponding FP equation is

$$\frac{\partial}{\partial\tau}P(x,\theta;\tau)=\left[\frac{1}{x}\frac{\partial}{\partial x}x^2\left(1-\frac{A}{1+x^2}\right)+qS\left(\frac{1}{x}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right)\right]P(x,\theta;\tau), \tag{2.25}$$

with $q=(1+\sigma)/4\sigma_t$. The last column in Tables I and II gives the intensity and intensity fluctuations evaluated numerically with the stationary solution of Eq. (2.25). From Tables I and II we conclude that our simplified master equation gives the closest approximation to the exact results, although it is obvious that the absolute difference between the four sets of results is small. We made this comparison between four different equations because the problem is fairly simple and clear for the model we considered in this section. This will not be the case in Sec. III. Let us mention that, for all stationary properties discussed in this paragraph, the effect of detuning may be incorporated most easily by replacing everywhere the polarization relaxation time $\gamma_\perp$ by $\gamma_\perp[1+(\omega-\nu)^2/\gamma_\perp^2]$.

In order to see what the connection is between the three descriptions we have discussed until now, let us derive from Eq. (2.22) two asymptotic forms. Well below threshold the stationary distribution is peaked around the origin $x=0$. Furthermore, it is a domain of low intensity so that the coupling between the field and the atoms may be considered to be weak. Hence we linearize the drift and diffusion coefficients with respect to $|g|^2$ (which is identical to expanding these coefficients around $x=0$). This yields the linearized FP equation:

$$\frac{\partial}{\partial\tau} P(x,\theta;\tau)=\left[\frac{1}{x}\frac{\partial}{\partial x}x^2(1-A)+qS\left(\frac{1}{x}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right)\right]P(x,\theta;\tau), \quad (2.26)$$

whose exact solution is

$$P(x,\theta;\tau)=\sum_{n=0}^{\infty}\sum_{m=-\infty}^{+\infty}A(n,m)\exp[im\theta-\lambda(n,m)\tau]\times x^{|m|}e^{-z^2}L_n^{|m|}(z^2).$$

In this expression of the solution, the constants $A(n,m)$ are determined by the initial condition, the $L_n^{|m|}(z^2)$ are the associated Laguerre polynomials, and

$$z^2=x^2(1-A)/2qS,$$
$$\lambda(n,m)=(2n+|m|)(1-A).$$

The normalized stationary solution is

$$P(x)=\frac{1-A}{2\pi q}\exp-\frac{x^2(1-A)}{2qS}.$$

With these results it is elementary to calculate the stationary intensity, intensity fluctuation, and linewidth, which are given by:

$$\langle I\rangle=2qS/(1-A), \quad F(I)=1,$$
$$\Gamma=\lambda(0,1)=2qS/\langle I\rangle.$$

These results for $\langle I\rangle$ and Γ are in perfect agreement with the corresponding asymptotic expressions deduced from the SQ theory [Eqs. (2.16) and (2.18)]. On the other hand, the SC results in this domain are $\langle I\rangle=0$ and $\Gamma=0$ which are bad approximations.

Another asymptotic description can be deduced from Eq. (2.22) when the laser operates far above its threshold. In this domain the stationary distribution is sharply peaked around its maximum which is located in the vicinity of $A-1(\simeq\langle I\rangle)$. This suggests at once the quasilinearization procedure,[18] in which the evolution operator is linearized around $x_M^2=A-1$. Let $y=x^2-x_M^2$. The quasilinearized FP equation is

$$\frac{\partial}{\partial\tau}P(y,\theta;\tau)=\left(2\frac{A-1}{A}\frac{\partial}{\partial y}y+\frac{S}{\sigma_t}\frac{(A-1)(A+\sigma)}{A^2}\frac{\partial^2}{\partial y^2}+\frac{S}{4\sigma_t}\frac{A+\sigma}{A(A-1)}\frac{\partial^2}{\partial\theta^2}\right)P(y,\theta;\tau), \quad (2.27)$$

and its solution is

$$P(y,\theta;\tau)=\sum_{n=0}^{\infty}\sum_{m=-\infty}^{+\infty}A(n,m)\exp[im\theta-\lambda(n,m)\tau]\times e^{-\beta^2y^2}H_n(\beta y),$$

where the functions $H_n(\beta y)$ are Hermite polynomials $\beta=\sigma_t/[S(1+\sigma_t)]$ and

$$\lambda(n,m)=2n\frac{A-1}{A}+m^2\frac{S}{4\sigma_t}\frac{A+\sigma}{A(A-1)}.$$

The stationary solution is

$$P(y)=\mathfrak{N}e^{-\beta^2y^2}=\mathfrak{N}\exp[-\beta^2(x^2-x_M^2)],$$

where $\mathfrak{N}$ is the normalization constant. The three typical functions we are interested in are easily evaluated by using the results:

$$\mathfrak{N}=2\beta S[\pi^{3/2}\operatorname{erfc}(-z)]^{-1},$$
$$\langle I\rangle=x_M^2\left(1+\frac{e^{-z^2}}{z\sqrt{\pi}\operatorname{erfc}(-z)}\right),$$
$$\langle I^2\rangle=x_M^2\langle I\rangle+\tfrac{1}{2}\beta^2,$$

where $z=\beta x_M^2=\beta(A-1)$. Because $z\gg1$, we may approximate $\langle I\rangle\simeq x_M^2$ so that we finally have

$$\langle I\rangle\simeq A-1, \quad F(I)\simeq\tfrac{1}{2}z^2\ll1,$$
$$\Gamma=\lambda(0,1)=\frac{qS}{x_M^2}\left[1-\left(1-\frac{\sigma_t}{\sigma}\right)\frac{\sigma}{\sigma+1}\right]\ll1.$$

Here too we recover for $\langle I\rangle$ are Γ the asymptotic results of the SQ theory. Furthermore the agreement with SC results is very good. Therefore we may conclude that: (a) below threshold, the SQ theory and the linearized FP equation give asymptotically exact results; (b) above threshold, the SQ and SC theories as well as the quasilinearized FP equation give asymptotically exact results; and (c) in the threshold region, the SQ theory gives a qualitative picture, but only the simplified master equation (2.22) reproduces with sufficient precision the exact result.

Such a conclusion holds only because the normal laser displays a behavior which closely resembles a (out-of-equilibrium) second-order phase transition. It is the uniqueness of the intensity (versus pump parameter) which makes it possible for moment theories like the SC and the SQ theories to give good to accurate predictions.

III. LASER WITH SATURABLE ABSORBER

In this second part we consider the problem of a monomode solid-state laser with saturable absorber. Physically as well as mathematically the difference between this problem and the usual (monatomic) laser is that, besides the nonlinear amplification and linear absorption, we now describe nonlinear absorption. It was guessed by many authors[4,19] that such an addition would induce drastic changes and in particular first-order phase transition-like behavior. However, none realized the full extent of the modifications brought about by the nonlinear absorption, which was only recently discovered.[14(d)]

A. Semiclassical theory

We use as a starting point Eq. (2) and the SC factorization assumptions (2.1) and (2.2). This yields in a direct way the basic equations:

$$[i(\partial_t+\kappa)-\nu]\langle\beta\rangle = Ng^*\langle a\rangle + \bar{N}\bar{g}^*\langle A\rangle ,$$
$$[i(\partial_t+\gamma_\perp)-\nu]\langle a\rangle = -gD(t)\langle\beta\rangle ,$$
$$i(\partial_t+\gamma_\parallel)D(t) = i\gamma_\parallel\sigma + 2g\langle a^\dagger\rangle\langle\beta\rangle - 2g^*\langle a\rangle\langle\beta\rangle^* ,$$
$$[i(\partial_t+\bar{\gamma}_\perp)-\nu]\langle A\rangle = -\bar{g}\bar{D}(t)\langle\beta\rangle ,$$
$$i(\partial_t+\bar{\gamma}_\parallel)\bar{D}(t) = i\bar{\gamma}_\parallel\bar{\sigma} + 2\bar{g}\langle A^\dagger\rangle\langle\beta\rangle - 2\bar{g}^*\langle A\rangle\langle\beta\rangle^* ,$$

where we have already introduced the oversimplified atomic models for the active and passive atoms. For our purpose it is sufficient to consider the stationary solutions of the SC equations, for which we introduce the decomposition

$$\langle\beta\rangle = Ee^{-i\Omega t} .$$

Let us define

$$\Delta = (\Omega-\nu)/\gamma_\perp ,\quad I = SE^2 ,\quad A = \sigma/\sigma_t = |g|^2 N\sigma/\kappa\gamma_\perp ,$$
$$1-C = \bar{\sigma}/\bar{\sigma}_t ,\quad a = \bar{S}/S ,\quad d = \gamma_\perp/\kappa ,$$
$$\bar{d} = \bar{\gamma}_\perp/\kappa ,\quad b = (\gamma_\perp/\bar{\gamma}_\perp)^2 .$$

In terms of these variables the stationary field amplitude and frequency satisfy the set of coupled equations:

$$I^{1/2} = I^{1/2}\left(\frac{A}{1+\Delta^2+I} + \frac{1-C}{1+\Delta^2 b + aI}\right), \tag{3.1}$$

$$I^{1/2}(\Omega-\nu) = -I^{1/2}(\Omega-\nu)\left(\frac{Ad^{-1}}{1+\Delta^2+I} + \frac{(1-C)\bar{d}^{-1}}{1+\Delta^2 b + aI}\right). \tag{3.2}$$

These equations admit three classes of solutions:

(i) Class A: $\begin{cases} I = 0 , \\ \Omega = \nu ; \end{cases}$

(ii) Class B: $\begin{cases} I = I_\pm = \dfrac{1}{2a}\left(a(A-1) - C \pm \{[a(A-1)-C]^2 - 4a(C-A)\}^{1/2}\right), \\ \Omega = \nu ; \end{cases}$

(iii) Class C: $\begin{cases} I = \tilde{I} = f(a,b) , \\ \Omega_\pm = \nu \pm \gamma_\perp [f(b,a)]^{1/2} ; \end{cases}$

with

$$f(a,b) = \frac{1}{a-b}\left[b - 1 + (\bar{d}-d)\left(\frac{1-C}{\bar{d}(1+d)} + \frac{Ab}{d(1+\bar{d})}\right)\right].$$

To be complete we ought to give a full discussion of the various domains of existence of the five solutions and a stability analysis. This is already in itself a huge piece of work and will be published separately.[20] The only result we need for our discussion is that all solutions, except I_-, have a finite stability domain in parameter space. Hence within the SC frame they do exist. The most peculiar property of the class C solutions is that they oscillate at a frequency $\Omega \neq \nu$ although we assumed perfect tuning: $\nu = \omega = \bar{\omega}$. This can be related directly to the nonlinear absorption. Because both the field amplitude and the frequency vary as functions of the pump parameter A, both functions can display first- and second-order phase transition-like behavior. Figs. 1–7 display some typical situations.

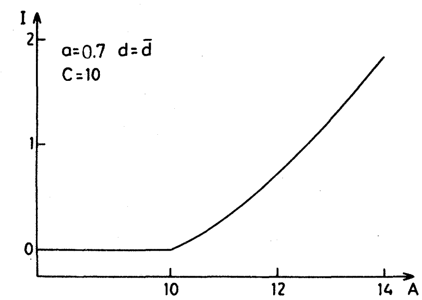

FIG. 1. Semiclassical intensity versus pump parameter. Class A and B solutions. Active atoms saturate more easily than passive atoms.

B. Semiquantum theory

As in the first part of this paper we investigate the SQ description of the laser with saturable absorber by starting with Eq. (2) and using the SQ factorization assumption (2.2) only. The set of

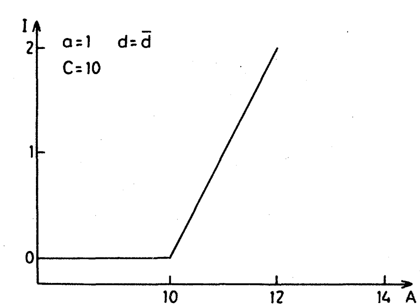

FIG. 2. Same as in Fig. 1 but with equal saturation parameters.

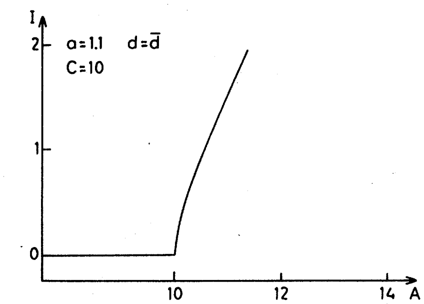

FIG. 3. Same as in Fig. 1 but passive atoms saturate more easily than active atoms ($a > 1$).

equations which determine the intensity is

$$i(\partial_t + 2\kappa)n(t) = N(g^*\langle \alpha\beta^*\rangle - \text{c.c.}) + \bar{N}(\bar{g}^*\langle A\beta^*\rangle - \text{c.c.})\,,$$

$$i(\partial_t + \gamma_\perp + \kappa)g^*\langle \alpha\beta^*\rangle = -\,|g|^2\{n(t)D(t) + \tfrac{1}{2}[1 + D(t)]\}\,,$$

$$i(\partial_t + \bar{\gamma}_\perp + \kappa)\bar{g}^*\langle A\beta^*\rangle = -\,|\bar{g}|^2\{n(t)\bar{D}(t) + \tfrac{1}{2}[1 + \bar{D}(t)]\}\,,$$

$$i(\partial_t + \gamma_\parallel)D(t) = i\gamma_\parallel\sigma - 2(g^*\langle \alpha\beta^*\rangle - \text{c.c.})\,,$$

$$i(\partial_t + \bar{\gamma}_\parallel)\bar{D}(t) = i\bar{\gamma}_\parallel\bar{\sigma} - 2(\bar{g}^*\langle A\beta^*\rangle - \text{c.c.})\,.$$

We recognize in the right-hand side of the second and third equations the occurrence of spontaneous emission through the terms which are independent of the intensity. The existence of these contributions will have far-reaching consequences. Let us introduce two auxiliary parameters which are related to the spontaneous emission gain:

$$F = S(\sigma + 1)/2\sigma_t\,, \quad \bar{F} = S(\bar{\sigma} + 1)/2\bar{\sigma}_t\,.$$

Then the stationary SQ intensity is the solution of

$$I = \frac{AI + F}{1 + I + \frac{1}{2}S} + \frac{(1 - C)I + \bar{F}}{1 + a(I + \frac{1}{2}S)}\,. \tag{3.3}$$

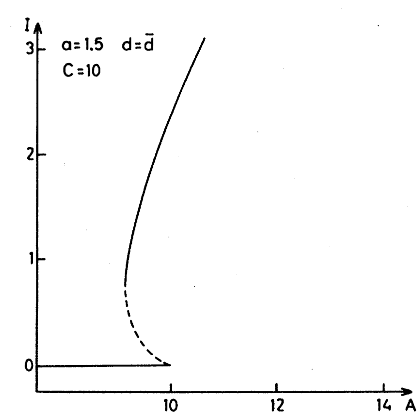

FIG. 4. Example of bistable domain with class A and B solutions.

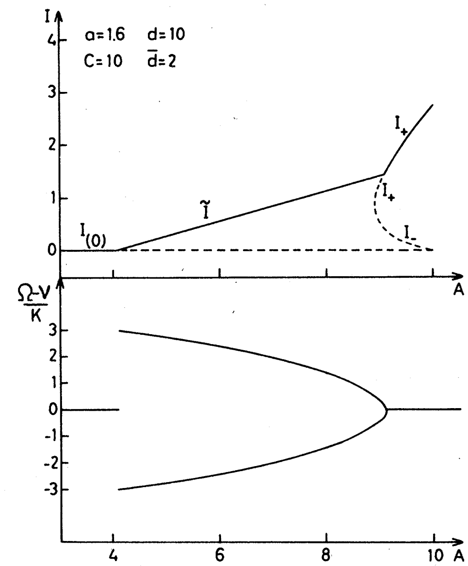

FIG. 5. Solutions of class A, B, and C without hysteresis cycle.

This leads to a cubic equation which can have one or three real roots. They generalize the SC solutions of classes A and B in exactly the same way as the SQ intensity (2.13) generalizes the SC result (2.9): sharp bifurcations are replaced by smooth variations. The SC zero solution is now replaced by a finite solution which, to first order in the spontaneous gain, is given by

$$I(0) \simeq \frac{S}{2}\left(\frac{[(1+\sigma)/\sigma_t + (1+\bar{\sigma})/\bar{\sigma}_t]}{(C - A)}\right).$$

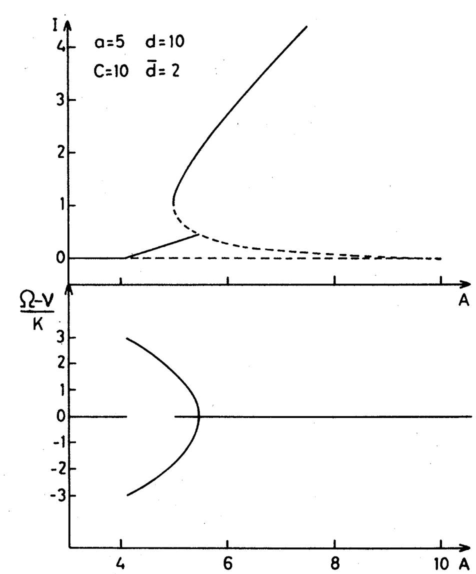

FIG. 6. Solutions of class A, B, and C with simple hysteresis cycles for the intensity and the frequency.

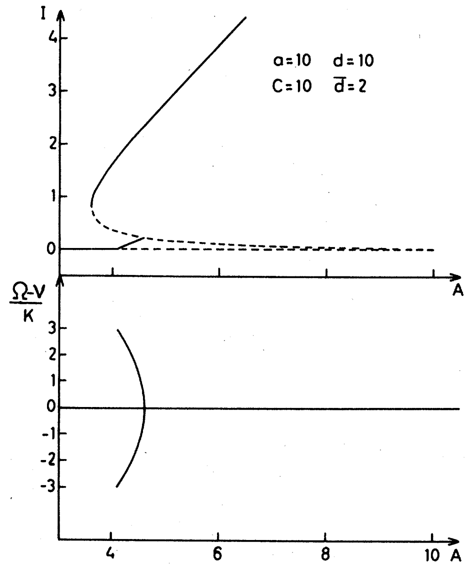

FIG. 7. Solutions of class A, B, and C with composite hysteresis cycle between three solutions for the intensity and simple hysteresis cycle for the frequency.

In this respect spontaneous emission again induces a singular perturbation of the bifurcation. However, there is no room left in this description for the class C solutions. They cannot exist in a true stationary state according to the SQ theory (and the FQ theory as well). Therefore we can say that for class C solutions, spontaneous emission induces a singular destruction of the solutions. To use the language of bifurcation theory,[10] the stationary SC equations can be written in compact vector notation as

$$f(y;A)=0\,, \tag{3.4}$$

and one studies the branching solutions of the nonlinear equation $f=0$. The SQ theory leads formally to the study of the nonlinear equation

$$F(y;A;\delta)=0\,, \tag{3.5}$$

where the additional parameter δ measures the magnitude of spontaneous gain (i.e., we multiply the spontaneous gain in the SQ equations by δ). Hence $\delta=1$ corresponds to the SQ equations. When $\delta\to 0$ Eq. (3.5) reduces to Eq. (3.4) and so do its solutions:

$$y(A;0)=y(A),\quad F(y(A;0);A;0)=f(y(A);A)\,.$$

The converse is not true: to each solution of (3.4) there does not correspond a solution of (3.5). In other words, Eqs. (3.4) (i.e., the SC equations) have an accidental degeneracy which is destroyed by the spontaneous process. This degeneracy can be traced back to the zero linewidth, which implies the equality $\langle|\beta|^2\rangle=|\langle\beta\rangle|^2$. We now show how this degeneracy is responsible for the singular destruction of the class C solutions. Let us consider the SQ equations for the field:

$$[i(\partial_t+\kappa)-\nu]\langle\beta\rangle=Ng^*\langle a\rangle+\bar{N}\bar{g}^*\langle A\rangle\,,$$

$$[i(\partial_t+\gamma_\perp)-\nu]g^*\langle a\rangle=-|g|^2\langle\beta\rangle D(t)\,,$$

$$[i(\partial_t+\bar{\gamma}_\perp)-\nu]\bar{g}^*\langle A\rangle=-|g|^2\langle\beta\rangle\bar{D}(t)\,.$$

The difference with the SC description is that we add to these three equations the two equations for the atomic inversions $D(t)$ and $\bar{D}(t)$ which are functions of $\langle|\beta|^2\rangle$ and not of $|\langle\beta\rangle|^2$. Introducing the decomposition $\langle\beta\rangle=E(t)e^{-i\Omega(t)}$ into the SQ field equations yields in the long time limit:

$$\left(\frac{d}{dt}+\kappa\right)E(t)=E(t)\left(\frac{|g|^2ND(t)}{\gamma_\perp[1+\Delta^2(t)]}+\frac{|\bar{g}|^2\bar{N}\bar{D}(t)}{\bar{\gamma}_\perp[1+\Delta^2(t)b]}\right), \tag{3.6}$$

$$E(t)[\dot{\Omega}(t)-\nu]=-E(t)[\dot{\Omega}(t)-\nu]\left(\frac{1}{\gamma_\perp}\frac{|g|^2ND(t)}{\gamma_\perp[1+\Delta^2(t)]}+\frac{1}{\bar{\gamma}_\perp}\frac{|\bar{g}|^2\bar{N}\bar{D}(t)}{\bar{\gamma}_\perp[1+b\Delta^2(t)]}\right), \tag{3.7}$$

where $\Delta(t)=[\nu-\dot{\Omega}(t)]/\gamma_\perp$. These equations clearly show where the paradox lies: if we (incorrectly) assume that $\langle|\beta|^2\rangle=|\langle\beta\rangle|^2$, then Eqs. (3.6) and (3.7) become closed nonlinear equations for $E(t)$ and $\Omega(t)$; in the stationary case they will have other solutions than the trivial solution $E=0$. These other solutions correspond to class B and C solutions. On the contrary, if we keep $\langle|\beta|^2\rangle\neq|\langle\beta\rangle|^2$, we have a linear differential equation for the field amplitude of the form

$$\frac{dE(t)}{dt}=-\gamma(t)E(t)\,,$$

where $\gamma(t)$ is no longer a function of $E(t)$. The only stationary solution of this equation is the trivial solution $E=0$. This result is consistent with the existence of a linewidth.

Should we conclude that the class C solutions do not exist? Not really. We know from other examples that whenever a stable SC solution is predicted, it corresponds either to a true stationary state or to a long-lived (metastable) state. These metastable states have a finite lifetime because the linewidth is finite, but they decay very slowly because the linewidth is very small. The ultimate test for the existence of stable or metastable class C solutions will be experimental.

C. Fully quantum theory

Because there is such a discrepancy between the predictions of the SC and the SQ theories, it is highly advisable that we seek the FQ description of the laser with saturable absorber. Proceeding along the same lines as in the first part of this paper [see also Ref. 14(a)], we easily get in the long time limit a set of three coupled equations which link the field distribution function (P) and two atomic functions defined by

$$P(\beta;t)=e^{i\mathfrak{L}_F t}\,\mathrm{Tr}W(t)\,,$$
$$C(\beta;t)=2e^{i\mathfrak{L}_F t}\,\mathrm{Tr}a^\dagger aW(t)\,,$$
$$\bar{C}(\beta;t)=2e^{i\mathfrak{L}_F t}\,\mathrm{Tr}A^\dagger AW(t)\,.$$

This master set of equations is

$$\left[\frac{\partial}{\partial\tau}-\left(1+\frac{1}{\sigma_t}+\frac{1}{\bar\sigma_t}\right)\frac{1}{x}\frac{\partial}{\partial x}x^2\right]P(x,\theta;\tau)=\frac{1}{\sigma_t}\left[\frac{S}{4}\left(\frac{1}{x}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right)-\frac{1}{x}\frac{\partial}{\partial x}x^2\right]C(x,\theta;\tau)+\frac{1}{\bar\sigma_t}\left[\frac{S}{4}\left(\frac{1}{x}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right)-\frac{1}{x}\frac{\partial}{\partial x}x^2\right]\bar C(x,\theta;\tau)\,, \tag{3.8}$$

$$(1+\sigma+x^2)P(x,\theta;\tau)=\left(1+x^2-\frac{Sx}{4}\frac{\partial}{\partial x}\right)C(x,\theta;\tau)\,, \tag{3.9}$$

$$(1+\bar\sigma+ax^2)P(x,\theta;\tau)=\left(1+ax^2-\frac{aSx}{4}\frac{\partial}{\partial x}\right)\bar C(x,\theta;\tau)\,, \tag{3.10}$$

for the oversimplified atomic models. Fortunately, here too the exact stationary solution is known. It can be expressed in terms of hypergeometric functions. Because this solution is quite complicated to define, the interested reader is referred to the original paper [Ref. 14(b)] for analytic expressions. By means of this stationary solution, Dembinski *et al.*[16] have been able to compute numerically the mean intensity and intensity fluctuation. They are given in the first columns of Tables III and IV.

In order to get a simplified master equation for the field distribution function, we apply the method devised in the first part of this paper for the derivation of Eq. (2.22). It amounts to inverting Eqs. (3.9) and (3.10) and expressing the functions $C(x,\theta;\tau)$ and $\bar C(x,\theta;\tau)$ in power series of S. These series are introduced in Eq. (3.8) and all zero- and first-order contributions in S are retained. This leads to

$$\frac{\partial}{\partial\tau}P(x,\theta;\tau)=\left[\frac{1}{x}\frac{\partial}{\partial x}x^2\left(1-\frac{A}{1+x^2}-\frac{1-C}{1+ax^2}\right)+\frac{S}{4x^2}\frac{\partial^2}{\partial\theta^2}\left(\frac{1}{\sigma_t}\frac{1+\sigma+x^2}{1+x^2}+\frac{1}{\bar\sigma_t}\frac{1+\bar\sigma+ax^2}{1+ax^2}\right)+\frac{S}{4\sigma_t}\frac{1}{x}\frac{\partial}{\partial x}\frac{x}{1+x^2}\frac{\partial}{\partial x}\frac{1+\sigma+x^2}{1+x^2}+\frac{S}{4\bar\sigma_t}\frac{1}{x}\frac{\partial}{\partial x}\frac{x}{1+ax^2}\frac{\partial}{\partial x}\frac{1+\bar\sigma+ax^2}{1+ax^2}\right]P(x,\theta;\tau)\,. \tag{3.11}$$

The stationary solution of this equation can be derived analytically; because its expression is rather lengthy, we leave its explicit evaluation for the Appendix. Numerical results obtained by using the stationary solution of Eq. (3.11) are reported in the second column of Tables III and IV. Here again we reach a very good agreement between the two groups of results. This indicates that Eq. (3.11) might be a very good approximation of the master set of equations. To give a reference point, the last column of Tables III and IV display the results deduced from the FP approximation with constant diffusion coefficient. This equation is

$$\frac{\partial}{\partial\tau}P(x,\theta;\tau)=\left[\frac{1}{x}\frac{\partial}{\partial x}x^2\left(1-\frac{A}{1+x^2}-\frac{1-C}{1+ax^2}\right)+qS\left(\frac{1}{x}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right)\right]P(x,\theta;\tau)\,, \tag{3.12}$$

where $q=\frac{1}{4}[(1+\sigma)/\sigma_t+(1+\bar\sigma)/\bar\sigma_t]$. The corresponding stationary solution is

$$P(x)=\mathfrak{N}[\,e^{-x^2}(1+x^2)^A(1+ax^2)^{(1-C)/a}]^\epsilon\,, \tag{3.13}$$

with $\epsilon=(2qS)^{-1}$.

Recently Casagrande and Lugiato[13] derived a renormalized Fokker-Planck equation to describe the laser with saturable absorber. We do not try to make any comparison with their results for two reasons. First of all, they approximate their stationary solution by a superposition of Gaussian functions. Hence it is impossible to trace the origin of the descrepancies which arise with the exact results. Secondly, they introduce the parameter $\epsilon=(2qS)^{-1}$ but assume it is a constant. This is in contradiction with the fact that q, and therefore ϵ, is a function of the pump parameter.

A characteristic of all the FQ stationary solutions discussed in this second part of the paper is that there is a correspondence between the

TABLE III. Mean stationary intensity vs pump parameter A (1) with the solution of Eqs. (3.8)–(3.10); (2) with the solution of Eq. (3.11); (3) with the solution of Eq. (3.13). The parameters are $\sigma_t=\bar{\sigma}_t=10^{-2}$, $S=10^{-4}$, $a=4/3$, and $C=20$.

A	(1)	(2)	(3)
18.44	6.6851×10^{-3}	6.6855×10^{-3}	6.7949×10^{-3}
18.46	6.7814×10^{-3}	6.7821×10^{-3}	6.8947×10^{-3}
18.48	6.8929×10^{-3}	6.8936×10^{-3}	6.9976×10^{-3}
18.50	7.6417×10^{-3}	7.6600×10^{-3}	7.1047×10^{-3}
18.52	4.5090×10^{-2}	4.6166×10^{-2}	7.2215×10^{-3}
18.53	2.7244×10^{-1}	2.7928×10^{-1}	7.2934×10^{-3}
18.54	1.1072	1.1222	7.3975×10^{-3}
18.56	1.9983	1.9991	8.0863×10^{-3}
18.58	2.0585	2.0585	1.3531×10^{-2}
18.60	2.0952	2.0952	6.3189×10^{-2}
18.65	2.1830	2.1830	1.9498
18.70	2.2672	2.2672	2.2746

number of extrema they have and the number of SC or SQ solutions. This property was discussed at length previously [Ref. 14(a)]. More precisely, it usually turns out that stable (unstable) states correspond to maxima (minima) of the FQ distribution, i.e., to most (least) probable values. A property of the FQ stationary solutions of Eqs. (3.8), (3.11), and (3.12) is that their extrema correspond to SC solutions of class A and B when they exist. However, there does not seem to be any relation whatsoever between the FQ distributions and the class C solutions. This substantiates our conclusion that, when first-order phase transition-like situations are studied, the SQ theory provides a more faithful picture of the system and that the class C solutions do not exist in the true stationary state of the field.

TABLE IV. Same as in Table III but for the stationary fluctuation of intensity.

A	(1)	(2)	(3)
18.44	1.0491	1.0495	1.0688
18.46	1.0507	1.0609	1.0709
18.48	1.5285	1.5408	1.0731
18.50	2.2380×10^{1}	2.2852×10^{1}	1.1105
18.51	6.6667×10^{1}	6.6951×10^{1}	1.1787
18.52	3.5701×10^{1}	3.4998×10^{1}	1.3770
18.53	6.0795	5.9106	1.9581
18.54	7.9429×10^{-1}	7.7057×10^{-1}	3.6571
18.55	1.0148×10^{-1}	9.8436×10^{-2}	8.4922
18.56	1.5882×10^{-2}	1.5500×10^{-2}	2.1035×10^{1}
18.58	4.1421×10^{-3}	4.1364×10^{-3}	6.8091×10^{1}
18.60	3.7650×10^{-3}	3.7629×10^{-3}	2.8727×10^{1}
18.65	3.3027×10^{-3}	3.3027×10^{-3}	1.3591×10^{-1}
18.70	2.9408×10^{-3}	2.9408×10^{-3}	1.0812×10^{-2}

Simplified but still valuable information can be gained if we consider the asymptotic behavior of Eq. (3.11). Below the transition region we may linearize the simplified master equation around the origin which is the most probable stationary value of x. This leads to

$$\frac{\partial}{\partial\tau}P(x,\theta;\tau)=\left[\frac{1}{x}\frac{\partial}{\partial x}x^2(C-A)+qS\left(\frac{1}{x}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}+\frac{1}{x^2}\frac{\partial^2}{\partial\theta^2}\right)\right]P(x,\theta;\tau).$$

The only differences between this equation and Eq. (2.26) is that $1-A$ is replaced by $C-A$ and the diffusion coefficient is now defined by

$$q=\frac{1}{4}\left(\frac{1+\sigma}{\sigma_t}+\frac{1+\bar{\sigma}}{\bar{\sigma}_t}\right).$$

Thus we may directly use the results derived from Eq. (2.26). This yields for the three important functions:

$$\langle I\rangle=2qS/(C-A),\quad F(I)=1,\quad \Gamma=2qS/\langle I\rangle.$$

At the other extreme, namely, above the transition region, we adopt the quasilinearization procedure which amounts to linearizing Eq. (3.11) around the SC solution I_+ (for all practical purposes, the corresponding SQ solution is the same in this domain). Let $z=x^2-I_+$ and

$$d=\frac{A}{(1+I_+)^2}+\frac{a(1-C)}{(1+aI_+)^2},$$

$$q(y)=\frac{1}{4}\left(\frac{1}{\sigma_t}\frac{1+\sigma+y}{1+y}+\frac{1}{\bar{\sigma}_t}\frac{1+\bar{\sigma}+ay}{1+ay}\right),$$

$$p(y)=\frac{1}{\sigma_t}\frac{1+\sigma+y}{(1+y)^2}+\frac{1}{\bar{\sigma}_t}\frac{1+\bar{\sigma}+ay}{(1+ay)^2}.$$

With these notations, the quasilinearized FP equation takes the form

$$\frac{\partial}{\partial\tau}P(z,\theta;\tau)=\left(2I_+d\frac{\partial}{\partial z}z+\frac{Sq(I_+)}{I_+}\frac{\partial^2}{\partial\theta^2}+p(I_+)SI_+\frac{\partial^2}{\partial z^2}\right)P(z,\theta;\tau),$$

which is analogous to Eq. (2.27) so that we may write at once the general solution and therefore the three functions of interest:

$$\langle I\rangle \simeq I_+ ,$$

$$F(I) \simeq [Sp(I_+)/d]^2/2I_+ \ll 1 ,$$

$$\Gamma = Sq(I_+)/I_+ = (qS/I_+)G(I_+) ,$$

where $G(x) = q(x)/q(0)$. Hence, in the whole domain of variation of the pump parameter A, we may write

$$\Gamma = qS\alpha/\langle I\rangle , \quad (3.14)$$

with $\alpha = 2$ well below the transition region and $\alpha = 1$ well above the transition region. This is the same behavior as in the normal (i.e., monatomic) laser case. However, there is a deep difference in the behavior of the function $\alpha = \alpha(A)$ in the transition region. For the monatomic laser the function α decreases monotonically from two to one. On the contrary, for a laser with saturable absorber, α has a large maximum in the transition region. This behavior is similar to that of the fluctuations of intensity $F(I)$. This relation can be partially elucidated in the framework of the SQ theory, where it is easy to show that the linewidth is given by

$$\Gamma = \left(\frac{1+D}{\sigma_t} + \frac{1+\overline{D}}{\overline{\sigma}_t}\right)\frac{S}{4\langle I\rangle}\frac{1}{a(1)} ,$$

where $D(\overline{D})$ is the stationary inversion of active (passive) atoms and

$$a(n) = \langle x^n\rangle / \langle x^{n-2}\rangle \langle x^2\rangle .$$

This functions as a direct characterization of the field fluctuations. For blackbody radiation we have $a(n) = n/2$, whereas for a coherent state $a(n) = 1$. Furthermore, the intensity fluctuations are related to this function by the simple relation $F(I) = a(4) - 1$. The inverse of the function $a(1)$ may be interpreted as a measure of the field amplitude fluctuation.

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APPENDIX

The stationary solution of Eq. (3.11) may be written as

$$P(x) = \mathfrak{N} \exp\left(-\frac{2\sigma_t}{Sa(a+1)} U(x^2)\right) ,$$

where $\mathfrak{N}$ is the normalization constant and

$$\frac{U(z)}{a(a+1)} = \int dz \frac{(1+z)^3(1+az)^3 - A(1+az)^3[\frac{1}{2}S + (1+z)^2] - (1-C)(1+z)^3[\frac{1}{2}aS + (1+az)^2]}{(1+z)(1+az)[(1+\sigma+z)(1+az)^2 + (1+\overline{\sigma}+az)(1+z)^2]} .$$

Let

$$\varphi(z) = (1+\sigma+z)(1+az)^2 + (1+\overline{\sigma}+az)(1+z)^2$$
$$= a(a+1)(z+\alpha)(z+\beta)(z+\gamma) .$$

In terms of the three roots α, β, and γ, we define three sets of coefficients. The first set is

$$D(1) = (\beta-\gamma)/D, \quad D(2) = (\gamma-\alpha)/D ,$$
$$D(3) = (\alpha-\beta)/D ,$$
$$D = \alpha\beta(\alpha-\beta) + \alpha\gamma(\gamma-\alpha) + \gamma\beta(\beta-\gamma) ,$$

with the properties:

$$\sum_i D(i) = 0, \quad \sum_i \delta(i)D(i) = 0 ,$$
$$\sum_i D(i)\delta^2(i) = 1 ,$$

where, for the sake of compactness, we defined $\delta(1) = \alpha$, $\delta(2) = \beta$, and $\delta(3) = \gamma$. These coefficients come from the reduction $a(a+1)\varphi^{-1}(z) = \sum_i D(i)/[z+\delta(i)]$. Another necessary reduction is

$$\frac{a(a+1)}{\varphi(z)(1+az)} = \sum_i \frac{F(i)}{[z+\delta(i)]} + \frac{F(4)}{(1+az)} ,$$

where

$$F(1) = (\gamma-\beta)[a^2\beta\gamma - a(\beta+\gamma) + 1]/F ,$$
$$F(2) = (\alpha-\gamma)[a^2\alpha\gamma - a(\alpha+\gamma) + 1]/F ,$$
$$F(3) = (\beta-\alpha)[a^2\alpha\beta - a(\alpha+\beta) + 1]/F ,$$
$$F(4) = a^3[\beta\gamma(\beta-\gamma) + \alpha\gamma(\gamma-\alpha) + \alpha\beta(\alpha-\beta)]/F ,$$

with

$$F = \beta\gamma(\gamma-\beta)[a^2\beta\gamma - a(\beta+\gamma) + 1]$$
$$+ \alpha\gamma(\alpha-\gamma)[a^2\alpha\gamma - a(\alpha+\gamma) + 1] + \alpha\beta(\beta-\alpha)[a^2\alpha\beta - a(\alpha+\beta) + 1] + a^3\alpha\beta\gamma[\beta\gamma(\beta-\gamma) + \alpha\gamma(\gamma-\alpha) + \alpha\beta(\alpha-\beta)] .$$

The properties of these coefficients are

$$a^2 \sum_i F(i)\delta(i) + F(4) = 0, \quad a \sum_i F(i) + F(4) = 0 .$$

The last set of coefficients arises from the reduction

$$\frac{a(a+1)}{(1+z)\varphi(z)} = \sum_i \frac{E(i)}{[z+\delta(i)]} + \frac{E(4)}{(z+1)} .$$

The $E(i)$ are obtained by taking the corresponding coefficients $F(i)$ and replacing a by one.

We may then write

$$U(z) = \mu(2)z^2 + \mu(1)z + \sum_i \nu(i) \ln[z + \delta(i)] + \nu(4)\ln(1+z) + \nu(5)\ln(1+az) ,$$

with

$$\mu(2) = a^2/2, \quad \mu(1) = 2a(a+1) - a^2A - a(1-C) - a^2 \sum_i D(i)\delta^3(i) ,$$

$$\begin{aligned}\nu(i) = {} & a^2\delta^4(i)D(i) + [-2a(a+1) + a^2A + a(1-C)]D(i)\delta^3(i)\\ & + \delta^2(i)[D(i)(a^2+4a+1) - a^2A\tfrac{1}{2}SE(i) - Aa(a+2)D(i) - a(1-C)\tfrac{1}{2}SF(i) - (2a+1)BD(i)]\\ & + \delta(i)[-D(i)2(a+1) + aASE(i) + (1+2a)AD(i) + aBSF(i) + (a+2)BD(i)]\\ & + D(i) - A\tfrac{1}{2}SE(i) - AD(i) - aB\tfrac{1}{2}SF(i) - BD(i) ,\end{aligned}$$

$$\nu(4) = -(a-1)^2A\tfrac{1}{2}SE(4), \quad \nu(5) = -\frac{1}{a^2}(a-1)^2B\tfrac{1}{2}SF(4), \quad \text{where } B = 1 - C .$$

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