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Coherent dynamics of a free-electron laser with arbitrary magnet geometry. II. Conservation laws, small-signal theory, and gain-spread relations

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Generalized conservation laws and gain-spread relations in the free-electron laser are derived for variable wiggler configurations. The derivation follows from the general equations of paper I in this series. It is suggested that bandwidth limitations on the optimization of the small-signal gain-to-spread ratio of a storage ring FEL might be overcome by transverse velocity filtering of the electrons.

I. INTRODUCTION

Since the experimental realization of the free-electron laser (FEL),[1] this device has attracted considerable interest because of its tunability and potential for high average output power and high efficiency. In order to attain high efficiency, however, it is necessary to develop a good theoretical understanding of the processes which lead to gain and electron velocity spreading in the FEL. Our approach to this problem[2,3] has always stressed the coherent interaction of the electrons and the light field via a self-consistent solution of the coupled Boltzmann and Maxwell equations. Other workers (e.g., Colson[4] and Louisell[5]) have approached the problem by integrating the equations of motion of single electrons in a prescribed field and using energy arguments to calculate the growth of the fields. In a companion paper in this issue[6] (Paper I), we resolve discrepancies which had existed between these two approaches and develop a theory which combines the best features of both. Moreover, our new theory applies to a static magnetic field configuration with an arbitrary slowly varying amplitude and phase.

In this paper we apply these results to a general discussion of the issues of gain and spread in the FEL. We show how the gain-to-spread ratio can be optimized by using a technique directly inspired by the "zero-gain" electron-echo technique.[7] The limitations of this technique are discussed, and a possible way around the difficulties is suggested.

Since the multiple-scaling approach used in paper I is rather complex, we proceed by first deriving our working equations in a more transparent, although less general and rigorous manner. This approach considers the interaction of an electron and a single mode of the radiation field in a Lorentz frame of reference moving at the velocity v_s of the ponderomotive potential (the interference term between the laser and static fields). In this frame, the electron motion is nonrelativistic,[8] and in the Weizsäcker-Williams approximation, the electron motion and plane-wave amplitudes are governed by a simple Hamiltonian (the Bambini-Renieri-Stenholm Hamiltonian), which can be easily written in quantized form.

In Sec. II, we generalize the BRS Hamiltonian to a many-electron system and discuss constants of motion, equations of motion, gain, and velocity spread in the classical limit. Although this derivation does not apply in principle to the general case of a variable wiggler, we show in Sec. III that, formally, if the equations of motion derived in paper I are specialized to the case of steady-state operation, those obtained via the BRS model have the same mathematical structure. Thus, although the physical interpretation of the variables is different, we can use this formal analogy to readily extend the results of Sec. II to the general case. The details of the calculations are relegated to the Appendix.

In Sec. IV, we apply these results to a discussion of the gain-to-spread problem in a steady-state, variable-wiggler FEL. We show that by choosing an appropriate wiggler design reminiscent of the "echo" design, *both* the gain and gain-to-spread ratio can be made arbitrarily high. However, the higher the gain-to-spread ratio, the smaller the electron energy bandwidth over which this optimization applies. We discuss a possible way around this problem using transverse variations in the wiggler and a filamentation of the incident electrons. Section V is a summary and conclusion.

II. HAMILTONIAN THEORY IN A MOVING FRAME

A. Introduction

In a recent series of papers, Bambini, Renieri and Stenholm[8] have shown that, in a properly chosen moving frame, the FEL can be described in terms of a one-electron nonrelativistic Hamiltonian. This Hamiltonian can readily be generalized to an N-electron system to get:

$$H = H_0 + H_1 , \quad (2.1)$$

$$H_0 = \hbar\omega(a_l^\dagger a_l + a_w^\dagger a_w) , \quad (2.2)$$

$$H_1 = \sum_j \frac{p_j^2}{2M} + i\hbar g \sum_j (a_l^\dagger a_w e^{-2ikz_j} - \text{H.c.}) . \quad (2.3)$$

Here a_l, $a_l^\dagger$ and a_w, $a_w^\dagger$ are photon annihilation and creation operators which represent, respectively, the laser field and the wiggler pseudoradiation field in the Weizsäcker-Williams approximation. The electrons are represented by the canonically conjugate variables z_j and p_j, where $j = 1, \ldots, N$ and $[z_i, p_j] = i\hbar\delta_{ij}$. The coupling constant g is given by

$$g = 2\pi c r_e / kV , \quad (2.4)$$

where $r_e = e^2/4\pi\epsilon_0 mc^2$ is the classical electron radius, V is the quantization volume, and M is the electron mass multiplied by a relativistic correction ($M = m\Delta^{1/2}$ in the notation of paper I). The reference frame moves at a velocity v_s chosen so that the laser and wiggler frequencies coincide with $\omega_l = \omega_w = \omega = ck$. Equation (2.1) is the basic Hamiltonian of our discussion. It yields the following constants of motion:

$$H_0 = \hbar\omega(a_l^\dagger a_l + a_w^\dagger a_w) = \text{constant}, \quad (2.5)$$

$$P = \sum_j p_j + \hbar k(a_l^\dagger a_l - a_w^\dagger a_w) = \text{constant} . \quad (2.6)$$

Since H_0 is constant, it can be eliminated easily by going to a rotating frame at frequency ω in Hilbert space. Equation (2.6) states that the *total* electron-field momentum P is constant. (Remember that the laser and wiggler are counter-propagating waves.) Equations (2.5) and (2.6) can be combined to give

$$P_1 = \sum_j p_j + 2\hbar k a_l^\dagger a_l = \text{constant} . \quad (2.7)$$

Physically, this indicates that on the microscopic level, the interaction between electrons and the laser field proceeds via the exchange of $2\hbar k$ units of momentum. This discrete, quantum-mechanical process can be described as a continuous one if the gain width of the FEL is much larger than $2\hbar k$. In this limit, the system behaves classically. In this paper, we will consider the classical FEL only. Following standard procedures, we obtain this limit by first deriving the Heisenberg equations of motion of the system, and then treating all operators as c numbers.

B. Classical equations of motion

The Heisenberg equations of motion for z_j, p_j, and a_l corresponding to the Hamiltonian (2.1) are

$$\dot{z}_j = p_j/M , \quad (2.8)$$

$$\dot{p}_j = -2g\hbar k(a_l^\dagger a_w e^{-2ikz_j} + \text{H.c.}) , \quad (2.9)$$

$$\dot{a}_l = g a_w \sum_j e^{-2ikz_j} , \quad (2.10)$$

where we have assumed that the wiggler is so strong that any intrinsic time variation in a_w can be neglected; i.e., we regard a_w as an externally imposed arbitrary function of time.

In the classical limit, all operators in Eqs. (2.8)–(2.10) are regarded as c numbers. It is well known[2,3] that the relevant physical quantity for the dynamics of the FEL is the relative phase θ_j between the individual electron j and the ponderomotive potential. Depending upon θ_j, a given electron will be either accelerated or decelerated, leading to a bunching of the electron distribution. Gain or absorption will follow depending upon the difference between the number of accelerated and decelerated electrons.

In order to make the physics more apparent in the equations of motion, we thus reexpress Eqs. (2.8)–(2.10) in terms of this phase θ_j. In the BRS frame, we have

$$\theta_j(t) = 2kz_j(t) . \quad (2.11)$$

We assume for simplicity that all electrons initially have the same momentum p_0 and make the transition to a continuum of electrons by labeling $\theta(t, \theta_0)$ by its initial value θ_0 and assuming that the electrons are initially evenly distributed in θ_0. Because of the physical equivalence of electrons entering the ponderomotive potential with phase differences $2n\pi$, only electrons within one period $0 \leq \theta_0 \leq 2\pi$ need be summed over in Eq. (2.10). Thus one obtains

$$\dot{\theta}(t, \theta_0) = (2k/M)p(t, \theta_0) , \quad (2.12)$$

$$\dot{p}(t, \theta_0) = -2g\hbar k(a_l^* a_w e^{-i\theta} + \text{c.c.}) , \quad (2.13)$$

$$\dot{a}(t) = g a_w N \langle e^{-i\theta(t,\theta_0)} \rangle . \quad (2.14)$$

In Eq. (2.14) we use pointed brackets to denote the average over θ_0:

$$\langle X(\theta_0) \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta_0 X(\theta_0) .$$

C. Gain calculation

In the small-signal regime the force on the electrons perturbs their motion only slightly. In this case the electron motion may be described by letting

$$\theta = \theta_0 + 2kp_0 t/M + \delta\theta \quad (2.15)$$

in Eq. (2.14) and treating $\delta\theta$ as a small quantity. That is, the relative phase between the electrons and the ponderomotive potential does not change significantly, and the electron distribution becomes only slightly bunched. In this way we get

$$\dot{a}_l(t,\theta_0) = -igNa_w \exp(-i2kp_0/M)\langle e^{-i\theta_0}\delta\theta\rangle . \quad (2.16)$$

The phase angle change $\delta\theta$, obtained by integrating Eq. (2.12), is

$$\delta\theta(t,\theta_0) = (2k/M)\int_0^t dt'[p(t',\theta_0) - p_0] . \quad (2.17)$$

The momenta $p(t,\theta_0)$ may be obtained to lowest order by neglecting the bunching $\delta\theta$ in Eq. (2.13) and integrating to obtain

$$p(t,\theta_0) = p_0 - [K_1(t)e^{i\theta_0} + \text{c.c.}] , \quad (2.18)$$

where

$$K_1(t) \equiv \int_0^t dt' K(t') \quad (2.19)$$

and

$$K(t) \equiv 2g\hbar k a_l(t) a_w^*(t)\exp(i2kp_0 t/M) . \quad (2.20)$$

Inserting Eq. (2.18) into (2.17), and then (2.17) into (2.16) gives

$$\delta\theta(t) = -[K_2(t)e^{i\theta_0} + \text{c.c.}] \quad (2.21)$$

and

$$\dot{a}_l(t) = igNa_w \exp(-i2kp_0/M)K_2(t) , \quad (2.22)$$

where

$$K_2(t) \equiv (2k/M)\int_0^t dt' K_1(t') . \quad (2.23)$$

The rate of change of the dimensionless laser intensity $a_l^* a_l$ is found by multiplying Eq. (2.22) by a_l^* and adding the complex conjugate. Using Eq. (2.20), we find that

$$\frac{d}{dt}(a_l^* a_l) = i\left(\frac{N}{2\hbar k}\right)[K^*(t)K_2(t) - \text{c.c.}] . \quad (2.24)$$

From Eqs. (2.19) and (2.23) we have

$$\frac{d^2 K_2}{dt^2} = \left(\frac{2k}{M}\right)K(t) , \quad (2.25)$$

which permits us to eliminate $K(t)$ from Eq. (2.24). We obtain

$$\begin{aligned}\frac{d}{dt}|a_l|^2 &= i\left(\frac{NM}{4\hbar k^2}\right)[\ddot{K}_2^*(t)K_2(t) - \text{c.c.}]\\ &= i\left(\frac{NM}{4\hbar k^2}\right)\frac{d}{dt}[\dot{K}_2^*(t)K_2(t) - \text{c.c.}]\\ &= i\left(\frac{N}{2\hbar k}\right)\frac{d}{dt}[K_1^*(t)K_2(t) - \text{c.c.}] . \quad (2.26)\end{aligned}$$

Integrating Eq. (2.26) finally gives the gain in the interaction time T:

$$\begin{aligned}G &\equiv [\,|a_l(T)|^2 - |a_l(0)|^2]/|a_l(0)|^2\\ &= i[N/2\hbar k|a_l(0)|^2][K_1^*(T)K_2(T) - \text{c.c.}] . \quad (2.27)\end{aligned}$$

If the gain is small, one may treat a_l as constant in the integrals (2.19) and (2.23). In this case Eq. (2.27) reduces to

$$G = i(4Ng^2\hbar k^2/M)\left(\int_0^T dt'' a_w(t'')\exp(-i2kp_0 t''/M)\int_0^T dt\int_0^t dt' a_w^*(t')\exp(i2kp_0 t'/M) - \text{c.c.}\right) . \quad (2.28)$$

D. Gain-spread relationships

As was shown in our previous work,[2,3] the small-signal regime of the FEL is characterized to first order by a large spread of the electron distribution and to second order only by gain (in the constant wiggler limit). Thus, in order to optimize the system, it is necessary to consider both of these effects, or, equivalently, the gain and gain-to-spread ratio.

We define the spread in the electron momenta by

$$(\Delta p)^2 = \langle (p - \langle p\rangle)^2\rangle . \quad (2.29)$$

In the small-signal regime, where $\langle p\rangle \simeq p_0$, we insert Eq. (2.18) into (2.29) to obtain

$$(\Delta p)^2 = 2|K_1(t)|^2 . \quad (2.30)$$

In particular, if the gain is small, Eq. (2.30) becomes

$$(\Delta p)^2 = 8g^2\hbar^2 k^2 |a_l|^2 \left|\int_0^T dt\, a_w^*(t)\exp(i2kp_0 t/M)\right|^2 . \quad (2.31)$$

We may then write the ratio of gain to spread from (2.28) and (2.31) as

$$\frac{G}{(\Delta p)^2} = \alpha\frac{i[\int_0^T dt'' a_w(t'')\exp(-i2kp_0 t''/M)\int_0^T dt\int_0^t dt' a_w^*(t')\exp(i2kp_0 t'/M) - \text{c.c.}]}{|\int_0^T dt\, a_w^*(t)\exp(i2kp_0 t/M)|^2} , \quad (2.32)$$

where α is a constant of proportionality. This is the most important result of this section. As will be shown in Sec. III, a formally identical relationship between recoil and gain can be derived in the case of arbitrary slowly varying wigglers. In order to improve the performance of a FEL, both the gain G and $G/(\Delta p)^2$ will have to be maximized. We will show that this can be done by using techniques closely related to the echo technique discussed in Ref. 7.

Before proceeding along these lines, we conclude this section by rederiving a theorem due to Madey and which relates the recoil and spread of a FEL in the small-signal regime.

We first recall that, at the microscopic level, the interaction between the electrons and electromagnetic field involves the exchange of $2\hbar k$ units of momentum, as shown by the constant of motion, Eq. (2.7). In the classical limit this may be written as

$$P_1 = N\langle p\rangle + 2\hbar k |a_l|^2 = \text{constant}. \tag{2.33}$$

If we let

$$\delta P_l = \hbar k[\,|a(t)|^2 - |a(0)|^2] = \hbar k |a(0)|^2 G \tag{2.34}$$

be the change in the field momentum, then Eq. (2.33) becomes

$$\delta p = -(2/N)\delta P_L = -(2\hbar k/N)|a_l(0)|^2 G, \tag{2.35}$$

where $\delta p \equiv \langle p - p_0\rangle$ is the electron recoil. Equation (2.35) shows explicitly that the electron recoil is of second order in a_l. [This is why it cannot be obtained from Eq. (2.18), which is only correct to first order in a_l.] The spread Δp, however, is first order in a_l as seen from Eq. (2.31), and is much larger than the recoil in the small-signal regime.[2] In the small-signal regime, we have, using Eqs. (2.19) and (2.20),

$$\frac{dK_1(T)}{dp_0} = \left(\frac{i2k}{M}\right)\int_0^T dt\, tK(t). \tag{2.36}$$

Differentiating Eq. (2.30) and using Eq. (2.36), we get

$$d(\Delta p)^2/dp_0 = 2K_1^* dK_1/dp_0 + \text{c.c.}$$
$$= \left(\frac{4ik}{M}\right)\left(K_1^*(T)\int_0^T dt\, tK(t) - \text{c.c.}\right). \tag{2.37}$$

If we integrate by parts and use the definitions of K_1 and K_2, it follows that

$$\int_0^T dt\, tK(t) = TK_1(T) - \left(\frac{M}{2k}\right)K_2(T). \tag{2.38}$$

Combining these last two equations, we obtain

$$d(\Delta p)^2/dp_0 = -2i[K_1^*(T)K_2(T) - \text{c.c.}]. \tag{2.39}$$

At this point let us consider the right-hand side of this equation and its relation to electron recoil. In the small-signal regime we use Eq. (2.27) in Eq. (2.35) to get

$$\delta p = -i[K_1^*(T)K_2(T) - \text{c.c.}]. \tag{2.40}$$

Now from Eqs. (2.39) and (2.40) it is clear that

$$d(\Delta p)^2/dp_0 = 2\delta p, \tag{2.41}$$

which is Madey's result.[9]

Equation (2.41) relates the recoil (gain) of the FEL to the derivative of the spread of the electron distribution. Since $(\Delta p)^2$ is positive definite, it is impossible to design a system which exhibits gain but no spread. Intuitively, this is reasonable since the gain is due to the bunching of the electron distribution, and bunching clearly requires that some electrons are accelerated while others are decelerated. However, relation (2.41) still leaves much freedom to optimize the design of a FEL, for instance by having a spread curve with a very steep slope near the point $(\Delta p)^2 \simeq 0$. This will be discussed in detail in Sec. IV.

Finally, it is interesting to contrast Madey's relation with the law of energy conservation. If the wiggler field a_w does not depend explicitly on time, then the Hamiltonian H_1 in Eq. (2.3) is a constant of the motion. Inserting Eq. (2.10) into Eq. (2.3), we obtain, in the classical limit

$$(N/2M)\langle p^2\rangle + i\hbar(a_l^*\dot{a}_l - \text{c.c.}) = \text{constant}. \tag{2.42}$$

If we write a_l as $a_l = |a_l|e^{-i\phi}$, then Eq. (2.42) becomes

$$(N/2M)\langle p^2\rangle + 2\hbar\bar{\omega}|a_l|^2 = \text{constant}, \tag{2.43}$$

where $\bar{\omega} = \dot{\phi}$. This shows that the variation of the electron energy is connected to the variation of the laser phase. Equation (2.42) relates $\langle p^2\rangle$ to changes in the laser field a_l. This relation is valid even in the strong-signal regime, but only provided that a_w is constant. By contrast, Madey's relation (2.41) is valid for arbitrary variations in a_w, but only in the small-signal, small-gain regime.

III. THE cw FREE-ELECTRON LASER AMPLIFIER

In the preceding section, we described the FEL in a moving reference frame in terms of a one-mode model. The major advantage of this approach is that it gives a straightforward physical picture of the FEL, and several important relationships between spread and gain can be derived in a particularly simple way.

However, one would like to consider more general situations than single-mode operation with a constant wiggler. In particular, we want a theory which accounts for slow variations in the amplitude of the wiggler field and laser operation with large gain per pass. In paper I, a formalism was introduced which allows for a rigorous treatment of these problems.

A remarkable feature of the general equations derived in I is that, in the case of cw operation, they have *exactly* the same mathematical structure as those derived in Sec. II. Thus, these last results not only give an excellent physical insight into the problem, but can also readily be applied to the more general case by an appropriate change in the interpretation of the variables. In this section, we proceed to generalize the results of Sec. II to the case of a variable wiggler. For the sake of clarity, the details of the algebra are relegated to the Appendix.

We first note that in the cw cold-beam limit, the equations of motion (40), (41), and (50) of paper I reduce to

$$d\hat{\theta}(\xi,\theta_0)/d\xi=\hat{\mu}(\xi,\theta_0)\,, \quad (3.1)$$

$$d\hat{\mu}(\xi,\theta_0)/d\xi=-\kappa\{A_q^*(\xi)E_s(\xi)\exp[i\hat{\theta}(\xi,\theta_0)]+\text{c.c.}\}\,, \quad (3.2)$$

$$dE_s(\xi)/d\xi=D_0A_q(\xi)\langle\exp[-i\hat{\theta}(\xi,\theta_0)]\rangle\,. \quad (3.3)$$

Here, ξ is the generalized position coordinate

$$\xi\equiv\int_0^z dz'\Delta(z')=\int_0^z dz'(1+e^2|\hat{A}_q|^2/m^2c^2)\,, \quad (3.4)$$

where $e^2|\hat{A}_q|^2/m^2c^2$ is the usual electron mass shift. Note that ξ reduces to z when the mass shift is negligible. A_q and E_s are the complex slowly varying amplitudes of the wiggler field and the laser field. The position of the electron with respect to the ponderomotive potential is described by the phase angle

$$\hat{\theta}(\xi,\theta_0)=(k_s/2\gamma_0^2)\xi-\omega_s\tau\,, \quad (3.5)$$

where $\tau=t-z/c$ is the retarded time and θ_0 the initial value of the phase angle. The energy detuning $\hat{\mu}(\xi,\theta_0)$ between the electron and ponderomotive potential is defined through the identity

$$\gamma\equiv\gamma_0(1+\gamma_0^2\hat{\mu}/k_s)\,. \quad (3.6)$$

The constants κ and D_0 are given, respectively, by

$$\kappa=e^2k_s/2m^2c^2\gamma_0^4 \quad (3.7)$$

and

$$D_0=\frac{eI}{2mc^2\gamma_0\epsilon_0A_0}\,, \quad (3.8)$$

where A_0 is the laser mode area and I the electron current. All other quantities are defined in paper I.

Equations (3.1)–(3.3) evidently have the same mathematical structure as Eqs. (2.11)–(2.13), although they describe different physical systems. The former equations describe a single-mode laser in a moving frame; the latter equations describe a laser amplifier in the rest frame, possible with large gain per pass. The former do not treat variable magnet geometry in a rigorous fashion, since $M=m\Delta^{1/2}$ and v_s are regarded as constant; the latter describe an FEL with a magnet whose spatial frequency is chirped to compensate for variable Δ. The former make use of an arbitrary quantization volume V; the latter are expressed in terms of measurable quantities, although the beam area A_0 is treated phenomenologically. In the Hamiltonian model of Sec. II it is assumed that the electrons are unbunched at time $t=0$ in the moving frame; in Eqs. (3.1)–(3.3) the electrons are taken to be unbunched at the position $z=\xi=0$ where they enter the magnet. The interaction time T in the former case is analogous to the effective magnet length L' in the latter. However, we can use the formal mathematical equivalence between these two sets of equations to immediately obtain general forms for the gain, spread, etc.

As mentioned earlier, the relevant quantities for the optimization of the FEL are the gain and gain-to-spread ratio. We find that the gain per pass is given in the small-signal regime by [see Eq. (A21)]

$$G=\left((D_0\kappa)i\int_0^{L'}d\xi\int_0^{\xi}d\xi'A_q^*(\xi')\int_0^{L'}d\xi''A_q(\xi'')+\text{c.c.}\right)\,. \quad (3.9)$$

Here L' is the "renormalized" length of the magnet,

$$L'\equiv\int_0^L\Delta(z)dz\,, \quad (3.10)$$

and $A_q(\xi)$ is related to the wiggler amplitude $\overline{A}_q(\xi)$ via

$$A_q(\xi)=\overline{A}_q(\xi)e^{-i\mu_0\xi}\,, \quad (3.11)$$

where μ_0 is the initial detuning between the electrons and the wiggler. By using A_q instead of $\overline{A}_q$, μ_0 is included in the static field. For a standard magnet ($\overline{A}_q=$constant), Eq. (3.9) reduces to the usual antisymmetric expression for the FEL gain.

The gain-to-spread ratio for a variable wiggler may be written in the small-signal, small-gain regime as

$$\frac{G}{(\Delta\mu)^2}\propto\frac{i\int_0^{L'}d\xi\int_0^{\xi}d\xi'A_q^*(\xi')\int_0^{L'}d\xi''A_q(\xi'')+\text{c.c.}}{|\int_0^{L'}d\xi\,A_q^*(\xi)|^2}\,, \quad (3.12)$$

[compare to Eq. (2.32)]. In the next section, we use this relation to optimize the gain-to-spread characteristics of a FEL. Before doing that, however, we note that in the variable wiggler case, Madey's theorem

$$d(\Delta\mu)^2/d\mu_0=2\delta\mu \qquad (3.13)$$

also applies if the input energy distribution is not sharp, provided that the derivative is taken with respect to the mean electron energy and the shape of the input distribution is held constant.

IV. APPLICATION TO THE PROBLEM OF VELOCITY NARROWING IN THE FEL

Recently we have considered the possibility that a technique analogous to conventional photon echo applied to the FEL could reduce the spread in the electron velocity distribution.[7] In that work, we showed that the insertion of a "time-reversing magnet" could eliminate spreading of the velocity distribution. While we felt that this example was instructive, it was incomplete because, as we noted in that article, the effects of gain were not taken into account. Instead, we emphasized that the main cause of spread is random injection times, and spread is therefore largely gain independent. Therefore, one can obtain interesting insights into these random effects without considering gain. However, the question remains: In the presence of finite gain is it possible to minimize spread? That is, to what extent can we tailor the wiggler field so as to maximize the gain-spread ratio? With Eq. (3.12) at hand, we are now in a position to answer this question.

Let us return to Eq. (3.12) and write the complex wiggler field amplitude which appears there in terms of real and imaginary parts:

$$A_q(\xi)\equiv\overline{A}_q(\xi)e^{-i\mu_0\xi}=A_1+iA_2\,. \qquad (4.1)$$

Inserting Eq. (4.1) into Eq. (3.12), we obtain

$$\frac{G}{(\Delta\mu)^2}\propto\frac{\int_0^{L'}d\xi\int_0^{\xi}d\xi'\int_0^{L'}d\xi''[A_1(\xi')A_2(\xi'')-A_1(\xi'')A_2(\xi')]}{(\int_0^{L'}d\xi\,A_1(\xi))^2+(\int_0^{L'}d\xi\,A_2(\xi))^2}\,. \qquad (4.2)$$

The numerator $\mathfrak{N}$ of Eq. (4.2) can be simplified by noting that (see Fig. 1)

$$\int_0^{L'}d\xi\int_0^{\xi}d\xi'=\int_0^{L'}d\xi'\int_{\xi'}^{L'}d\xi\,. \qquad (4.3)$$

Substituting (4.3) into (4.2) and interchanging the dummy variables ξ' and ξ'' in the second term of the integrand, we obtain for the numerator

$$\mathfrak{N}=\left(\int_0^{L'}d\xi'\int_0^{L'}d\xi''\int_{\xi'}^{L'}d\xi-\int_0^{L'}d\xi'\int_0^{L'}d\xi''\int_{\xi''}^{L'}d\xi\right)A_1(\xi')A_2(\xi'')\,. \qquad (4.4)$$

The integral over ξ can now be evaluated trivially and, replacing ξ'' by y and ξ' by x, one finds that the gain-to-spread ratio is proportional to the simple expression

$$\frac{G}{(\Delta\mu)^2}\propto\frac{\int_0^{L'}dx\int_0^{L'}dy(y-x)A_1(x)A_2(y)}{(\int_0^{L'}A_1(x)dx)^2+(\int_0^{L'}A_2(x)dx)^2}\,. \qquad (4.5)$$

In order to discuss the implications of Eq. (4.5), it is instructive to consider a concrete example. The gain-to-spread ratio takes a particularly simple form if $A_2(\xi)$ is an odd function of $\xi-L'/2$.

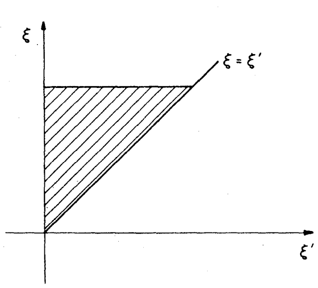

FIG. 1. Domain of integration in Eq. (4.3).

In that case Eq. (4.5) reduces to

$$\frac{G}{(\Delta\mu)^2}\propto\frac{\int_0^{L'}dx\;xA_2(x)}{\int_0^{L'}dxA_1(x)}\,. \qquad (4.6)$$

Thus, one sees that the gain (or spread), and gain-to-spread ratio may be independently controlled by an appropriate choice of $A_1(x)$ and $A_2(x)$.

It is worth mentioning that the magnet design considered here is closely related to the echo configuration discussed in Ref. 7. Although in the present discussion, we do not have a drift region, the sinusoidal shape of A_2 is reminiscent of the FEL plus time reversing magnet, since the phase of the ponderomotive potential in the second part of the magnet is reversed with respect to the first part of the magnet. (Remember that in the echo discussion, one of the possible solutions did not involve a drift region.) Thus, one sees that the echo concept still applies in the presence of gain, although a complete rephasing of the electron distribution is not possible. This, however, is not surprising, since the extraction of optical energy from the system represents an energy loss or dissipation as viewed by the electrons. The fluctuation-dissipation theorem indicates that such an interaction is irreversible; that is, we would not expect arguments based on time reversal to apply. One can draw an analogy between the process of gain in the FEL and atomic lifetime T_1 in the usual optical problem. In both systems, the presence of an irreversible process prohibits a complete rephasing of the medium. However, the analogy should not be carried too far, since in the FEL A_1 and A_2 are functions of the detuning μ_0, so that a magnet design which optimizes the gain-to-spread ratio for a given velocity group may not (and in general will not) be adequate for other detunings.

We discuss this aspect of the problem for the specific example of Fig. 2. Here, we take

$$A_q(\xi)=e^{-i\mu_0\xi}(A_1+iA_2\cos\pi\xi/L')\,, \tag{4.7}$$

with A_1 and A_2 constant. Clearly, for $\mu_0=0$ the gain-to-spread ratio reduces to (4.6) and can be optimized by taking $A_2 \gg A_1$. The question that we address is then to determine the gain-to-spread ratio for other values of μ.

As seen in Eq. (3.12) [see also Eq. (A20)], the spread $(\Delta\mu)^2$ is given by

$$(\Delta\mu)^2 \propto \left|\int_0^{L'} d\xi\, A_q(\xi)\right|^2 , \tag{4.8}$$

which reduces for the example of Eq. (4.7) to

$$(\Delta\mu)^2 \propto \left(\frac{2A_1}{\mu_0}\sin\frac{\mu_0 L'}{2}+\frac{2A_2\mu_0}{\mu_0^2+(\pi/L')^2}\cos\frac{\mu_0 L'}{2}\right)^2 . \tag{4.9}$$

From the recoil-spread theorem [Eq. (3.13)], we see that the gain is proportional to the derivative of $(\Delta\mu)^2$ with respect to μ_0:

$$G \propto \left(\frac{A_1}{\mu_0}\sin\frac{\mu_0 L'}{2}+\frac{A_2\mu_0}{\mu_0^2+(\pi/L')^2}\cos\frac{\mu_0 L'}{2}\right) \times\left\{-\left(\frac{2A_1}{\mu_0^2}+\frac{A_2\mu_0 L'}{\mu_0^2+(\pi/L')^2}\right)\sin\left(\frac{\mu_0 L'}{2}\right)+\left[\frac{A_1L'}{\mu_0}+\frac{2A_2}{\mu_0^2+(\pi/L')^2}-A^2\left(\frac{2\mu_0}{\mu_0^2+(\pi/L')^2}\right)^2\right]\cos\left(\frac{\mu_0L'}{2}\right)\right\}. \tag{4.10}$$

For $\mu_0 \simeq 0$, we obtain

$$G \propto A_1A_2L'^3/\pi^2\,, \tag{4.11}$$

and with Eq. (4.9)

$$G/(\Delta\mu)^2 \propto \frac{1}{\pi^2}\frac{A_1A_2L'^3}{(2A_1L')^2} \propto \frac{A_2L'}{A_1}\,, \tag{4.12}$$

consistently with Eq. (4.6). As $|\mu_0|$ increases, however, the terms proportional to A_2 become dominant in Eq. (4.10), if we choose $A_1 \ll A_2$ in order to optimize (4.12).

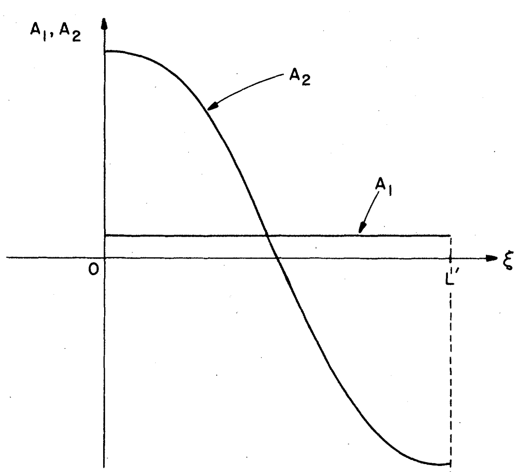

FIG. 2. Magnet geometry used in Eq. (4.7) to illustrate optimization of gain-to-spread ratio.

We can define a critical detuning μ_c as the detuning such that for $|\mu|>\mu_c$, the terms proportional to A_2 become dominant in Eq. (4.9). If one considers only values of μ much smaller than the gain bandwidth π/L', then one finds

$$\mu_c=\left(\frac{\pi}{2}\right)\left(\frac{\pi}{L'}\right)\left(\frac{A_1}{A_2}\right). \tag{4.13}$$

(Note that this value is consistent with $\mu<\pi/L'$ for $A_1 \ll A_2$.) For $|\mu| \gg \mu_c$, we can neglect the first term in Eq. (4.9). Both the gain and spread are then proportional to $|A_2|^2$; that is, the gain-to-spread ratio is independent of the choice of A_1/A_2. Consequently, a wiggler design which optimizes the gain and gain-to-spread ratio at $\mu_0=0$ is ineffective for $|\mu|>\mu_c$.

Since μ_c scales as (A_1/A_2), one sees then that the product (gain divided by spread) times (optimized gain bandwidth) is roughly constant, independent of A_1 and A_2. In other words, if one chooses a strong optimization for a given μ_0, it will be effective for a very small range of detunings only. If, on the other hand, one is interested in a wiggler effective for a wide range of detunings, then the gain-to-spread ratio cannot be improved significantly as compared to that for a standard wiggler.

However, we have another degree of freedom to work with, namely the transverse dimension. This gives an extra possibility of improving the output of the FEL. To get a sense of how this might help, let us consider Eq. (3.11) and replace $\overline{A}_q(\xi)$ by $\overline{A}_q(\xi,\vec{r})$, where $\vec{r}$ is the transverse dimension of the magnet (i.e., we now consider transverse variation of the wiggler). Also let us define $\psi(\xi,\vec{r})$ to be the phase of $\overline{A}_q(\xi,\vec{r})$. Equation (3.11) now becomes

$$A_q(\xi,\vec{r}) = |\overline{A}_q(\xi,\vec{r})| \exp[-i\mu_0\xi + \psi(\xi,\vec{r})] . \qquad (4.14)$$

It is easily seen that by injecting electrons with various initial energies at various transverse positions (see Fig. 3), so that $\mu_0(\vec{r})$ is a function of $\vec{r}$, we can design a wiggler such that all electrons interact with an optimum gain-to-spread ratio; that is, we choose ψ such that

$$-i\mu_0(\vec{r})\xi + \psi(\xi,\vec{r}) = -i\mu_{\text{opt}}\,\xi , \qquad (4.15)$$

where μ_{opt} is the "velocity" at which the gain-to-spread ratio is optimized. One way of carrying out the optimization physically is to use a magnet of constant period, but with a transverse gradient in the field strength. By adjusting the transverse dependence of the mass shift $\Delta(\vec{r})$, one can largely eliminate initial spread in the z component of the electron velocities. This is related to the gain-expansion technique proposed by Madey *et al.*[10] An alternative approach is to introduce an actual transverse dependence in the magnet period, as indicated in Fig. 3. However, it remains to be seen whether such a scheme can be developed into a practical device.

In conclusion, then, we see that by combining the echo technique with a gain-expansion wiggler design, or with a transversely varying wiggler period, it should be possible to obtain a magnet configuration for which both the gain (or spread) and the gain-to-spread ratio can be optimized for a wide range of electron energies. We are currently initiating a detailed study of the complete three-dimensional FEL problem in order to investigate this possibility in detail.

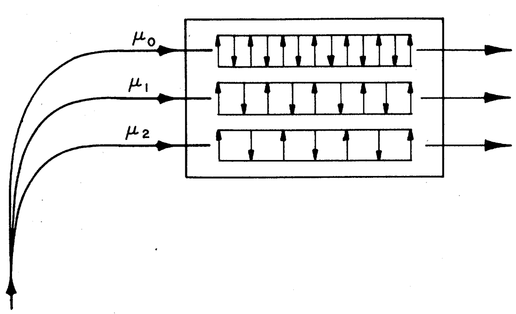

FIG. 3. Schematic diagram of how a transversely varying wiggler period could be used to compensate for initial velocity spread.

V. SUMMARY

In this paper, we have used the multiple-scaling analysis developed in paper I to discuss the issue of gain and spread in a variable wiggler FEL. We have shown that one can obtain an intuitive understanding of the general cw FEL equations by considering the simple Bambini-Renieri-Stenholm approach[8] and generalizing it to an N-electron problem. The rigorous results for the variable-wiggler cw case can then be rederived trivially by a simple change of interpretation of the BRS variables. The electron-echo technique discussed in a previous paper[7] has been shown to essentially still apply in a system exhibiting gain. However, the irreversibility due to the extraction of energy from the electrons limits the extent to which this technique is applicable. We have used a simple example to show that the product (gain divided by spread) times (optimized gain bandwidth) is constant, so that only a narrow band of electron energies will interact in an optimum way with the ponderomotive potential if the gain-to-spread ratio is strongly optimized.

However, the FEL case offers an extra degree of freedom, since electrons with different energies can be injected at different transverse locations of the wiggler field. Thus, by the use of transverse gradients in the magnetic field (gain expansion)[10] or of transverse variations of the wiggler period, one should be able to optimize the gain-to-spread ratio for a significant range of electron velocities.

In order to assess the limitations of the proposed technique, it will be necessary to study in detail the complete three-dimensional FEL. This analysis will also have the advantage of providing a detailed description of the transverse electromagnetic mode geometry and of diffraction effects.

ACKNOWLEDGMENT

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APPENDIX

In Sec. II we described the FEL in a moving reference frame in terms of a one-mode Hamiltonian model. In this appendix we connect this analysis to the results of paper I. In paper I, we derived coupled Maxwell and generalized pendulum equations for the FEL, using a multiple-scaling technique. These equations are very general, in that they allow for coherent transient behavior, an arbitrary initial energy distribution, arbitrary slow variation in the amplitude of the static field, and diffractive beam spreading. Now we restrict these equations [Eqs. (40), (41), and (50) of paper

I] to the case of cw operation in the cold-beam limit, neglecting diffraction. In this case the variables τ, τ_0, μ_0 may be suppressed and these equations become

$$d\hat{\theta}(\xi,\theta_0)/d\xi = \hat{\mu}(\xi,\theta_0)\,, \tag{A1}$$

$$d\hat{\mu}(\xi,\theta_0)/d\xi = -\kappa\{A_q^*(\xi)E_s(\xi)\exp[i\hat{\theta}(\xi,\theta_0)] + \text{c.c.}\}\,, \tag{A2}$$

$$dE_s(\xi)/d\xi = D_0 A_q(\xi)\langle\exp[-i\hat{\theta}(\xi,\theta_0)]\rangle\,. \tag{A3}$$

The constant D_0 in Eq. (A3) is given by

$$D_0 = \alpha I/\sigma^2 = eI/2mc^2\gamma_0\epsilon_0 A_0\,, \tag{A4}$$

and all other quantities are defined in paper I.

Equations (A1)–(A3) evidently have the same mathematical structure as Eqs. (2.10)–(2.13). Therefore, the analysis carried out in Sec. II goes through in the same way for Eqs. (A1)–(A3). In this appendix we merely state the principal results without repeating the detailed proofs.

The law of conservation of energy is obtained by multiplying Eq. (A3) by $\sigma^2 E^*$, adding the complex conjugate and inserting Eq. (A2):

$$\frac{d}{d\xi}|\sigma E_s|^2 = -\frac{d}{d\xi}\left(\frac{\alpha I}{\kappa}\langle\hat{\mu}(\xi,\theta_0)\rangle\right) = -\frac{d}{d\xi}\left(\frac{I}{e}\frac{mc^2\gamma_0^3}{k_s}\langle\hat{\mu}(\xi,\theta_0)\rangle\right)\,. \tag{A5}$$

We have used Eqs. (47) and (18) of paper I in obtaining the last step. The left-hand side is the rate of change of the optical power, I/e is the number of electrons per second, and [from Eq. (12) of paper I] $mc^2\gamma_0^3\langle\hat{\mu}\rangle/k_s$ is the average recoil energy per electron. This conservation law is the counterpart of Eq. (2.33). Thus, conservation of energy in the rest frame corresponds to conservation of momentum in the moving frame. This is consistent with the ultrarelativistic nature of the electron motion.

The initial conditions associated with Eqs. (A1) and (A2) are

$$\hat{\theta}(0,\theta_0) = \theta_0\,, \tag{A6}$$

$$\hat{\mu}(0,\theta_0) = \mu_0\,. \tag{A7}$$

The normalized electron velocity distribution is given by

$$w(\xi,\mu) = \langle\delta(\mu - \hat{\mu}(\xi,\theta_0))\rangle\,. \tag{A8}$$

If one performs the transformation $\hat{\mu}' = \hat{\mu} - \mu_0$, $\hat{\theta}' = \hat{\theta} - \mu_0\xi$, $A_q'(\xi) = A_q(\xi)\exp(-i\mu_0\xi)$, then both the unprimed and primed variables satisfy Eqs. (A1)–(A3), but Eq. (A7) is replaced by $\hat{\mu}'(0,\theta_0) = 0$. Thus, it is possible (and sometimes convenient) to include the energy detuning in the static field amplitude and thus take the initial electron detunings to be zero. We adopt this frame of reference now in discussing the small-signal regime. Note that, if we do this, we have

$$dA_q/d\mu_0 = -i\mu_0 A_q\,. \tag{A9}$$

The coupled linear equations describing the FEL in the small-signal regime are

$$dE_s(\xi)/d\xi = iD_0 A_q(\xi)K_2(\xi)\,, \tag{A10}$$

$$dK_2(\xi)/d\xi = K_1(\xi)\,, \tag{A11}$$

$$dK_1(\xi)/d\xi - \kappa A_q^*(\xi)E_s(\xi)\,, \tag{A12}$$

where

$$K_2 = -\langle e^{-i\theta_0}\delta\theta\rangle\,, \tag{A13}$$

$$K_1 = -\langle e^{-i\theta_0}\hat{\mu}\rangle\,, \tag{A14}$$

and the changes in phase angle $\delta\theta(\xi,\theta_0) = \hat{\theta} - \theta_0$ are assumed to be small. The electron velocities are given by

$$\hat{\mu}(\xi,\theta_0) = -[K_1(\xi)e^{i\theta_0} + \text{c.c.}]\,, \tag{A15}$$

and the velocity spread is given by

$$(\Delta\mu)^2 = \langle\hat{\mu}^2\rangle = 2|K_1(\xi)|^2\,. \tag{A16}$$

The explicit form of the velocity distribution may be found by substituting Eq. (A15) into Eq. (A8) and carrying out the integration over θ_0. We obtain

$$w(\xi,\mu) = (1/\pi)(4|K_1|^2 - \mu^2)^{-1/2}\,, \quad |\mu| < 2|K_1|\,. \tag{A17}$$

The distribution has singularities, as is generally the case in the cw cold-beam limit. It is therefore simpler, and just as physically illuminating, to deal with $\hat{\mu}(\xi,\theta_0)$ rather than $w(\xi,\mu)$.

Although Eq. (A15) does not give recoil, we can nevertheless calculate linear gain by using the Maxwell equation (A10). We get

$$\begin{aligned} d|E_s|^2/d\xi &= iD_0 A_q E_s^* K_2 + \text{c.c.} \\ &= \left(\frac{iD_0}{\kappa}\right)K_2\frac{dK_1^*}{d\xi} + \text{c.c.} \\ &= \left(\frac{iD_0}{\kappa}\right)\left(K_2\frac{d^2K_2^*}{d\xi^2} - K_2^*\frac{d^2K_2}{d\xi^2}\right) \\ &= \left(\frac{iD_0}{\kappa}\right)\frac{d}{d\xi}\left(K_2\frac{dK_2^*}{d\xi} - K_2^*\frac{dK_2}{d\xi}\right) \\ &= \left(\frac{iD_0}{\kappa}\right)\frac{d}{d\xi}(K_2K_1^* - K_2^*K_1)\,. \end{aligned} \tag{A18}$$

We may integrate Eq. (A18) to get the gain per pass:

$$\begin{aligned} G &= |E_s(L')|^2/|E_s(0)|^2 - 1 \\ &= \{iD_0/[\kappa|E_s(0)|^2]\}[K_2(L')K_1^*(L') - K_2^*(L')K_1(L')]\,. \end{aligned} \tag{A19}$$

One can see from Eq. (A16) that the spread vanishes at the end of the magnet if and only if $K_1(L')=0$. However, in this case G is also zero. In the small-gain limit E_s is practically constant and we may write explicit expressions for the spread and gain:

$$(\Delta\mu)^2 = 2\,|\kappa E_s|^2 \left|\int_0^{L'} d\xi\, A_q^*(\xi)\right|^2 , \tag{A20}$$

$$G = (D_0\kappa)\left(i\int_0^{L'} d\xi \int_0^{\xi} d\xi' A_q^*(\xi') \times \int_0^{L'} d\xi'' A_q(\xi'') + \text{c.c.}\right) . \tag{A21}$$

For the standard magnet, where there is a constant $\overline{A}_q$ such that

$$A_q(\xi) = \overline{A}_q e^{-\mu_0 \xi} , \tag{A22}$$

we have

$$(\Delta\mu)^2 = 2\,|\kappa E_s \overline{A}_q^* L'|^2 (\sin\eta/\eta)^2 \tag{A23}$$

and

$$G = (D_0 \kappa L'^3)\,|\overline{A}_q|^2 (\sin\eta)(\eta^{-3}\sin\eta - \eta^{-2}\cos\eta) , \tag{A24}$$

where

$$\eta = \mu_0 L'/2 . \tag{A25}$$

Equation (A24) is the usual antisymmetric expression for the FEL gain. The maximum gain is near $\eta = 1.3$.

With an arbitrary helical magnet geometry, the energy detuning still enters $A_q(\xi)$ as in Eq. (A22), except that $\overline{A}_q$ is a function of ξ. In particular, if we differentiate Eq. (A20) with respect to μ_0 and make use of Eqs. (A9), (A21), and (A5), we obtain Madey's theorem

$$d(\Delta\mu)^2/d\mu_0 = 2\delta\mu \tag{A26}$$

by the same method as was used in Sec. II.

It can be shown that Eq. (A26) also applies if the input energy distribution is not sharp, provided that the derivative is taken with respect to the mean electron energy and the shape of the input distribution is held constant.

If we integrate Eq. (A15), we get

$$\delta\theta = -[K_2(\xi)e^{i\theta_0} + \text{c.c.}] . \tag{A27}$$

The small-signal theory is valid provided that $|K_2(\xi)|$ is, everywhere, much less than one. By integrating Eqs. (A12) and (A11), we see that this is true if

$$\kappa\,|A_q^* E_s|L'^2 \ll 1 . \tag{A28}$$

This is not a necessary condition, however. If A_q oscillates appreciably, as it will if $|\mu_0| \gtrsim \pi L'$, then the electrons undergo quasifree motion and the condition

$$\kappa\,|A_q^* E_s|L'/\mu_0 \ll 1 \tag{A29}$$

is sufficient and less restrictive.

In Sec. II we obtained a conservation-of-energy law [Eq. (2.42)] if the wiggler a_w was independent of time. In the present context, this law may be obtained by assuming that A_q is constant (i.e., a standard magnet). In this case we use the frame of reference in which Eq. (A7) applies. Thus,

$$\begin{aligned}\frac{d}{d\xi}\langle\hat\mu^2(\xi,\theta_0)\rangle &= 2\left\langle\hat\mu\frac{d\hat\mu}{d\xi}\right\rangle \\ &= 2\kappa\left\langle\frac{d\hat\theta}{d\xi}\,A_q E_s^* e^{-i\hat\theta}\right\rangle + \text{c.c.} \\ &= -2i\kappa A_q E_s^* \frac{d}{d\xi}\langle e^{-i\hat\theta}\rangle + \text{c.c.} \\ &= -\left(\frac{2i\kappa}{D_0}\right)E_s^* \frac{d^2 E_s}{d\xi^2} + \text{c.c.} \\ &= -\left(\frac{2i\kappa}{D_0}\right)\frac{d}{d\xi}\left(E_s^*\frac{dE_s}{d\xi} - E_s\frac{dE_s^*}{d\xi}\right) .\end{aligned} \tag{A30}$$

We have obtained the second line by substituting from Eqs. (A1) and (A2), and the fourth line by substituting from Eq. (A3). By integrating Eq. (A30), we obtain a relation between electron spread $\Delta\mu = \langle(\hat\mu - \langle\hat\mu\rangle)^2\rangle^{1/2}$, recoil $\delta\mu = \langle\hat\mu - \mu_0\rangle$, and the laser field:

$$(\Delta\mu)^2 = -\delta\mu(2\mu_0 + \delta\mu) - \left(\frac{2i\kappa}{D_0}\right)\left(E_s^*\frac{dE_s}{d\xi} - E_s\frac{dE_s^*}{d\xi}\right) . \tag{A31}$$

Note that this relation is valid even in the strong-signal regime.

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