Coherent dynamics of a free-electron laser with arbitrary magnet geometry. I. General formalism

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Coupled equations describing the propagation of the laser field (Maxwell equation) and the evolution of the electron distribution (generalized pendulum equations) are derived classically for the case of ultrarelativistic electrons moving in a helical magnet with an arbitrary slowly varying amplitude. The equations apply to coherent transient phenomena, such as picosecond pulse propagation, as well as to continuous-wave laser operation.

I. INTRODUCTION

In previous works,[1-5] we developed a self-consistent classical theory of the free-electron laser (FEL)[6,7] in terms of coupled Maxwell and Boltzmann equations. In order to eliminate rapid oscillations from the equations, the Boltzmann distribution was expanded in terms of harmonics of the frequency of the bunching potential, and harmonics other than the first two were neglected. This led to a set of "quasi-Bloch" equations describing the electron distribution, which provided a good qualitative description of FEL physics. In particular, it provided the first calculation of nonlinear saturation of the device,[2] clearly showed the importance of bunching, as well as first demonstrating velocity spread in the FEL. Furthermore, by means of these equations, we were able to show that effects of laser lethargy are important in the FEL and to provide a good explanation for such puzzles as the power-tuning curves and the electron energy distribution.[5]

Independently Colson, Louisell, Lam, and Copeland[8,9] developed a theory of the FEL in which pendulum equations are used to describe the motion of the electrons. While their theory provides a more accurate description of the FEL in the strongly saturated regime, it is not a self-consistent theory and does not apply to the pulsed regime of the present Stanford experiment.[7] Instead, the amplitude of the laser field is taken to be constant, and gain is inferred by using conservation of energy. Thus, their theory applies only to single-mode operation with small gain per pass.

In this paper we combine the best features of both of the above approximations, i.e., the present theory is self-consistent and applies to high laser powers.[10] In order to accomplish this we use a multiple-scaling perturbation expansion to carry out the slowly varying amplitude and phase approximation without recourse to the "harmonic approximation". Our equations apply either to coherent transient phenomena, such as the picosecond pulse regime of the Stanford experiment, or to continuous wave (cw) operation. In the case of small-gain cw operation with a helical magnet of constant amplitude, our equations for the electrons (generalized pendulum equations) reduce to the pendulum equations used by Louisell *et al.*[9]

Our theory can be used to deal with geometries in which the static magnetic field is a function of axial position. In Sec. II we formulate the Maxwell and Boltzmann equations for an arbitrary magnet geometry. We use the multiple-scaling technique in Sec. III to eliminate rapid oscillations from these equations in the case of a helical magnetic field with slowly varying amplitude. Section IV introduces single-particle variables for the electrons and reformulates the treatment of the helical magnet in terms of coupled Maxwell and generalized pendulum equations. Finally, in Sec. V we compare the present theory to our previous work which used the quasi-Bloch equations.

The present paper is the first in a series exploring various aspects of FEL physics. In a companion paper[11] in this issue (paper II) we apply the results of the present paper to the conservation laws of the FEL, the small-signal regime, and the relationship between electron recoil and velocity spreading in an arbitrary helical magnet.

In additional papers now in preparation, we will present other results based on the theoretical approach to the FEL given here. In particular, we will present numerical solutions of the coupled Maxwell and generalized pendulum equations, treating phenomena such as pulse propagation and coherent transients in the FEL. We also will generalize the results of the present paper to the case of an arbitrary periodic magnet. This leads to equations governing nonlinear optical harmonic generation in the FEL.

II. FEL DYNAMICS WITH AN ARBITRARY MAGNET GEOMETRY

As is appropriate to the Stanford experiment, we neglect Coulomb repulsion between the electrons[12] and also neglect incoherent radiation by the electrons. Then, assuming that the filling factor (the electron beam area divided by the laser mode area)

$$F = a_0/A_0 \tag{1}$$

is small, the motion of the electrons is governed by the one-dimensional Hamiltonian

$$H = mc^2\gamma = c\,[p_z^2 + m^2c^2 + e^2A^2(z,t)]^{1/2}. \tag{2}$$

The transverse canonical momentum is zero for a properly aligned FEL, so that the transverse kinetic momentum is $-e\vec{A}$, where

$$\vec{A} = \vec{A}_{\text{static}}(z) + \vec{A}_{\text{laser}}(z,t) \tag{3}$$

is the sum of the vector potentials of the magnet and the laser fields evaluated on axis. These fields are assumed to be transverse:

$$\vec{A}_{\text{static}}(z) = 2^{-1/2}\hat{e}_-\hat{A}_q(z) + \text{c.c.}, \tag{4}$$

$$\vec{A}_{\text{laser}}(z,t) = 2^{-1/2}\hat{e}_-\hat{A}_s(z,t) + \text{c.c.}. \tag{5}$$

For the present $\hat{A}_q(z)$ is taken to be arbitrary. We assume that the electrons propagate in the positive z direction. $\hat{A}_s(z,t)$ is also assumed to propagate in the positive z direction. In the FEL, light propagating in the negative z direction does not interact resonantly with the electrons, so we can ignore it. From Eqs. (3)–(5) we obtain

$$A^2 = |\hat{A}_q|^2 + |\hat{A}_s|^2 + (\hat{A}_s\hat{A}_q^* + \text{c.c}). \tag{6}$$

We assume that $|\hat{A}_s| \ll |\hat{A}_q|$. The Boltzmann equation corresponding to Eq. (2) is

$$\frac{\partial h}{\partial t} + v_z\frac{\partial h}{\partial z} - \frac{e^2}{2m\gamma}\frac{\partial A^2}{\partial z}\frac{\partial h}{\partial p_z} = 0. \tag{7}$$

The $|\hat{A}_q|^2$ term in Eq. (6) is in general dominant. However, the cross term involving the product of the static wiggler field and the laser field leads to bunching of the electrons at optical frequencies, and is therefore crucial to the operation of the FEL. The term $|\hat{A}_s|^2$ is small and will be neglected. In order to account for the large-scale effects of the $|\hat{A}_q|^2$ term on the electron motion, we note that, since the Hamiltonian (2) is time independent when $\hat{A}_s = 0$, it follows that in this case γ is constant, although p_z and v_z in general are not. This fact suggests that we use γ as an independent variable in the Boltzmann equation, rather than p_z. This transformation yields

$$\frac{\partial h}{\partial t} + v_z\frac{\partial h}{\partial z} + \frac{e^2}{2m^2c^2\gamma}\frac{\partial A^2}{\partial t}\frac{\partial h}{\partial \gamma} = 0, \tag{8}$$

where

$$v_z = c[1 - (1 + e^2A^2/m^2c^2)/\gamma^2]^{1/2}. \tag{9}$$

Note that $\partial A^2/\partial t$, rather than $\partial A^2/\partial z$, appears in Eq. (8). However, the dominant term $|\hat{A}_q|^2$ in Eq. (6) does not contribute to $\partial A^2/\partial t$, since $\hat{A}_q$ is independent of time. Using Eq. (6) in Eq. (8) and neglecting the term of order $|A_s|^2$, we get

$$\frac{\partial h}{\partial z} + \frac{1}{v_z}\frac{\partial h}{\partial t} - \frac{e^2}{2m^2c^2\gamma v_z}(\hat{E}_s\hat{A}_q^* + \text{c.c.})\frac{\partial h}{\partial \gamma} = 0, \tag{10}$$

where the electric field amplitude is given by

$$\hat{E}_s = -\partial\hat{A}_s/\partial t. \tag{11}$$

To proceed further, we assume that the electrons have a narrow distribution of energies centered at $mc^2\gamma_0$. We define a detuning parameter μ such that

$$\gamma = \gamma_0(1 + \gamma_0^2\mu/k_s), \tag{12}$$

where k_s is a constant to be defined later. The second term in parentheses in Eq. (12) is assumed to be much less than one. We assume that $\gamma_0 \gg 1$, so that v_z may be replaced by c in the *third* term of Eq. (10). This approximation is *not* adequate in the second term of Eq. (10), since we wish to describe a process which is resonant in the electron velocity. However, if we define the function

$$\Delta(z) \equiv 1 + e^2|\hat{A}_q|^2/m^2c^2, \tag{13}$$

then we can get a good approximation to $1/v_z$ by using Eqs. (9), (12), and (13):

$$\begin{aligned} 1/v_z &\simeq (1 - \Delta/\gamma^2)^{-1/2}/c \\ &\simeq (1 + \Delta/2\gamma^2)/c \\ &\simeq [1 + \Delta(1 - 2\gamma_0^2\mu/k_s)/2\gamma_0^2]/c. \end{aligned} \tag{14}$$

In our earlier work[4] we defined a relativistic mass $M = m\,\Delta^{1/2}$, so we call Δ the mass shift. It is a measure of the relativistic nature of the transverse motion of the electrons.

We next define new independent variables according to

$$\tau = t - z/c, \tag{15}$$

$$\xi = \int_0^z \Delta(z')dz', \tag{16}$$

where τ is the retarded time and ξ is a new position variable useful in that it includes the effect of relativistic mass shift. Transforming Eq. (10) using Eqs. (12), (15), and (16), and inserting Eq. (14), we obtain

$$\frac{\partial h}{\partial \xi} + \frac{1}{c}\left(\frac{1}{2\gamma_0^2} - \frac{\mu}{k_s}\right)\frac{\partial h}{\partial \tau} - \kappa\left(\hat{E}_s\frac{\hat{A}_q}{\Delta} + \text{c.c.}\right)\frac{\partial h}{\partial \mu} = 0, \tag{17}$$

where

$$\kappa = e^2 k_s / 2m^2 c^3 \gamma_0^4. \qquad (18)$$

Equation (17) is our basic equation describing the electron distribution.

Let us next turn our attention to the Maxwell wave equation. In our earlier work[1-4] we showed that, if we neglect diffraction, $\vec{A}_{\text{laser}}$ obeys the wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}_{\text{laser}} = \frac{e^2 F}{m c^2 \epsilon_0}\vec{A}\int_{-\infty}^{\infty}\frac{dp_z}{\gamma}h, \qquad (19)$$

provided that h is normalized such that $\int dz \int dp_z h$ is equal to the number of electrons in the magnet divided by the electron beam area. In Appendix A we describe an approximate way of putting diffractive spreading of the beam back into the Maxwell equation. Although the total field $\vec{A}$ appears on the right-hand side of Eq. (19), the contribution from $\vec{A}_{\text{laser}}$ is in practice negligible. Therefore, introducing Eqs. (4) and (5) and transforming to the variables τ, ξ, μ, we obtain

$$\left(\Delta\frac{\partial}{\partial \xi}\Delta\frac{\partial}{\partial \xi} - \frac{2}{c}\Delta\frac{\partial}{\partial \xi}\frac{\partial}{\partial \tau}\right)\hat{A}_s = \frac{e^2 F \gamma_0^2}{c\epsilon_0 k_s}\hat{A}_q\int d\mu\, h. \qquad (20)$$

However, since $\hat{A}_s$ is an optical field propagating to the right, variations in $\hat{A}_s$ as a function of $c\tau$ occur over an optical wavelength. By contrast, variations in $\hat{A}_s$ as a function of ξ are much slower (on the order of the magnet length L). Therefore, we may drop the first term on the left-hand side of Eq. (20) and use Eq. (11) to obtain

$$\frac{\partial \hat{E}_s}{\partial \xi} = D\frac{\hat{A}_q}{\Delta}\int d\mu\, h, \qquad (21)$$

where

$$D = e^2 F \gamma_0^2 / 2\epsilon_0 k_s. \qquad (21')$$

Equation (21) is our basic equation for the laser field which, together with (17), provides a self-consistent description of FEL physics.

III. FEL DYNAMICS WITH A HELICAL MAGNET OF SLOWLY VARYING AMPLITUDE

At this stage of the development $\hat{A}_q$ is completely arbitrary, and Eqs. (17) and (21) contain fields which oscillate at optical frequencies. We now specialize to the physically reasonable case of a circularly polarized (helical) magnet with a slowly varying complex amplitude $A_q(\xi)$. We define $A_q(\xi)$ and k_s by the equation

$$\hat{A}_q/\Delta = A_q(\xi)\exp(-ik_s\xi/2\gamma_0^2). \qquad (22)$$

With this magnet geometry Δ is also slowly varying. Note that the magnet wave vector

$$k_q = k_s \Delta / 2\gamma_0^2 \qquad (23)$$

is a function of position unless Δ is constant; i.e., unless $|A_q|$ is constant. A magnet of the type given by Eq. (22) generates nearly monochromatic radiation with a wave vector near k_s. We define a slowly varying complex electric field amplitude E_s according to

$$\hat{E}_s = E_s(\xi, \tau)e^{-i\omega_s \tau}, \qquad (24)$$

where $\omega_s = ck_s$.

At this point it is useful to introduce scaled independent variables

$$\bar{\xi} = \xi/L', \quad \bar{\mu} = L'\mu, \quad \bar{\tau} = 2c\gamma_0^2\tau/L',$$

where

$$L' = \int_0^L \Delta(z)dz \qquad (25)$$

is the effective magnet length in terms of the ξ variable. The time unit $L'/2c\gamma_0^2$ is typically on the order of picoseconds in present experiments. In particular, it is much longer than an optical period. Inserting Eqs. (22) and (24) into Eqs. (21) and (17), our working equations now read

$$\partial E_s/\partial \bar{\xi} = DA_q e^{-i\theta}\int d\bar{\mu}\, h, \qquad (26)$$

$$\frac{\partial h}{\partial \bar{\xi}} + (1 - \epsilon\bar{\mu})\frac{\partial h}{\partial \bar{\tau}} = \kappa L'^2(E_s A_q^* e^{i\theta} + \text{c.c.})\frac{\partial h}{\partial \bar{\mu}}. \qquad (27)$$

Here we define

$$\theta = k_s\xi/2\gamma_0^2 - \omega_s\tau = (\bar{\xi} - \bar{\tau})/\epsilon. \qquad (27')$$

The angle θ is the phase of the bunching potential (i.e., the interference term in A^2), and ϵ is given by

$$\epsilon = 2\gamma_0^2/k_s L' = 1/(2\pi N), \qquad (28)$$

where N is the number of turns in the magnet. For the Stanford experiment, we have $\epsilon \simeq 10^{-3}$.

Because of the rapidly oscillating factors $e^{i\theta}$ in Eqs. (26) and (27), it is not feasible to solve these equations as they stand. In order to deal with these rapid oscillations, we wish to develop a perturbation expansion of Eqs. (26) and (27) in the limit of small ϵ. We do this by means of multiple-scaling perturbation theory. This is a well-known technique in the theory of nonlinear oscillations,[13] though it has not been much applied to laser theory. Further description of the multiple-scaling technique is given in Appendix B.

There are three main stages in the use of multiple-scaling perturbation theory. First, variables describing the rapid and slow processes in the system are identified and treated as independent variables. A small parameter ϵ is introduced to describe the relative rates of the slow and fast processes. Second, the dependent variables are expanded in a power series in ϵ and solved order by order. Third, the arbitrariness which results from regarding the fast and slow variables as independent must be removed at each step in the perturbation by requiring that the solutions have no secular (unbounded) growth in the fast variable. In the present case θ describes the rapid oscillations in the electron distribution, whereas $\bar{\xi}$ and $\bar{\tau}$ describe the slowly varying envelope of the distribution, as depicted in Fig. 1. Following the usual procedure of multiple-scaling perturbation theory, we now regard $E_s(\bar{\xi}, \bar{\tau}, \theta)$ and $h(\bar{\xi}, \bar{\tau}, \bar{\mu}, \theta)$ as depending on θ as an extra independent variable. This means that we use the chain rule to transform the partial derivatives in Eqs. (26) and (27) according to $\partial/\partial\bar{\xi} \to \partial/\partial\bar{\xi} + \epsilon^{-1}\partial/\partial\theta$, $\partial/\partial\bar{\tau} \to \partial/\partial\bar{\tau} - \epsilon^{-1}\partial/\partial\theta$. We thus write Eqs. (26) and (27) as

$$\frac{\partial E_s}{\partial \theta} + \epsilon \frac{\partial E_s}{\partial \bar{\xi}} = \epsilon D A_q(\bar{\xi}) e^{-i\theta} \int d\bar{\mu}\, h, \tag{29}$$

$$\frac{\partial h}{\partial \bar{\xi}} + (1 - \epsilon\bar{\mu}) \frac{\partial h}{\partial \bar{\tau}} + \bar{\mu} \frac{\partial h}{\partial \theta} = \kappa L'^2 (E_s A_q^*(\bar{\xi}) e^{i\theta} + \text{c.c.}) \frac{\partial h}{\partial \bar{\mu}}. \tag{30}$$

Next we assume a perturbation expansion

$$E_s = E^{(0)} + \epsilon E^{(1)} + \cdots, \tag{31}$$

$$h = h^{(0)} + \epsilon h^{(1)} + \cdots. \tag{32}$$

We insert the expansions (31) and (32) into Eqs. (29) and (30) and obtain equations relating the coefficients of the various powers of ϵ. In the limit $\epsilon \to 0$ we are really only interested in $E^{(0)}$ and $h^{(0)}$. The zeroth order of Eq. (29) is

$$\partial E^{(0)}/\partial\theta = 0. \tag{33}$$

This says that $E^{(0)}$ is slowly varying, which confirms that ω_s may be identified as the carrier frequency of the light. The zeroth order of Eq. (30) is

$$\frac{\partial h^{(0)}}{\partial \bar{\xi}} + \frac{\partial h^{(0)}}{\partial \bar{\tau}} + \bar{\mu} \frac{\partial h^{(0)}}{\partial \theta} = \kappa L'^2 (E^{(0)} A_q^* e^{i\theta} + \text{c.c.}) \frac{\partial h^{(0)}}{\partial \bar{\mu}}. \tag{34}$$

If the incident electrons are unbunched, $h^{(0)}(0, \tau, \mu, \theta)$ is independent of θ. Since θ enters Eq. (34) only through the periodic function $e^{i\theta}$, Eq. (34) clearly implies that $h^{(0)}(\bar{\xi}, \bar{\tau}, \bar{\mu}, \theta)$ is a periodic function of θ. Thus, $h^{(0)}(\bar{\xi}, \bar{\tau}, \bar{\mu}, \theta)$ may be expanded in a Fourier series:

$$h^{(0)}(\bar{\xi}, \bar{\tau}, \bar{\mu}, \theta) = \sum_{n=-\infty}^{\infty} c_n(\bar{\xi}, \bar{\tau}, \bar{\mu}) e^{in\theta}. \tag{35}$$

The first order of Eq. (29) is

$$\frac{\partial E^{(1)}}{\partial \theta} = D A_q e^{-i\theta} \int d\bar{\mu}\, h^{(0)} - \frac{\partial E^{(0)}}{\partial \bar{\xi}}. \tag{36}$$

In order that $E^{(1)}$ remain bounded as a function of θ, the right-hand side of Eq. (36) must not have a harmonic component independent of θ. Thus,

$$\frac{\partial E^{(0)}}{\partial \bar{\xi}} = D A_q \int d\bar{\mu}\, c_1(\bar{\xi}, \bar{\tau}, \bar{\mu}). \tag{37}$$

Equation (37) is a necessary and sufficient condition that $\epsilon E^{(1)}$ remain small compared to $E^{(0)}$ for all θ. From Eqs. (37) and (34), we obtain the desired slowly varying coupled Maxwell and Boltzmann equations

$$\frac{\partial E_s(\xi, \tau)}{\partial \xi} = D A_q \frac{1}{2\pi} \int_0^{2\pi} d\theta\, e^{-i\theta} \int d\mu\, h(\xi, \tau, \mu, \theta), \tag{38}$$

$$\frac{\partial h}{\partial \xi} + \frac{1}{2\gamma_0^2 c} \frac{\partial h}{\partial \tau} + \mu \frac{\partial h}{\partial \theta} = \kappa (E_s A_q^* e^{i\theta} + \text{c.c.}) \frac{\partial h}{\partial \mu}. \tag{39}$$

We have dropped the superscripts for simplicity and have restored the units of the independent variables. In the analysis below we again omit the factors of $1/2\gamma_0^2 c$. The harmonic approximation (see Sec. V) used in our previous work is obtained by keeping only the terms $n = -1, 0, 1$ in Eq. (35). This is a good approximation in the weakly saturated regime, but breaks down if there is strong saturation.[9]

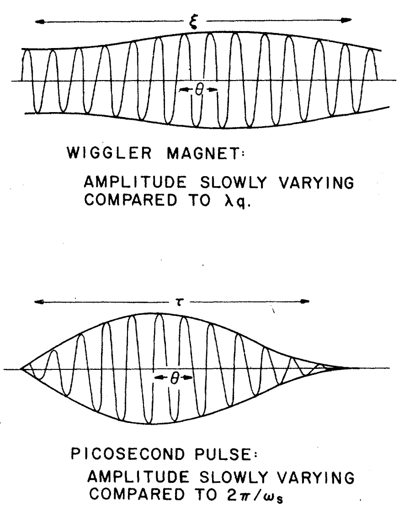

FIG. 1. Roles played by the slow variables ξ, τ and the fast variable θ in the description of the FEL.

IV. THE GENERALIZED PENDULUM EQUATIONS

Equations (38) and (39) can be converted to a more tractable form by introducing the Lagrangian (single-particle) variables $\hat{\theta}(\xi, \tau_0, \mu_0, \theta_0)$ and $\hat{\mu}(\xi, \tau_0, \mu_0, \theta_0)$. These are defined to obey the differential equations (generalized pendulum equations)

$$d\hat{\theta}(\xi, \tau_0, \mu_0, \theta_0)/d\xi = \hat{\mu}(\xi, \tau_0, \mu_0, \theta_0), \tag{40}$$

$$d\hat{\mu}(\xi, \tau_0, \mu_0, \theta_0)/d\xi = -\kappa\{A_q^*(\xi)E_s(\xi, \tau_0+\xi) \times \exp[i\hat{\theta}(\xi, \tau_0, \mu_0, \theta_0)] + \text{c.c.}\}, \tag{41}$$

together with the initial conditions

$$\hat{\theta}(0, \tau_0, \mu_0, \theta_0) = \theta_0, \tag{42}$$

$$\hat{\mu}(0, \tau_0, \mu_0, \theta_0) = \mu_0. \tag{43}$$

The variable τ_0 is the time at which the electrons enter the magnet. The derivatives in Eqs. (40) and (41) are of course really partial derivatives in which τ_0, μ_0, and θ_0 are held constant. However, we write them as total derivatives to emphasize the single-particle nature of the equations. It is easily shown that Eq. (39) has the formal solution

$$h(\xi, \tau, \mu, \theta) = \int d\mu_0 \int_0^{2\pi} d\theta_0 h(0, \tau-\xi, \mu_0, \theta_0)\delta(\mu - \hat{\mu}(\xi, \tau-\xi, \mu_0, \theta_0))\delta(\theta - \hat{\theta}(\xi, \tau-\xi, \mu_0, \theta_0)). \tag{44}$$

If we insert Eq. (44) into Eq. (38), we may carry out the integrations over θ and μ to obtain

$$\partial E_s(\xi, \tau)/\partial\xi = DA_q(\xi) \int d\mu_0 \frac{1}{2\pi} \int_0^{2\pi} d\theta_0 h(0, \tau-\xi, \mu_0, \theta_0)\exp[-i\hat{\theta}(\xi, \tau-\xi, \mu_0, \theta_0)]. \tag{45}$$

From Eq. (39) we see that $h(\xi, \tau, \mu, \theta)$, when integrated over θ and μ, is a function of $\tau - \xi$. This means that the temporal distribution of the electrons does not change appreciably during propagation through the magnet. Taking into account the definition of D [Eq. (21′)] and the normalization of h [given following Eq. (19)], we have

$$D \int d\mu \frac{1}{2\pi} \int_0^{2\pi} d\theta\, h(\xi, \tau, \mu, \theta) = \left(\frac{\alpha}{\sigma^2}\right) I(\tau - \xi), \tag{46}$$

where $I(\tau-\xi)$ is the electron current and we define

$$\alpha = e/2mc\gamma_0, \tag{47}$$

$$\sigma = (\epsilon_0 A_0 c)^{1/2}. \tag{48}$$

Recall that A_0 is the laser mode area. Assuming that the electrons are unbunched at $\xi = 0$, so that $h(0, \tau, \mu, \theta)$ is independent of θ, Eq. (46) suggests that we define $\mathcal{I}(\tau, \mu)$ to be the current per unit interval in μ entering the magnet:

$$Dh(0, \tau, \mu, \theta) = (\alpha/\sigma^2)\mathcal{I}(\tau, \mu). \tag{49}$$

Substituting Eq. (49) into Eq. (45) gives

$$\sigma^*(\xi)\partial[\sigma(\xi)E_s(\xi, \tau)]/\partial\xi = \alpha A_q(\xi) \int d\mu_0 \mathcal{I}(\tau-\xi, \mu_0)\frac{1}{2\pi} \int_0^{2\pi} d\theta_0 \exp[-i\hat{\theta}(\xi, \tau-\xi, \mu_0, \theta_0)]. \tag{50}$$

In writing Eq. (50) we have allowed the possibility that $\sigma(\xi)$ is a function of ξ so as to approximately account for diffractive spreading of the laser beam. This generalization is of some practical importance. Further discussion of diffraction is given in Appendix A. We note that the beam power P (see Appendix A of Ref. 4) is given by

$$P = \epsilon_0 A_0 c|E_s|^2 = |\sigma E_s|^2. \tag{51}$$

Equations (50) and (40)–(43) are the fundamental equations for a helical magnet with slowly varying amplitude. These equations may be solved numerically by conventional techniques. Solutions for the pulsed regime of the FEL will be discussed in a future publication. There are several cases in which these equations simplify.

A. cw operation. This limit is obtained by assuming that all variables are independent of τ. Even if $\mathcal{I}$ is independent of τ, cw operation may not result, because of the large number ($\sim 10^4$) of modes within the gain bandwidth. However, presumably cw operation can be achieved by using an intracavity etalon. In cw operation, the equations do not determine the frequency of the field selfconsistently. This means that one cannot center $\mathscr{I}(\mu_0)$ at $\mu_0 = 0$ without loss of generality, unless one compensates by redefining $A_q(\xi)$. This is explained further in paper II.

B. Small-diffraction limit. In this case we may take σ as constant.

C. Cold-beam limit. If the electrons all enter with the same energy, we have

$$\mathscr{I}(\tau, \mu_0) = I(\tau)\delta(\mu_0). \tag{52}$$

In this case one may suppress the variable μ_0 in the equations.

D. Small-gain limit. If $\mathscr{I}$ is sufficiently small, gain of the field can be neglected. One can nevertheless investigate the motion of single electrons as they traverse the magnet and calculate effects such as energy spreading and recoil.

E. Standard magnet. If A_q is constant or is of the form $A_q(\xi) = A_q e^{-i\bar{\mu}\xi}$, then we have the standard constant-amplitude field used in the Stanford experiment. However, it may be possible to improve the performance of the FEL by considering more general magnet geometries. In particular, we are investigating the possibility that other geometries may help to minimize the problem of energy spread.

F. Small-signal limit. If the laser field is sufficiently weak, the angles $\delta\theta = \hat{\theta} - (\theta_0 + \mu_0\xi)$ are small, and it is possible to linearize the equations. Further details are given in paper II. If one considers the case where A, B, D, and E, all apply, then one recovers the pendulum-equation description of the FEL which has been developed by Colson, Louisell, etc.[8,9]

V. COMPARISON WITH PREVIOUS THEORY

In this section we compare our present results with our previous theory,[1-5] regarding notation and accuracy.

Most of the notational changes in this paper were made in order to be able to deal with arbitrary magnet geometries. This general approach led to the realization that the electric field, rather than the vector potential, is the natural variable with which to describe the laser field. In addition, we have found it convenient to change the normalization of the field amplitudes and the sign of the detuning parameter μ. The previous theory applies in the case of a helical magnet with constant Δ. The transition from new notation to old notation is then as follows: $\Delta^{-1}\xi \to z$, $\Delta^{-1/2}\gamma_0 \to \gamma_s$, $\Delta^{1/2}m \to M$, $-\mu\Delta \to \mu$, $c(1-\Delta/2\gamma_0^2) \to v_s$, $E_s \to 2^{1/2}i\omega_s A_s$, $\Delta A_q \to 2^{1/2}A_i$, $\Delta^{-1}D \to \omega_s D$, $2^{3/2}\omega_s\Delta\kappa \to \kappa$, $c_0(\xi, \tau, \mu) \to n(z, \mu, \tau)$, $c_1(\xi, \tau, \mu) \to -ig_1^*(z, \mu, \tau)$.

The harmonic approximation used in our previous theory may be obtained by substituting the expansion (35) into Eq. (39), keeping only the terms $n = -1, 0, 1$, and neglecting products which lead to higher harmonics than the first. In this way we obtain the coupled quasi-Bloch equations

$$\frac{\partial c_0}{\partial \xi} + \frac{\partial c_0}{\partial \tau} = \kappa\left(E_s^* A_q \frac{\partial c_1}{\partial \mu} + \text{c.c.}\right), \tag{53}$$

$$\frac{\partial c_1}{\partial \xi} + \frac{\partial c_1}{\partial \tau} + i\mu c_1 = \kappa E_s A_q^* \frac{\partial c_0}{\partial \mu}. \tag{54}$$

These equations, together with the Maxwell equation (37), are structurally the same as the basic equations of our old theory. The bunching amplitude c_1, obeys the initial condition $c_1(0, \tau, \mu) = 0$. If we had kept the higher harmonics, then the equation for c_1 would have been driven by c_0 and c_2, and the equation for c_2 would have been driven by c_1 and c_3, etc. If $|c_1| \ll |c_0|$, then it should be a good approximation to neglect the higher harmonics. The small-signal limit is obtained by replacing c_0 on the right side of Eq. (54) by the input distribution $c_0(0, \tau, \mu)$. Equations (54) and (37) then form a closed set of linear equations for the field E_s and the bunching amplitude c_1. This procedure gives the correct small-signal behavior of the field, but has the disadvantage of neglecting electron energy spreading *ab initio*.

To illustrate the equivalence of our old and new theories in the small-signal regime, let us consider the diffractionless cw cold-beam limit. In this case we may put

$$c_0 = (\alpha I/\sigma^2 D)\delta(\mu) \tag{55}$$

in Eq. (54), so that

$$\frac{\partial c_1}{\partial \xi} + i\mu c_1 = \left(\frac{\kappa\alpha I}{\sigma^2 D}\right) E_s A_q^* \delta'(\mu). \tag{56}$$

In order to solve Eq. (56), we make the ansatz

$$c_1(\xi, \mu) = (\alpha I/\sigma^2 D)[iK_2(\xi)\delta(\mu) + K_1(\xi)\delta'(\mu)]. \tag{57}$$

Substituting Eq. (57) into (56), using the identity $\mu\delta'(\mu) = -\delta(\mu)$, and equating coefficients of $\delta(\mu)$ and $\delta'(\mu)$, we obtain

$$dK_2/d\xi = K_1, \tag{58}$$

$$dK_1/d\xi = \kappa A_q^* E_s. \tag{59}$$

Furthermore, substituting Eq. (57) into (37), we obtain

$$dE_s/d\xi = (i\alpha I/\sigma^2)K_2. \tag{60}$$

Equations (58)–(60) are the same as Eqs. (A11), (A12), and (A10) derived in paper II using our new theory. It is not hard to show that this equivalence of our old and new theories holds also for the pulsed FEL and for a broad incident energy distribution.

Louisell *et al.*[9] have calculated saturation-detuning curves for the single-mode FEL, using the pendulum-equation approach. These results differ at high power levels from our less accurate results obtained using the quasi-Bloch equations.[2] Louisell's results were obtained in the cold-beam limit, and ours were close to this limit. The quasi-Bloch approach should be more accurate with a broad incident distribution than it is in the cold-beam limit. This is because δ-function singularities in c_0 give rise to singularities in c_1 [see, for example Eq. (57)], so that one cannot assert that $|c_1| \ll |c_0|$. In this case the neglect of higher harmonics does not seem to be justified.

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APPENDIX A: DIFFRACTIVE BEAM SPREADING

In Eq. (50) we allow for the possibility that $\sigma(\xi)$ may be a complex function of ξ in order to account approximately for diffractive spreading of the laser beam. The way in which $\sigma(\xi)$ is included in the Maxwell equation (50) is motivated by the requirement that energy be conserved. Recall that E_s is the on-axis field. The optical energy $\mathcal{E}_{\text{opt}}$ of a light pulse passing the point ξ is found by integrating Eq. (51) over τ:

$$\mathcal{E}_{\text{opt}}(\xi) = \int d\tau |\sigma E_s|^2 . \tag{A1}$$

In order to obtain conservation of energy, we differentiate Eq. (A1) and insert Eq. (50) [multiplied by $E_s^*(\xi, \tau)$] to get

$$d\mathcal{E}_{\text{opt}}(\xi)/d\xi = \alpha \int d\tau \int d\mu_0 \mathcal{I}(\tau - \xi, \mu_0) \frac{1}{2\pi} \int_0^{2\pi} d\theta_0 \{A_q(\xi) E_s^*(\xi, \tau) \exp[-i\hat{\theta}(\xi, \tau - \xi, \mu_0, \theta_0)] + \text{c.c.}\}. \tag{A2}$$

Converting the integral over τ to an integral over $\tau_0 = \tau - \xi$ and inserting Eq. (41), we obtain

$$d\mathcal{E}_{\text{opt}}(\xi)/d\xi = -(\alpha/\kappa) \frac{d}{d\xi} \int d\tau_0 \int d\mu_0 \mathcal{I}(\tau_0, \mu_0) \frac{1}{2\pi} \int_0^{2\pi} d\theta_0 \hat{\mu}(\xi, \tau_0, \mu_0, \theta_0) . \tag{A3}$$

It follows from Eqs. (12), (18), and (47) that the right side of Eq. (A3) is the rate of change of the net recoil energy of the electrons passing the point ξ. Thus, total energy is conserved.

The ratio $A_q(\xi)/\sigma^*(\xi)$ is the function of principal importance in determining the dependence of FEL behavior on magnet and laser geometry. To some extent at least, it should be possible to affect the behavior of the FEL in the same way either by changing the magnet geometry or by putting optical elements (lenses, apertures, etc.) into the beam. The latter method would obviously be experimentally simpler and more versatile.

It remains to be seen how $\sigma(\xi)$ should be determined. In general, this is not clear. However, if the mirror losses are small, it should generally be a good approximation to use the function $\sigma(\xi)$ corresponding to Gaussian beam propagation[14] in the bare resonator. If the beam waist w_0 is located in the center of the magnet, then we take

$$\sigma = (\tfrac{1}{2}\pi\epsilon_0 c)^{1/2} w_0 [1 + i(2z - L)/w_0^2 k_s] . \tag{A4}$$

Note that the function σ accounts for both the beam spreading and the optical phase shift associated with diffraction. The phase shift acts to decrease the frequency of the laser radiation by an amount typically on the order of the gain bandwidth. Also note that in the limit of zero current, Eqs. (50) and (A4) give the correct z dependence of the on-axis field for the fundamental Gaussian mode of the resonator.

We may derive the Maxwell equation (50) corresponding to laser beam propagation in the fundamental Gaussian mode by starting with the paraxial wave equation[14]

$$(\partial/\partial z + \nabla_T^2/2ik_s) E_s(\xi, \tau, \vec{r}) = u(\vec{r}) J(\xi, \tau) . \tag{A5}$$

Here ∇_T^2 is the transverse Laplacian, $u(\vec{r})$ is the normalized transverse distribution of the electrons, and $J(\xi, \tau)$ is defined by

$$J(\xi, \tau) = \left(\frac{\alpha \Delta}{\epsilon_0 c}\right) A_q(\xi) \int d\mu_0 \mathcal{I}(\tau - \xi, \mu_0) \times \frac{1}{2\pi} \int_0^{2\pi} d\theta_0 \exp[-i\hat{\theta}(\xi, \tau - \xi, \mu_0, \theta_0)] . \tag{A6}$$

Equation (A5) provides a more exact treatment of diffraction than Eq. (50), but is considerably more complicated to solve. We now show that Eq. (50) may be obtained by projecting Eq. (A5) onto the fundamental Gaussian mode of the resonator.

The resonator modes $\psi_n(z, \vec{r})$ are a complete orthonormal set of solutions of the homogeneous paraxial wave equation

$$(\partial/\partial z + \nabla_T^2/2ik_s)\psi_n(z, \vec{r}) = 0\,. \tag{A7}$$

In particular, the fundamental mode is

$$\psi_0(z, \vec{r}) = (\pi/2)^{-1/2}[1/w(z)]\exp[-r^2/w_0 w(z)]\,, \tag{A8}$$

where

$$w(z) = w_0[1 + i(2z - L)/w_0^2 k_s]\,. \tag{A9}$$

We may expand $E_s(\xi, \tau, \vec{r})$ in terms of the mode functions $\psi_n(z, \vec{r})$ as

$$E_s(\xi, \tau, \vec{r}) = \sum_{n=0}^{\infty} C_n(z, \tau)\psi_n(z, \vec{r})\,. \tag{A10}$$

Inserting Eq. (A10) into (A5) and using (A7), we obtain

$$\sum_{n=0}^{\infty}\left(\frac{\partial C_n}{\partial z}\right)\psi_n(z, \vec{r}) = u(\vec{r})J(\xi, \tau)\,. \tag{A11}$$

If we multiply Eq. (A11) by $\psi_m^*(z, \vec{r})$ and integrate over the transverse coordinates, we obtain

$$\partial C_m(z, \tau)/\partial z = \int d^2r\,\psi_m^*(z, \vec{r})u(\vec{r})J(\xi, \tau)\,. \tag{A12}$$

Let us consider the term $m = 0$ in Eq. (A12). Since the filling factor is assumed small, we may evaluate $\psi_0^*(z, \vec{r})$ at $\vec{r} = 0$ on the right-hand side of (A12) and carry out the transverse integration to get

$$\partial C_0(z, \tau)/\partial z = \psi_0^*(z, 0)J(\xi, \tau)\,. \tag{A13}$$

At this point we assume that we can truncate the sum in Eq. (A10) and only retain the term for $n = 0$. Thus, we have

$$E_s(\xi, \tau) = E_s(\xi, \tau, 0) = C_0(z, \tau)\psi_0(z, 0)\,. \tag{A14}$$

Using Eq. (A14) in (A13), we obtain

$$\partial[E_s(\xi, \tau)/\psi_0(z, 0)]/\partial z = \psi_0^*(z, 0)J(\xi, \tau)\,. \tag{A15}$$

Comparing Eqs. (A8), (A9), and (A4), we find that

$$\psi_0(z, 0) = (\epsilon_0 c)^{1/2}/\sigma(\xi)\,, \tag{A16}$$

so that

$$\sigma^*(\xi)\partial[\sigma(\xi)E_s(\xi, \tau)]/\partial z = \epsilon_0 c J(\xi, \tau)\,, \tag{A17}$$

which is equivalent to Eq. (50).

In situations where diffractive effects are more complicated than simple Gaussian beam spreading, it is not clear whether Eq. (50) gives a good description of diffraction, or, if so, how one should choose $\sigma(\xi)$. For instance, it may be necessary to use the paraxial equation (A5) to account for the results of experiments at Stanford[15] in which the cavity losses of the FEL were adjusted by aperturing one of the resonator mirrors. We have done numerical calculations of the fundamental mode of the bare resonator and obtain substantial disagreement with Eq. (A4) when such aperturing is present.

APPENDIX B: A SIMPLE EXAMPLE OF MULTIPLE SCALING

Although the techniques of multiple-scaling perturbation theory are described in the textbooks,[13] few laser physicists seem to be familiar with the theory. Therefore, we believe it is justified to include a simple example here for pedagogical purposes. Consider the equation

$$\ddot{x}(t) + \omega^2(t)x(t) = 0\,, \tag{B1}$$

where $\omega(t)$ is a prescribed slowly varying positive-valued function such that $|\dot{\omega}| \ll \omega^2$. We wish to use multiple-scaling perturbation theory to find an approximate solution to Eq. (B1) valid uniformly in t.

We begin by noting that, if ω is constant, Eq. (B1) has the solution

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t}\,. \tag{B2}$$

The phase angle of the oscillator is $\theta = \omega t$. If ω is not constant, we define the phase angle by

$$\theta = \int_0^t dt'\,\omega(t')\,. \tag{B3}$$

We use θ as the "fast" variable of the system. To describe the slow changes in ω we define a "slow" variable

$$s = \epsilon t\,, \tag{B4}$$

where ϵ is a small parameter. We regard $x(s, \theta)$ as depending on s and θ independently. However, $\omega(s)$ is regarded as depending only on s. From the chain rule we have

$$\frac{d}{dt} = \omega(s)\frac{\partial}{\partial\theta} + \epsilon\frac{\partial}{\partial s}\,. \tag{B5}$$

Inserting Eq. (B5) into Eq. (B1), we obtain

$$\omega^2\left(\frac{\partial^2 x}{\partial\theta^2} + x\right) = -2\epsilon\omega\frac{\partial^2 x}{\partial\theta\,\partial s} - \epsilon\omega'\frac{\partial x}{\partial\theta} - \epsilon^2\frac{\partial^2 x}{\partial s^2}\,, \tag{B6}$$

where $\omega'(s) = d\omega/ds$. We next assume a perturbation expansion

$$x=\sum_{n=0}^{\infty}\epsilon^n x^{(n)} . \tag{B7}$$

Using this expansion, the zeroth order of Eq. (B6) is simply

$$\frac{\partial^2 x^{(0)}}{\partial\theta^2}+x^{(0)}=0 , \tag{B8}$$

which has the solution

$$x^{(0)}(s,\theta)=A(s)e^{i\theta}+B(s)e^{-i\theta} . \tag{B9}$$

The first order of Eq. (B6) is

$$\begin{aligned}\omega^2\left(\frac{\partial^2 x^{(1)}}{\partial\theta^2}+x^{(1)}\right)&=-2\omega\frac{\partial^2 x^{(0)}}{\partial\theta\partial s}-\omega'\frac{\partial x^{(0)}}{\partial\theta}\\&=-2\omega(iA'e^{i\theta}-iB'e^{-i\theta})\\&\quad-\omega(iAe^{i\theta}-iBe^{-i\theta}) .\end{aligned} \tag{B10}$$

This equation is that of a driven harmonic oscillator. However, in order to prevent secular growth in $x^{(1)}$, the oscillator must not be driven on resonance. Therefore, the coefficients of $e^{i\theta}$ and $e^{-i\theta}$ on the right side must vanish. Thus, we get

$$\begin{aligned}&2\omega A'+\omega' A=0 ,\\&2\omega B'+\omega' B=0 .\end{aligned} \tag{B11}$$

These equations have the solution

$$\begin{aligned}&A(s)=A_0\omega^{-1/2}(s) ,\\&B(s)=B_0\omega^{-1/2}(s) .\end{aligned} \tag{B12}$$

Inserting Eqs. (B12) and (B3) in (B9), we obtain the lowest-order solution

$$\begin{aligned}x(t)=&A_0\omega^{-1/2}(t)\exp\left(i\int_0^t dt'\,\omega(t')\right)\\&+B_0\omega^{-1/2}(t)\exp\left(-i\int_0^t dt'\,\omega(t')\right).\end{aligned} \tag{B13}$$

For this simple example the mulitple-scaling technique is a more powerful tool than is actually required. Equation (B13) is, in fact, the familiar WKB solution to the linear equation (B1). However, the multiple-scaling technique can be applied successfully to many complex nonlinear problems, such as the FEL, where more elementary methods are ineffective.

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