Infrared radiation in electron-atom scattering

Leonard Rosenberg
Department of Physics, New York University, New York, New York, 10003
(Received 8 February 1980)

The theory of electron-atom scattering in a radiation field, either spontaneously produced or external, is formulated in such a way that the effectively strong interaction of the electron with soft photons (those with frequencies lying below some cutoff ω_s) is properly accounted for at the outset. Effects of the residual interaction can then be included using standard perturbation theory. This approach follows that taken some years ago by Bloch and Nordsieck in their fundamental treatment of infrared radiation in potential scattering. The fact that the target is taken here to be a compound system which itself interacts with the field introduces additional complications. A gauge transformation is introduced which helps considerably in the analysis of these multiparticle effects. Particularly simple results are obtained by treating both the cutoff frequency ω_s and the residual interaction strength as quantities of first order and neglecting second-order corrections. The cross section obtained in this way, when summed over all final states containing soft photons, is expressed in terms of the cross section for scattering in the absence of any interaction with the field. The first-order correction term, not present in the earlier Bloch-Nordsieck version, can be interpreted in terms of the classical bremsstrahlung radiation emitted as the result of an instantaneous collision. The low-frequency approximation for the scattering amplitude derived here contains as a limiting case the result obtained by Low and others for single-photon bremsstrahlung. It also reduces in the appropriate limit to the recently derived approximation for electron-atom scattering in a low-frequency laser field.

I. INTRODUCTION

A nonperturbative treatment of the interaction of a free electron with the low-frequency modes of the radiation field, which is required to avoid infrared divergences in the calculation of transition probabilities, was given some time ago by Bloch and Nordsieck.[1,2] In the physical picture presented by Bloch and Nordsieck one recognizes that for any given measuring apparatus it will be impossible to detect individual photons with frequencies which lie below some limiting value, ω_s, say. The cross section should then be summed over all final states which contain these unobservable (or "soft") photons. It was shown[1] that for scattering by a short-range potential, with ω_s taken to be sufficiently small, this summation could be readily performed. The result is just the cross section for scattering in the absence of any electron–soft-photon interaction. It still remains to include the effect of the interaction of the electron with hard photons, those with frequencies exceeding ω_s. Here, however, the standard techniques of renormalized perturbation theory are available.

In a previous paper[3] (referred to in the following as I) the Bloch-Nordsieck sum rule was generalized, in the context of a nonrelativistic potential scattering model, to include correction terms of first order in ω_s arising from soft-photon interactions. These correction terms take a rather simple form; they appear as energy shifts in the cross section formula and can be interpreted in terms of the unobserved classical bremsstrahlung radiation which accompanies the scattering process. The radiation energy is quite small under normal circumstances and this provides a check on the underlying low-frequency approximation. The expression for the scattering amplitude which was derived in I includes as special cases the nonrelativistic version of Low's approximation[4] for spontaneous single-photon bremsstrahlung and the more recently derived low-frequency approximation for scattering in a laser field.[5] Thus much of the earlier work on potential scattering in a low-frequency radiation field can be viewed as particular aspects of a single, more general formulation of the scattering problem.

The result would be more satisfying if the restriction to potential scattering were lifted. The chief motivation of the present work is to extend the analysis given in I to the case where the target is a neutral atom. (Electron scattering by an ion carrying a net charge is explicitly excluded in order to avoid, at the present time, the additional complications arising from a long-range Coulomb tail.) It might be thought at first that the effect of the internal structure of the target could be accounted for by introducing an effective electron-atom potential and then applying the methods used earlier for potential scattering. However, the potential was required to be local in the previous work, while the effective potential is in general nonlocal. Furthermore, the field interacts with all of the electrons, not only the projectile, and this must be accounted for in some approximation. Rather than attempting to overcome these obstacles by modifying the approach of I we take a different approach here, based on the use of a gauge transformation. The power and efficiency of gauge techniques in the derivation of low-frequency theorems has been demonstrated by Low[4] and others,[6] and it proves to be a useful tool in the present problem as well.

In Sec. II, as a first step in setting up a scattering formalism which accounts for infrared radiation, we introduce solutions of the Schrödinger equation for an electron interacting only with the low-frequency modes of the radiation field. These so-called coherent states play the role of modified plane waves. The discussion of the properties of these states follows along the lines of that given in I, but is more general, in that effects of electron recoil are considered. (One might have expected these effects to modify the form of the first-order correction term in the cross section. It turns out that the modification is minimal; it merely introduces Doppler shifts in the soft-photon frequencies.) As in I we include only the $\vec{p}\cdot\vec{A}$ contribution to the interaction, with the A^2 term ignored. This is justified in the Appendix through a detailed consideration of the effect of the A^2 term on the structure of the asymptotic states. Section II concludes with a derivation of several addition formulas involving the expansion coefficients of the coherent states; these will be useful in the subsequent analysis. A gauge transformation, which enables one to extract the effects of the soft photons from the scattering operator in a particularly simple way, is introduced in Sec. III. This procedure also allows for a treatment of the effect of the soft photons on the target system in initial and final states. The formalism is then applied in Sec. IV to the derivation of an approximation, valid to first order in ω_s, which relates the physical scattering amplitude to that which would be obtained in the absence of any soft-photon interactions. The result is then used to derive the generalized sum rule for the cross section. I conclude by pointing out how the low-frequency approximation for single-photon bremsstrahlung in electron-atom scattering,[7,8] as well as various versions of the low-frequency approximation for scattering in a laser field,[9,10] can be obtained as special cases of the result derived here.

II. PROPAGATION OF AN ELECTRON IN A RADIATION FIELD

States of the free radiation field can be characterized by specifying an occupation number for each mode of the field. A state $|n\rangle$, say, will correspond to the set $\{n_0, n_1, \ldots, n_N\}$. The number of modes is taken to be finite (for finite quantization volume L^3) in order to cut off the spectrum at the high-frequency end. This is necessary since the interaction of electrons with photons of arbitrarily high frequency cannot be consistently described in a nonrelativistic theory. Let $\vec{k}_i$ and $\vec{\lambda}_i$ represent the wave number and polarization vectors for the ith mode. The eigenvalue equation for the state $|n\rangle$ is

$$H_F|n\rangle = E_n|n\rangle\,, \tag{2.1a}$$

where with $\omega_i = k_i c$,

$$H_F = \sum_{i=0}^{N} \hbar\omega_i a_i^\dagger a_i \tag{2.1b}$$

and

$$E_n = \sum_{i=0}^{N} \hbar\omega_i n_i\,. \tag{2.1c}$$

In the absence of any interaction, and with the electron in a momentum state $|\vec{p}\rangle$, the electron-field state vector $|n;\vec{p}\rangle \equiv |n\rangle|\vec{p}\rangle$ satisfies

$$(p_e^2/2\mu + H_F)|n;\vec{p}\rangle = (p^2/2\mu + E_n)|n;\vec{p}\rangle\,. \tag{2.2}$$

The interaction may be introduced by replacing the electron kinetic energy operator $p_e^2/2\mu$ with $(\vec{p}_e - e\vec{A}/c)^2/2\mu$, the vector potential being represented as

$$\vec{A}(\vec{r}) = \sum_{i=0}^{N}\left(\frac{2\pi\hbar c^2}{\omega_i L^3}\right)^{1/2} \vec{\lambda}_i (a_i e^{i\vec{k}_i\cdot\vec{r}} + a_i^\dagger e^{-i\vec{k}_i\cdot\vec{r}})\,. \tag{2.3}$$

An alternative procedure, however, allows us to focus more directly on the infrared radiation problem of immediate interest. That is, at this stage only interactions involving soft photons, with frequencies lying below ω_s, are considered. Hard-photon interactions must be included in a correct description of the asymptotic states but this can be done at a later stage using standard renormalization techniques. Such a renormalization procedure (which is a finite one because of the low-frequency and high-frequency cutoffs) is of no particular interest in the present study and will not be described here. We consider then, the soft-photon contribution

$$\vec{A}_s(\vec{r}) = \sum_{i=0}^{s}\left(\frac{2\pi\hbar c^2}{\omega_i L^3}\right)^{1/2} \vec{\lambda}_i (a_i e^{i\vec{k}_i\cdot\vec{r}} + a_i^\dagger e^{-i\vec{k}_i\cdot\vec{r}})\,, \tag{2.4}$$

to the vector potential and look for a solution of the Schrödinger equation

$$[(\vec{p}_e - e\vec{A}_s/c)^2/2\mu + H_F]\,|\psi_{n\vec{p}}\rangle = E_{n\vec{p}}\,|\psi_{n\vec{p}}\rangle\,; \tag{2.5}$$

the modified plane-wave state $|\psi_{n\vec{p}}\rangle$ is that which reduces to $|n;\vec{p}\rangle$ as the interaction is switched off. We attempt to represent the solution in the form

$$|\psi_{n\vec{p}}\rangle = W|n;\vec{p}\rangle, \tag{2.6}$$

with

$$E_{n\vec{p}} = E_n + p^2/2\mu + \Delta_{\vec{p}}. \tag{2.7}$$

Equation (2.5) is equivalent to

$$[(p_e^2/2\mu + H_F), W]|n;\vec{p}\rangle = (-H'_{es} + \Delta_{\vec{p}})W|n;\vec{p}\rangle. \tag{2.8}$$

In the following the A_s^2 contribution to the soft-photon interaction H'_{es} will be ignored, an approximation which is justified in the Appendix. We have then,[11]

$$H'_{es} = -(e/\mu c)\vec{p}_e \cdot \vec{A}_s. \tag{2.9}$$

Note that $\vec{p}_e$ fails to commute with W due to small recoil effects (which were neglected in I). Now the total momentum operator, which is $\vec{p}_e + \vec{p}_F$ with

$$\vec{p}_F = \sum_{i=0}^{N} \hbar\vec{k}_i a_i^\dagger a_i, \tag{2.10}$$

does commute with W so that

$$[\vec{p}_e, W] = -[\vec{p}_F, W]. \tag{2.11}$$

This is a quantity of first order in ω_s since

$$[a_i, W] = [a_i^\dagger, W] = 0, \quad \omega_i > \omega_s. \tag{2.12}$$

It follows that

$$[p_e^2/2\mu, W] = [\vec{p}_e, W]\cdot\vec{p}_e/2\mu + (\vec{p}_e/2\mu)\cdot[\vec{p}_e, W] \cong (\vec{p}_e/\mu)\cdot[\vec{p}_e, W], \tag{2.13}$$

ignoring second-order corrections. From Eqs. (2.10) and (2.11) we have

$$[\vec{p}_e, W] = -\left[\sum_{i=0}^{s} \hbar\vec{k}_i a_i^\dagger a_i, W\right], \tag{2.14}$$

so that

$$[(p_e^2/2\mu + H_F), W] = [\tilde{H}_F, W], \tag{2.15a}$$

with

$$\tilde{H}_F \equiv \sum_{i=0}^{N} \eta_i a_i^\dagger a_i. \tag{2.15b}$$

We have defined $\eta_i = \hbar\omega_i$ for $\omega_i > \omega_s$ and

$$\eta_i = \hbar\omega_i - \vec{p}_e\cdot\hbar\vec{k}_i/\mu, \quad \omega_i \leqq \omega_s. \tag{2.16}$$

Returning now to Eq. (2.8) we expand the commutator on the left-hand side as

$$[\tilde{H}_F, W] = \sum_{i=0}^{s} \eta_i (a_i^\dagger [a_i, W] + [a_i^\dagger, W]a_i). \tag{2.17}$$

This is evaluated using the commutation relations, assumed for $\omega_i \leqq \omega_s$,

$$[a_i, W] = -\rho_i e^{-i\vec{k}_i\cdot\vec{r}}W, \tag{2.18a}$$

$$[a_i^\dagger, W] = -\rho_i e^{i\vec{k}_i\cdot\vec{r}}W, \tag{2.18b}$$

with ρ_i defined by

$$\eta_i\rho_i = \left(\frac{2\pi\hbar e^2}{\mu^2\omega_i L^3}\right)^{1/2}(\vec{p}_e\cdot\vec{\lambda}_i). \tag{2.19}$$

In this notation we have

$$H'_{es} = \sum_{i=0}^{s} \eta_i\rho_i(a_i e^{i\vec{k}_i\cdot\vec{r}} + a_i^\dagger e^{-i\vec{k}_i\cdot\vec{r}}). \tag{2.20}$$

Equation (2.8) then reduces to[11]

$$-\sum_{i=0}^{s} \eta_i\rho_i^2 W|n;\vec{p}\rangle = \Delta_{\vec{p}}W|n;\vec{p}\rangle. \tag{2.21}$$

This of course cannot be satisfied in general. However, with ρ_i taken to be of zeroth order (a point discussed further below), $\eta_i\rho_i^2$ is seen to be of first order in ω_s and the commutator $[\eta_i\rho_i^2, W]$ is, in view of Eq. (2.14), of second order. To first order, then, $\eta_i\rho_i^2$ may be replaced by its eigenvalue $\eta_{i\vec{p}}\rho_{i\vec{p}}^2$ corresponding to the state $|\vec{p}\rangle$. The approximate solution is then defined by the commutation relations (2.18) and the expression

$$\Delta_{\vec{p}} = -\sum_{i=0}^{s} \eta_{i\vec{p}}\rho_{i\vec{p}}^2 \tag{2.22}$$

for the level shift.

Note that the electron–soft-photon coupling appears on the right-hand side of Eqs. (2.18) in the form of the operator ρ_i, defined in Eqs. (2.19) and (2.16). Thus, the effective strength of the interaction is determined by the ratio of the interaction energy to the photon energy. The effective coupling cannot be considered to be weak since perturbation theory gives unphysical results. Here ρ_i is treated as a quantity of zeroth order; the interaction energy and the soft-photon energy are treated as first-order quantities and second-order corrections are ignored. The failure of ordinary perturbation theory in a construction of the asymptotic states, i.e., the inapplicability of a finite expansion in powers of ρ_i in the solution of Eqs. (2.18), may be ascribed to the existence of near degeneracies in the energy levels of the electron–soft-photon system. With regard to the electron–soft-photon interaction in intermediate states of the scattering process there are no near degeneracies (assuming the absence of sharp resonances) and first-order perturbation theory will be adequate. The target-field interaction may be treated perturbatively for a similar reason.[12]

In the remainder of this section we review some of the properties of the wave operator W which will be useful later on. To begin with we note that W is unitary. This may be proved by using Eqs. (2.18) to show that $W^\dagger W$ commutes with a_i and $a_i^\dagger$; it is a c number in the space of photon states.

Since $W^\dagger W$ conserves both total momentum and photon momentum it must conserve electron momentum and hence must be a c number in electron momentum space. The relation $W^\dagger W = 1$ follows by choice of normalization. We then have the orthonormality relation

$$\langle\psi_{n\vec{p}'}|\psi_{n\vec{p}}\rangle = \langle n'; \vec{p}'|W^\dagger W|n; \vec{p}\rangle = \delta_{n'n}\delta(\vec{P}' - \vec{P}), \tag{2.23}$$

where $\vec{P}$ and $\vec{P}'$ represent the total momenta of the states $|n;\vec{p}\rangle$ and $|n';\vec{p}'\rangle$, respectively. Suppose now we represent W by the expansion

$$W|n;\vec{p}\rangle = \sum_l W_{ln}|l; \vec{P} - \vec{p}_l\rangle, \tag{2.24}$$

the sum running over all states of the field. (The coefficient W_{ln} should carry a label specifying the conserved total momentum $\vec{P}$, but it is omitted here for notational simplicity.) The unitarity property, when expressed in terms of the expansion coefficients, becomes

$$\sum_l W^*_{ln'}W_{ln} = \delta_{n'n}, \quad \vec{P}' = \vec{P}. \tag{2.25}$$

With electron recoil accounted for the various modes are not independent and in general an explicit closed-form solution of Eqs. (2.18) cannot be found. Fortunately, the unitarity property (2.25) allows us to determine the leading term in the expression for the cross section summed over soft-photon states; this is shown in Sec. IV. More detailed properties are required to determine the first-order correction to the cross section. Here, however, we may approximate the expansion coefficients appearing in Eq. (2.24) by neglecting the effect of electron recoil. This introduces first-order errors in the expansion coefficients and second-order errors in the cross section. The approximate version of Eqs. (2.18), appropriate to an electron in a fixed momentum state $|\vec{p}\rangle$, is

$$[a_i, W(\rho_{\vec{p}})] = [a_i^\dagger, W(\rho_{\vec{p}})] = -\rho_{i\vec{p}}W(\rho_{\vec{p}}). \tag{2.26}$$

The argument $\rho_{\vec{p}}$ of W is meant to represent the set of eigenvalues $(\rho_{0\vec{p}}, \rho_{1\vec{p}}, \ldots, \rho_{s\vec{p}})$. The commutation relations (2.26) are satisfied by the choice

$$W(\rho_{\vec{p}}) = \exp\left(\sum_{i=0}^{s} \rho_{i\vec{p}}(a_i - a_i^\dagger)\right). \tag{2.27}$$

The solution (2.27) may be verified with the aid of the relations[13]

$$e^{A+B} = e^A e^B e^{-1/2[A,B]}, \tag{2.28a}$$

$$[A, e^B] = [A,B]e^B, \tag{2.28b}$$

which hold when $[A,B]$ commutes with both A and B. Thus, using the relation (2.28a) we may rewrite Eq. (2.27) as

$$W(\rho_{\vec{p}}) = \exp\left(-\sum_{i=0}^{s} \rho_{i\vec{p}}a_i^\dagger\right) \times \exp\left(\sum_{i=0}^{s} \rho_{i\vec{p}}a_i\right) \exp\left(-\tfrac{1}{2}\sum_{i=0}^{s} \rho^2_{i\vec{p}}\right). \tag{2.29}$$

The commutation relations (2.26) then follow by application of Eq. (2.28b).

Several useful properties of W in the no-recoil (dipole) approximation are recorded here. Firstly, we have the relation

$$W^\dagger(\rho_{\vec{p}'})W(\rho_{\vec{p}}) = W(\rho_{\vec{p}} - \rho_{\vec{p}'}). \tag{2.30}$$

This follows directly from the representation (2.29) along with the identity [deduced from Eq. (2.28a)]

$$e^B e^A = e^A e^B e^{-[A,B]}, \tag{2.31}$$

for $[A,B]$ a c number. The analog of Eq. (2.24) is

$$W(\rho_{\vec{p}})|n\rangle = \sum_l W_{ln}(\rho_{\vec{p}})|l\rangle. \tag{2.32}$$

In terms of these expansion coefficients Eq. (2.30) becomes

$$\sum_l W^*_{ln'}(\rho_{\vec{p}'})W_{ln}(\rho_{\vec{p}}) = W_{n'n}(\rho_{\vec{p}} - \rho_{\vec{p}'}). \tag{2.33}$$

It will be necessary in the following to evaluate the sum $\sum_l W^*_{ln'}(\rho_{\vec{p}'})E_l W_{ln}(\rho_{\vec{p}})$. This can be done by writing it as the matrix element

$$\langle n'|W^\dagger(\rho_{\vec{p}'})H_F W(\rho_{\vec{p}})|n\rangle = E_n\langle n'|W^\dagger(\rho_{\vec{p}'})W(\rho_{\vec{p}})|n\rangle + \langle n'|W^\dagger(\rho_{\vec{p}'})[H_F, W(\rho_{\vec{p}})]|n\rangle. \tag{2.34}$$

The commutator $[H_F, W]$ may be evaluated with the aid of Eqs. (2.26). Then, using Eqs. (2.33), we find

$$\sum_l W^*_{ln'}(\rho_{\vec{p}'})E_l W_{ln}(\rho_{\vec{p}}) = E_n W_{n'n}(\rho_{\vec{p}} - \rho_{\vec{p}'}) + \sum_{i=0}^{s} \hbar\omega_i B_i, \tag{2.35}$$

where

$$B_i = \rho^2_{i\vec{p}}W_{n'n}(\rho_{\vec{p}} - \rho_{\vec{p}'}) - \rho_{i\vec{p}}\langle n'|W(\rho_{\vec{p}} - \rho_{\vec{p}'})[a_i + a_i^\dagger]|n\rangle. \tag{2.36}$$

Had we commuted H_F to the left rather than to the right in Eq. (2.34) we would have obtained the alternative form

$$\sum_l W^*_{ln'}(\rho_{\vec{p}'})E_l W_{ln}(\rho_{\vec{p}}) = E_{n'}W_{n'n}(\rho_{\vec{p}} - \rho_{\vec{p}'}) + \sum_{i=0}^{s} \hbar\omega_i B'_i, \tag{2.37}$$

with

$$B_i' = \rho_{i\vec{p}'}^2 W_{n'n}(\rho_{\vec{p}} - \rho_{\vec{p}'}) - \rho_{i\vec{p}'}\langle n' | [a_i + a_i^\dagger] W(\rho_{\vec{p}} - \rho_{\vec{p}'}) | n\rangle . \quad (2.38)$$

In a similar way we may perform the sum

$$\sum_l W_{ln'}^*(\rho_{\vec{p}'})\vec{p}_l W_{ln}(\rho_{\vec{p}}) = \vec{p}_n W_{n'n}(\rho_{\vec{p}} - \rho_{\vec{p}'}) + \sum_{i=0}^{s} \hbar \vec{k}_i B_i = \vec{p}_{n'} W_{n'n}(\rho_{\vec{p}} - \rho_{\vec{p}'}) + \sum_{i=0}^{s} \hbar \vec{k}_i B_i' . \quad (2.39)$$

III. GAUGE TRANSFORMATION

Having studied the effect of the interaction of soft photons with the incident and scattered electron we now examine its effect on the electron-atom system during the collision process, and with the atomic target in asymptotic states. As indicated earlier there are no near degeneracies to amplify these effects; they will be treated to first order only. Due to some fortunate cancellations the result obtained, embodied in Eq. (3.16) below, is quite simple. Perhaps the most convenient way to exhibit these cancellations is by means of a gauge transformation generated by the unitary operator e^g, where

$$g = \left(\frac{ie}{\hbar c}\right) \sum_j F(\vec{r}_j) . \quad (3.1)$$

The sum runs over all the electron coordinates. With $\vec{p}_j$ representing the momentum operator for the jth electron we have

$$\vec{p}_j - (e/c)\vec{A}(\vec{r}_j) = e^g\{\vec{p}_j - (e/c)[\vec{A}(\vec{r}_j) - \vec{\nabla}_j F(\vec{r}_j)]\} e^{-g}. \quad (3.2)$$

The transformation can be used to separate off the soft-photon contribution $\vec{A}_s$ if F can be chosen to satisfy $\vec{\nabla} F = \vec{A}_s$. Suppose now we introduce the dipole approximation $\vec{A}_s(\vec{r}_j) \cong \vec{A}_s(0)$, thereby ignoring the small effect of electron recoil. This is reasonable since we are calculating a first-order correction term and we do not attempt to keep track of second-order corrections. We may then take $F(\vec{r}) = \vec{A}_s \cdot \vec{r}$, so that[14]

$$g = \left(\frac{ie}{\hbar c}\right) \sum_j \vec{A}_s \cdot \vec{r}_j , \quad (3.3)$$

with $\vec{A}_s$ understood to be $\vec{A}_s(0)$ in the following.

In order to determine the effect of the transformation on the free-field Hamiltonian H_F we write

$$e^g H_F e^{-g} = H_F + [g, H_F] , \quad (3.4)$$

correct of first order in g. Since only the soft-photon part of H_F contributes to the commutator, that term is of first order in ω_s, as well as of first order in the interaction strength; it will be treated as a quantity of second order and will be dropped.[15] Thus H_F is unchanged by the transformation to first order. Now the total Hamiltonian may be expressed as

$$H = \sum_j \left(\vec{p}_j - \frac{e}{c} \vec{A}(\vec{r}_j)\right)^2 / 2\mu + H_F + V , \quad (3.5)$$

where V represents the (gauge-invariant) sum of the interparticle Coulomb potentials. According to the preceding discussion we have, to first order,

$$H = e^g \overline{H} e^{-g} , \quad (3.6)$$

where $\overline{H}$ is the modified Hamiltonian in which soft-photon interactions have been removed; i.e., it is obtained from Eq. (3.5) by replacing $\vec{A}$ with $\vec{A} - \vec{A}_s$. It follows that the resolvents $G(E) = (E - H)^{-1}$ and $\overline{G}(E) = (E - \overline{H})^{-1}$ are related by

$$G(E) = e^g \overline{G}(E) e^{-g} . \quad (3.7)$$

Equation (3.7) allows us to extract the effect of soft-photon interactions which occur during the collision. We must also consider soft-photon interactions with the target in asymptotic states. Let α represent the set of observables which define the asymptotic state $|\Phi_\alpha\rangle$. These include the photon occupation numbers, the electron momentum, and the quantum numbers specifying the atomic state of the isolated target. Since we are accounting for electron exchange (the antisymmetrized scattering amplitude is obtained by taking the appropriate linear combination of direct and exchange amplitudes in the usual way) we include in the set α a channel index specifying which of the electrons acts as the projectile. As discussed in Sec. II, we include only soft-photon interactions in the construction of the asymptotic states with the understanding that the effect of hard-photon interactions is to be included by a subsequent renormalization procedure. The Schrödinger equation for $|\Phi_\alpha\rangle$ then takes the form

$$(H_e + H_{es}' + H_F + H_T + H_{Ts}') |\Phi_\alpha\rangle = E_\alpha |\Phi_\alpha\rangle . \quad (3.8)$$

Here H_e is the kinetic energy operator for the projectile electron (channel labels will frequently be omitted to simplify notation), H_T is the Hamiltonian of the isolated target, and H_{Ts}' is the target–soft-photon interaction. The energy E_α is the sum of the electron-field energy $E_{n\vec{p}}$ [defined by Eq. (2.7)] and the energy E_T of the isolated target. A target level shift, arising from the emission and reabsorption of soft photons, contributes only to second order in the interaction H_{Ts}' and is therefore omitted.

Consider now the state $|\bar{\Phi}_\alpha\rangle \equiv |\chi\rangle|\psi_{n\vec{p}}\rangle$, where $|\psi_{n\vec{p}}\rangle$ is the electron-field state satisfying Eq. (2.5), and $|\chi\rangle$ is the state of the isolated target, a solution of

$$H_T|\chi\rangle = E_T|\chi\rangle . \tag{3.9}$$

The state $|\bar{\Phi}_\alpha\rangle$ satisfies

$$(H_e + H'_{es} + H_F + H_T)|\bar{\Phi}_\alpha\rangle = E_\alpha|\bar{\Phi}_\alpha\rangle , \tag{3.10}$$

in which the interaction H'_{Ts} does not appear. It is readily seen that in the approximation adopted here in which H'_{Ts} is treated to first order we have the relation

$$|\Phi_\alpha\rangle = e^{g_T}|\bar{\Phi}_\alpha\rangle , \tag{3.11}$$

where g_T is given by an expression of the form (3.3), but with the sum running over target electron coordinates only. Equation (3.11) is verified by using it to rewrite Eq. (3.10) as

$$e^{g_T}(H_e + H'_{es} + H_F + H_T)e^{-g_T}|\Phi_\alpha\rangle = E_\alpha|\Phi_\alpha\rangle . \tag{3.12}$$

This is of the required form (3.8), the effect of the transformation of the Hamiltonian in parentheses being simply to replace H_T by $H_T + H'_{Ts}$.

Having defined the asymptotic states we can, following the standard procedure of time-independent scattering theory,[16] write down an expression for the transition operator $T_{\alpha'\alpha}$. The final state $|\Phi_{\alpha'}\rangle$ corresponds to the target system in a discrete state, possibly different from its initial state; the final photon state may involve a change in the number of hard as well as soft photons. We have, with $E_{\alpha'} = E_\alpha \equiv E$,

$$T_{\alpha'\alpha} = \langle\Phi_{\alpha'}|(H - E)|\Phi_\alpha\rangle + \langle(H - E)\Phi_{\alpha'}|G(E)(H - E)|\Phi_\alpha\rangle . \tag{3.13}$$

As noted earlier a renormalization procedure leading to hard photon contributions to projectile and target level shifts must still be performed. This will remove disconnected parts (in which the projectile and target interact with the field but not with each other) appearing in the expression (3.13). Equations (3.6) and (3.7) may now be used to write

$$T_{\alpha'\alpha} = \langle e^{-g}\Phi_{\alpha'}|(\bar{H} - E)|e^{-g}\Phi_\alpha\rangle + \langle(\bar{H} - E)e^{-g}\Phi_{\alpha'}|\bar{G}(E)(\bar{H} - E)|e^{-g}\Phi_\alpha\rangle . \tag{3.14}$$

Setting $g = g_e + g_T$ and making use of Eq. (3.11) we have

$$e^{-g}|\Phi_\alpha\rangle = e^{-g_e}|\bar{\Phi}_\alpha\rangle . \tag{3.15a}$$

Similarly, we may write $g = g'_e + g'_T$, the primes indicating the identification of projectile and target appropriate to channel α', to obtain

$$e^{-g}|\Phi_{\alpha'}\rangle = e^{-g'_e}|\bar{\Phi}_{\alpha'}\rangle . \tag{3.15b}$$

Equation (3.14) then reduces to

$$T_{\alpha'\alpha} = \langle e^{-g'_e}\bar{\Phi}_{\alpha'}|(\bar{H} - E)|e^{-g_e}\bar{\Phi}_\alpha\rangle + \langle(\bar{H} - E)e^{-g'_e}\bar{\Phi}_{\alpha'}|\bar{G}(E)(\bar{H} - E)|e^{-g_e}\bar{\Phi}_\alpha\rangle . \tag{3.16}$$

Note that the effect of the target–soft-photon interaction H'_{Ts} has disappeared in this expression.

IV. LOW-FREQUENCY APPROXIMATION

The soft-photon component of the interaction appears in the amplitude (3.16) only through its effect on the projectile in asymptotic states. Further analysis enables us to evaluate this effect explicitly, the result being expressed in terms of the scattering amplitude obtained by omitting soft-photon interactions entirely. In carrying out this analysis we specialize to the case in which there are no hard photons in initial and final states; it should be clear from the discussion how to proceed in the more general case. Emission and reabsorption of hard photons in intermediate states of the scattering process lead to corrections of second order and are omitted here.

To begin with we simplify Eq. (3.16) by noting that

$$(\bar{H} - E)e^{-g_e}|\bar{\Phi}_\alpha\rangle = V_\alpha e^{-g_e}|\bar{\Phi}_\alpha\rangle , \tag{4.1}$$

where V_α is the electron-target interaction in channel α. We then have

$$T_{\alpha'\alpha} = \langle e^{-g'_e}\psi_{n'\vec{p}'}|t_{\alpha'\alpha}(E - H_F)|e^{-g_e}\psi_{n\vec{p}}\rangle , \tag{4.2}$$

with

$$t_{\alpha'\alpha}(E - H_F) = \left\langle\chi'\left|\left[V_\alpha + V_{\alpha'}\left(E - H_F - \sum_j p_j^2/2\mu - V\right)^{-1}V_\alpha\right]\right|\chi\right\rangle . \tag{4.3}$$

The physical significance of the operator $t_{\alpha'\alpha}$ can be understood by looking at the matrix representation, in the basis of unperturbed electron-field states,

$$\langle l' ; \vec{q}'|t_{\alpha'\alpha}(E - H_F)|l ; \vec{q}\rangle = \delta_{l'l}t_{\alpha'\alpha}(E - E_l ; \vec{q}', \vec{q}) . \tag{4.4}$$

Here $t_{\alpha'\alpha}(e ; \vec{q}', \vec{q})$ represents the off-shell electron-atom scattering amplitude in the absence of the radiation field; the physical amplitude is obtained by going on the energy shell according to the conditions

$$e = q^2/2\mu + E_T = q'^2/2\mu + E'_T . \tag{4.5}$$

Returning now to Eq. (4.2) we write (with second-order terms ignored)

$$e^{-\mathcal{S}_e} = 1 - i(e/\hbar c)\vec{A}_s \cdot \vec{r}\,. \tag{4.6}$$

Treating the projectile coordinate $\vec{r}$ as an operator in momentum space we have

$$e^{-\mathcal{S}_e}|\psi_{n\vec{p}}\rangle = |\psi_{n\vec{p}}\rangle + i(|e|/\hbar c)\vec{A}_s \cdot (-i\hbar\vec{\nabla}_{\vec{p}})|\psi_{n\vec{p}}\rangle \tag{4.7a}$$

and similarly

$$e^{-\mathcal{S}'_e}|\psi_{n'\vec{p}'}\rangle = |\psi_{n'\vec{p}'}\rangle + (|e|/c)\vec{A}_s \cdot \vec{\nabla}_{\vec{p}'}|\psi_{n'\vec{p}'}\rangle\,. \tag{4.7b}$$

The T matrix then becomes

$$T_{\alpha'\alpha} = T^{(1)}_{\alpha'\alpha} + T^{(2)}_{\alpha'\alpha}\,, \tag{4.8a}$$

where the leading term is

$$T^{(1)}_{\alpha'\alpha} = \langle\psi_{n'\vec{p}'}|t(E - H_F)|\psi_{n\vec{p}}\rangle\,. \tag{4.8b}$$

(To simplify notation the channel indices on the t amplitude are omitted here and in the following.) The first-order correction term takes the form

$$T^{(2)}_{\alpha'\alpha} = \sum_{i=0}^{s}\left(\frac{2\pi\hbar e^2}{\omega_i L^3}\right)^{1/2}\vec{\lambda}_i \cdot \{\vec{\nabla}_{\vec{p}}\langle\psi_{n'\vec{p}'}|t(E - H_F)(a_i + a_i^\dagger)|\psi_{n\vec{p}}\rangle + \vec{\nabla}_{\vec{p}'}\langle\psi_{n'\vec{p}'}|(a_i + a_i^\dagger)t(E - H_F)|\psi_{n\vec{p}}\rangle\}\,. \tag{4.8c}$$

According to Eqs. (2.6), (2.24), and (4.4) the leading term may be written as

$$T^{(1)}_{\alpha'\alpha} = \sum_l t(E - E_l, \vec{P}' - \vec{p}_l, \vec{P} - \vec{p}_l)W^*_{ln'}W_{ln}\,. \tag{4.9}$$

It will be convenient to express the amplitude $t(e;\vec{q}',\vec{q})$ as $t(\nu,\tau,\xi',\xi)$ with the new scalar variables defined as[8]

$$\begin{aligned}&\nu = \tfrac{1}{2}(q^2/2\mu + q'^2/2\mu)\,, \quad \tau = (\vec{q}' - \vec{q})^2\,,\\ &\xi = e - q^2/2\mu - E_T\,, \quad \xi' = e - q'^2/2\mu - E'_T\,.\end{aligned} \tag{4.10}$$

The on-shell condition is now simply $\xi = \xi' = 0$. Suppose we expand the t matrix in Eq. (4.9) about the values

$$\xi = \xi' = 0\,, \quad \nu = \tfrac{1}{2}(p^2/2\mu + p'^2/2\mu)\,, \quad \tau = (\vec{P}' - \vec{P})^2\,. \tag{4.11}$$

Then, to first order in ω_s,

$$\begin{aligned}t(E - E_l; \vec{P}' - \vec{p}_l, \vec{P} - \vec{p}_l) = t &+ \frac{1}{2\mu}\frac{\partial t}{\partial\nu}[\vec{p}' \cdot (\vec{p}_{n'} - \vec{p}_l) + \vec{p} \cdot (\vec{p}_n - \vec{p}_l)] + \frac{\partial t}{\partial\xi}\left(E_n - E_l - \frac{\vec{p} \cdot (\vec{p}_n - \vec{p}_l)}{\mu} + \Delta_{\vec{p}}\right)\\ &+ \frac{\partial t}{\partial\xi'}\left(E - E_l - \frac{p'^2}{2\mu} - E'_T - \vec{p}' \cdot \frac{(\vec{p}_{n'} - \vec{p}_l)}{\mu}\right).\end{aligned} \tag{4.12}$$

The amplitude t and its derivatives are here understood to be evaluated at the values of the scalar variables given in Eq. (4.11). The first term on the right-hand side of Eq. (4.12) is independent of l; its contribution to $T^{(1)}_{\alpha'\alpha}$ is $t(\sum_l W^*_{ln'}W_{ln})$. Since this term is of zeroth order in ω_s the addition formula (2.33), derived in a no-recoil approximation, is inapplicable here. [Nor can Eq. (2.25) be applied since the total momentum of the electron-field system is changed by the collision.] Now the remaining three terms in Eq. (4.12) are of first order in ω_s; their contribution to the sum in Eq. (4.9) may therefore be evaluated very easily using the approximate set of W coefficients, along with the addition formulas which they satisfy, obtained in Sec. II with neglect of electron recoil. A brief calculation gives the result

$$\begin{aligned}T^{(1)}_{\alpha'\alpha} = t\sum_l W^*_{ln'}W_{ln} &+ \frac{1}{2\mu}\frac{\partial t}{\partial\nu}\left(-\vec{p}' \cdot \sum_{i=0}^{s}\hbar\vec{k}_i B'_i - \vec{p} \cdot \sum_{i=0}^{s}\hbar\vec{k}_i B_i\right) + \frac{\partial t}{\partial\xi}\left(\Delta_{\vec{p}}W_{n'n}(\rho_{\vec{p}} - \rho_{\vec{p}'}) - \sum_{i=0}^{s}\eta_{i\vec{p}}B_i\right)\\ &+ \frac{\partial t}{\partial\xi'}\left(\Delta_{\vec{p}'}W_{n'n}(\rho_{\vec{p}} - \rho_{\vec{p}'}) - \sum_{i=0}^{s}\eta_{i\vec{p}'}B'_i\right);\end{aligned} \tag{4.13}$$

the quantities appearing in large parentheses have been defined in Sec. II.

In evaluating the correction term (4.8c) the momentum derivatives are expressed in terms of the scalar variables as

$$\begin{aligned}\vec{\nabla}_{\vec{p}}t &= \frac{\vec{p}}{2\mu}\frac{\partial t}{\partial\nu} - \frac{\vec{p}}{\mu}\frac{\partial t}{\partial\xi} - 2(\vec{p}' - \vec{p})\frac{\partial t}{\partial\tau}\,,\\ \vec{\nabla}_{\vec{p}'}t &= \frac{\vec{p}'}{2\mu}\frac{\partial t}{\partial\nu} - \frac{\vec{p}'}{\mu}\frac{\partial t}{\partial\xi'} + 2(\vec{p}' - \vec{p})\frac{\partial t}{\partial\tau}.\end{aligned} \tag{4.14}$$

With recoil effects neglected in this first-order term the electron-field states reduce to

$$|\psi_{n\vec{p}}\rangle = |\vec{p}\rangle W(\rho_{\vec{p}})|n\rangle\,. \tag{4.15}$$

We make use of the relations

$$\begin{aligned}&\langle n'|W^\dagger(\rho_{\vec{p}'})(a_i + a_i^\dagger)W(\rho_{\vec{p}})|n\rangle\\ &\quad= -2\rho_{i\vec{p}}W_{n'n}(\rho_{\vec{p}} - \rho_{\vec{p}'}) + \langle n'|W(\rho_{\vec{p}} - \rho_{\vec{p}'})(a_i + a_i^\dagger)|n\rangle\\ &\quad= -2\rho_{i\vec{p}'}W_{n'n}(\rho_{\vec{p}} - \rho_{\vec{p}'}) + \langle n'|(a_i + a_i^\dagger)W(\rho_{\vec{p}} - \rho_{\vec{p}'})|n\rangle\,,\end{aligned} \tag{4.16}$$

along with the definitions (2.22), (2.36), and (2.38) in the evaluation of (4.8c). The result, to first order, is

$$T^{(2)}_{\alpha'\alpha}=\left(\Delta_{\vec{p}}W_{n'n}(\rho_{\vec{p}}-\rho_{\vec{p}'})-\sum_{i=0}^{s}\eta_{i\vec{p}}B_i\right)\left(\frac{1}{2}\frac{\partial t}{\partial\nu}-\frac{\partial t}{\partial\xi}\right)$$
$$+\left(\Delta_{\vec{p}'}W_{n'n}(\rho_{\vec{p}}-\rho_{\vec{p}'})-\sum_{i=0}^{s}\eta_{i\vec{p}'}B_i'\right)$$
$$\times\left(\frac{1}{2}\frac{\partial t}{\partial\nu}-\frac{\partial t}{\partial\xi'}\right). \tag{4.17}$$

Combining this with Eq. (4.13) we find that

$$T_{\alpha'\alpha}=t\left(\sum_{l}W^*_{ln'}W_{ln}\right)$$
$$+\frac{1}{2}\frac{\partial t}{\partial\nu}\left((\Delta_{\vec{p}}+\Delta_{\vec{p}'})W_{n'n}(\rho_{\vec{p}}-\rho_{\vec{p}'})\right.$$
$$\left.-\sum_{i=0}^{s}\hbar\omega_i(B_i+B_i')\right). \tag{4.18}$$

Note that the off-shell derivatives have canceled so that only physical values of the field-free electron-atom scattering amplitude appear here. As is easily verified, Eq. (4.18) contains as a special case the result derived earlier for potential scattering in the dipole approximation.[3]

The quantity of physical significance is the cross section summed over all final states containing soft photons. Since the initial state $|n\rangle$ is assumed to contain no hard photons and since the expansion coefficient $W_{n'n}$ vanishes if $|n'\rangle$ contains one or more hard photons, no error is made when the cross section sum is extended over all final states of the field. The cross section of interest then is $(2\pi)^4(\mu\hbar^2/p)dQ$ with

$$dQ=\sum_{n'}\int d^3p'\delta(E'+E_{n'}-E)\left|T_{\alpha'\alpha}\right|^2. \tag{4.19}$$

Here $E=E_\alpha$ as before and we have written $E_{\alpha'}=E'+E_{n'}$, with $E'=p'^2/2\mu+\Delta_{\vec{p}'}+E_T'$. A useful sum rule could be obtained for the case where the initial state $|n\rangle$ represents a strong external radiation field. The derivation would be very similar to that given previously for the case of potential scattering.[17] (The external field problem is discussed further below.) Here $|n\rangle$ is taken to be the vacuum state. If we ignore $E_{n'}$ in the argument of the energy conserving δ function and keep only the first term in Eq. (4.18) we obtain the zeroth-order approximation

$$dQ\cong\int d^3p'\delta(E'-E)\left|t\right|^2$$
$$\times\sum_{n'}\left(\sum_{l'}W_{l'n'}W^*_{l'0}\right)\left(\sum_{l}W^*_{ln'}W_{l0}\right). \tag{4.20}$$

While we have not provided an explicit representation of the W coefficients with recoil effects accounted for, the sums over field states can nevertheless be performed with the aid of the unitarity relations

$$\sum_{n'}W^*_{ln'}W_{l'n'}=\delta_{ll'}\,, \tag{4.21}$$
$$\sum_{l}W^*_{l0}W_{l0}=1\,.$$

The expression (4.20) then reduces to the cross section for electron-atom scattering in the absence of the radiation field.

In calculating the contribution from the first-order correction terms the W coefficients may be replaced by those obtained in Sec. II in the no-recoil approximation since this introduces only second-order errors. The sum is then easily performed using the addition formulas of Sec. II. Since the calculation at this stage is essentially identical to that carried out previously for the case of potential scattering in the no-recoil approximation[3] we merely quote the final result:

$$dQ=\int d^3p'\delta(E'-E+R_{\vec{p}'\vec{p}})\left|t[\bar{\nu},(\vec{P}'-\vec{P})^2]\right|^2. \tag{4.22}$$

Here

$$R_{\vec{p}'\vec{p}}\equiv\sum_{i=0}^{s}\hbar\omega_i(\rho_{i\vec{p}'}-\rho_{i\vec{p}})^2 \tag{4.23}$$

can be interpreted classically as the bremsstrahlung radiation emitted by an electron which has undergone an instantaneous collision changing its momentum from $\vec{p}$ to $\vec{p}'$. The t-matrix element in (4.22) represents the on-shell electron-atom scattering amplitude corresponding to the squared momentum transfer $\tau=(\vec{P}'-\vec{P})^2$ and energy

$$\bar{\nu}\equiv p^2/2\mu+\Delta_{\vec{p}}-R_{\vec{p}}\,. \tag{4.24}$$

The increment $R_{\vec{p}}\equiv|\Delta_{\vec{p}}|$ can be thought of classically as the energy radiated by an electron with momentum $\vec{p}$ when brought suddenly to rest. Thus the effective energy $\bar{\nu}$, which the electron can deliver to the target, is less than the kinetic energy $p^2/2\mu+\Delta_{\vec{p}}$ by the radiation energy $R_{\vec{p}}$.

The result (4.22) generalizes that obtained by Bloch and Nordsieck in a potential scattering model[1] by allowing for a target with internal structure. It also demonstrates that a knowledge of the physical field-free scattering cross section is sufficient to enable one to include first-order corrections arising from soft-photon interactions. While the calculation of hard-photon interaction effects will be more difficult in general, it is possible to improve the treatment of the leading term (4.20) by including the effect of damping on the sum over photon states. This can be done very simply using the dipole approximation for the hard-photon interactions[13]; then Eq. (2.6) for the asymptotic state is replaced by

$$|\psi_{n\vec{p}}\rangle = W_h(\rho_{\vec{p}})W|n;\vec{p}\rangle, \tag{4.25}$$

with

$$W_h(\rho_{\vec{p}}) = \exp\left(\sum_{i=s+1}^{N} \rho_{i\vec{p}}(a_i - a_i^\dagger)\right). \tag{4.26}$$

This modification has the effect, in the sum over states in Eq. (4.20), of introducing the replacement

$$\sum_l W_{ln'}^* W_{l0} \to e^{-(1/2)D}\sum_l W_{ln'}^* W_{ln}, \tag{4.27a}$$

where

$$D = \sum_{i=s+1}^{N} (\rho_{i\vec{p}} - \rho_{i\vec{p}'})^2, \tag{4.27b}$$

as can be seen by using the appropriate version of Eq. (2.30) and recalling that $|n'\rangle$ contains no hard photons. Equation (4.20) then becomes, in view of Eqs. (4.21),

$$dQ \cong \int d^3p'\delta(E'-E)e^{-D}|t|^2. \tag{4.28}$$

The low-frequency approximation for the scattering amplitude Eq. (4.18) contains as a special case the result obtained some time ago[7,8] for single-photon bremsstrahlung. This can be seen explicitly by expressing the operator W as

$$W \cong 1 + \sum_{i=0}^{s}\rho_i(e^{i\vec{k}_i\cdot\vec{r}}a_i - e^{-i\vec{k}_i\cdot\vec{r}}a_i^\dagger), \tag{4.29}$$

which is correct to first order in the interaction strength.

The approximation (4.18) is also applicable to the problem of electron-atom scattering in a multimode low-frequency laser field. The problem simplifies if spontaneous omission is ignored and all modes of the external field have the same propagation direction; the operator W can then be constructed explicitly, with recoil included. The result in that case is

$$W = \exp\left(\sum_{i=0}^{s}\rho_i(e^{i\vec{k}_i\cdot\vec{r}}a_i - e^{-i\vec{k}_i\cdot\vec{r}}a_i^\dagger)\right), \tag{4.30}$$

where now ω_s represents a low-frequency cutoff in the spectrum of the external field. The commutation relations (2.18) may be verified for the above choice of W by following an argument similar to that given below Eq. (2.27). The identities (2.28) are applicable by virtue of the transversality condition $\vec{k}_i\cdot\vec{\lambda}_j = 0$ for all modes i and j, which implies that $e^{i\vec{k}_i\cdot\vec{r}}$ commutes with ρ_j for all i and j. The expansion coefficients, which are easily obtained from the representation (4.30), can be shown to satisfy an addition formula of the type (2.33) derived earlier in the no-recoil approximation. If the field is sufficiently strong so that photon depletion effects are negligible the expansion coefficients take the form of products of Bessel functions, a result obtained earlier[18] using quite different methods. It may be derived directly from Eq. (4.30) by expanding the exponential, with the commutator $[a_i, a_i^\dagger]$ taken to be zero, and with the identification

$$J_\nu(2\rho) = \rho^\nu\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+\nu)!}\rho^{2m}. \tag{4.31}$$

If one further specializes to the single-mode case, with recoil effects ignored, the low-frequency approximation derived recently by Mittleman[9] is regained.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grant No. PHY-7910413 and by the Office of Naval Research under Contract No. N00014-76-C-0317.

APPENDIX

We consider here the properties of the asymptotic states $|\psi_{n\vec{p}}\rangle = W|n;\vec{p}\rangle$ assuming the electron–soft-photon interaction

$$H'_{es} = -(e/\mu c)\vec{p}_e\cdot\vec{A}_s + (e^2/2\mu c^2)A_s^2. \tag{A1}$$

As shown in Sec. II the Schrödinger equation is equivalent to the relation

$$[\tilde{H}_F, W]|n;\vec{p}\rangle = (-H'_{es} + \Delta_{\vec{p}})W|n;\vec{p}\rangle. \tag{A2}$$

Since the effective interaction is the ratio of the interaction energy to the photon energy we may expect that the additional A_s^2 term in the interaction, nominally of second order, actually introduces first-order corrections and cannot be discarded at the outset. Nevertheless, the sum rule (4.22) for the cross section is unchanged since it is based on general properties of the operator W which are preserved, to the required order, with the inclusion of the A_s^2 term. Specifically, the zeroth-order contribution to the cross section Eq. (4.20), can be evaluated as before since the unitarity property (4.21) is preserved. Taking into account the first-order correction term we may determine the W coefficients to lowest order, ignoring both recoil effects and the A_s^2 contribution. The derivation of the sum rule then proceeds in a manner identical to that described in Sec. IV.

To verify these expectations concerning the properties of the operator W we look for the appropriate modification of the commutation relations (2.18) in the form

$$[a_i, W] = -C_i\vec{\lambda}_i\cdot[\beta\vec{p}_e - \gamma(e/2c)\vec{A}_s]e^{-i\vec{k}_i\cdot\vec{r}}W, \tag{A3a}$$

$$[a_i^\dagger, W] = -C_i \vec{\lambda}_i \cdot [\beta \vec{p}_e - \gamma(e/2c)\vec{A}_s] e^{i\vec{k}_i \cdot \vec{r}} W, \quad \text{(A3b)}$$

with

$$\eta_i C_i = \left(\frac{2\pi e^2 \hbar}{\mu^2 \omega_i L^3}\right)^{1/2}, \quad \text{(A4)}$$

and with β and γ to be determined. These assumed commutation relations are inserted into the right-hand side of Eq. (2.17). The first term is

$$\sum_{i=0}^{s} \eta_i a_i^\dagger [a_i, W] = -\sum_{i=0}^{s} \eta_i C_i \vec{\lambda}_i \cdot \left(\beta \vec{p}_e - \gamma \frac{e}{2c} \vec{A}_s\right) a_i^\dagger e^{-i\vec{k}_i \cdot \vec{r}} W + \gamma \sum_{i=0}^{s} \left(\frac{e^2}{2\mu c^2}\right)\left(\frac{2\pi \hbar c^2}{\omega_i L^3}\right) W. \quad \text{(A5)}$$

According to the rule

$$L^{-3} \sum_{\vec{k}} \to (2\pi)^{-3} \int d^3 k, \quad \text{(A6)}$$

the second term on the right-hand side of Eq. (A5) is proportional to ω_s^2 so that its contribution to the level shift may be ignored. We then have

$$[\tilde{H}_F, W] = -\sum_{i=0}^{s} \left(\beta \vec{p}_e - \gamma \frac{e}{2c} \vec{A}_s\right) \cdot \vec{\lambda}_i \eta_i C_i \times (a_i^\dagger e^{-i\vec{k}_i \cdot \vec{r}} W + e^{i\vec{k}_i \cdot \vec{r}} W a_i). \quad \text{(A7)}$$

We now move a_i to the left of W using Eq. (A3a) and make the identification

$$\frac{|e|}{\mu c} \vec{A}_s = \sum_{i=0}^{s} \vec{\lambda}_i \eta_i C_i (a_i e^{i\vec{k}_i \cdot \vec{r}} + a_i^\dagger e^{-i\vec{k}_i \cdot \vec{r}}). \quad \text{(A8)}$$

Equation (A7) then becomes

$$[\tilde{H}_F, W] = \frac{e}{\mu c} \sum_{i=0}^{s} \left(\beta \vec{p}_e \cdot \vec{A}_s - \gamma \frac{e}{2c} A_s^2\right) W + X, \quad \text{(A9)}$$

with the additional term

$$X = -\sum_{i=0}^{s} \left[\left(\beta \vec{p}_e - \gamma \frac{e}{2c} \vec{A}_s\right) \cdot \vec{\lambda}_i\right]^2 \eta_i C_i^2 W, \quad \text{(A10)}$$

arising from the commutator. As discussed in connection with Eq. (2.21) this term, which is of first order, may be evaluated with the operator $\vec{p}_e$ replaced by its eigenvalue on the state $|\vec{p}\rangle$ since recoil corrections will contribute to second order. The sum over soft-photon modes in Eq. (A10) may then be carried out using Eq. (A6) and the identity

$$\int d\Omega_{\hat{k}} \sum_{\hat{\lambda}} (\vec{U} \cdot \vec{\lambda})(\vec{V} \cdot \vec{\lambda}) = \tfrac{2}{3}(4\pi) \vec{U} \cdot \vec{V}, \quad \text{(A11)}$$

valid for arbitrary fixed vectors $\vec{V}$ and $\vec{U}$. The result is of the form

$$X = \left(-\sum_{i=0}^{s} \beta^2 \eta_{i\vec{p}} \rho_{i\vec{p}}^2 + \beta\gamma y \frac{e}{2c} \vec{p}_e \cdot \vec{A}_s - \gamma^2 z \frac{e^2}{2\mu c^2} A_s^2\right) W. \quad \text{(A12)}$$

The first-order parameters y and z are readily evaluated but the explicit values will not be required here. When Eqs. (A9) and (A12) are combined we see that (A2) will be satisfied with β and γ determined by

$$\beta + \beta\gamma y = 1, \quad \text{(A13)}$$

$$\gamma + \gamma^2 z = 1, \quad \text{(A14)}$$

Since β differs from unity by a quantity of first order, β^2 may be replaced by unity in the first term on the right-hand side of Eq. (A12). Then $\Delta_{\vec{p}}$ is again determined as in Eq. (2.22).

It follows from Eqs. (A3) that $W^\dagger W$ commutes with each of the a_i and $a_i^\dagger$. The expected unitarity property of W is an immediate consequence, the argument being identical to that given in Sec. II.

It was mentioned above that in calculating the first-order correction to the sum rule for the cross section it is sufficient to evaluate the W coefficients to lowest order. This implies, in particular, that in the commutation relations (A3) the parameters β and γ are to be replaced by unity and the recoil effect generated by the momentum translation operators $e^{\pm i\vec{k}_i \cdot \vec{r}}$ is to be ignored. Furthermore, since $(e/2c)\vec{A}_s$ is of first order relative to $\vec{p}$ the term involving $\vec{A}_s$ is dropped. This of course leads us back to Eqs. (2.26) and the solution (2.27), and from there to the completion of the derivation of Eq. (4.22).

---

[1] F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937); A. Nordsieck, *ibid.* 52, 59 (1937).

[2] Modern versions are described in J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Springer, New York, 1976).

[3] L. Rosenberg, Phys. Rev. A 21, 157 (1980).

[4] F. E. Low, Phys. Rev. 110, 974 (1958).

[5] N. M. Kroll and K. M. Watson, Phys. Rev. A 8, 804 (1973).

[6] See, for example, L. S. Brown and R. L. Goble, Phys. Rev. 173, 1505 (1968).

[7] Low's derivation in Ref. 4 was relativistic and was based on gauge invariance. A nonrelativistic version applicable to the scattering of a charged particle by a neutral compound system was derived subsequently by H. Feshbach and D. R. Yennie, Nucl. Phys. 37, 150 (1962). These authors made use of the effective potential formalism, and included the effects of electron recoil. A somewhat different derivation, restricted to the potential scattering problem, can be found in Ref. 8.

[8] L. Heller, Phys. Rev. 174, 1580 (1968).

[9]The Kroll-Watson result of Ref. 5 has recently been generalized to the case of electron-atom scattering by M. H. Mittleman, Phys. Rev. A 21, 79 (1980).

[10]L. Rosenberg, Phys. Rev. A 20, 275 (1979). This paper extends the Kroll-Watson result through the inclusion of electron recoil corrections and by allowing for laser fields of arbitrary polarization. It is restricted, however, to potential scattering.

[11]Here we have recognized that $(\vec{p}_e \cdot \vec{\lambda}_i, e^{i\vec{k}_i \cdot \vec{r}}) = 0$ since $\vec{k}_i \cdot \vec{\lambda}_i = 0$ in the Coulomb gauge.

[12]It is not difficult to modify the present calculational procedure to account for scattering resonances, or for resonant interactions between the target system and an external radiation field. However, we must restrict ourselves to external field strengths which are not too large since the present treatment breaks down when the probability of target ionization by the field is appreciable. In this connection see also Ref. 9.

[13]See, for example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), Sec. 17.10.

[14]Transformations of the type considered here have frequently been employed in the past. See, for example, R. G. Sachs and N. Austern, Phys. Rev. 81, 705 (1951).

[15]Note that $[g, H_F] = e\vec{E}_s \cdot \sum_j \vec{r}_j$, where $\vec{E}_s$ is the soft-photon contribution to the electric field at the origin. The approximation in which this term is neglected is based on the assumption that multiphoton effects, while significant asymptotically as a result of near degeneracies, play no role in intermediate stages of the scattering process. For a discussion of the dangers involved in extending approximations of this type beyond their domain of applicability, see C. Cohen-Tannoudji, J. Dupont-Roc, C. Fabre, and G. Grynberg, Phys. Rev. A 8, 2747 (1973).

[16]M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), Chap. 3.

[17]L. Rosenberg, Phys. Rev. A 20, 1352 (1979).

[18]E. J. Kelsey and L. Rosenberg, Phys. Rev. A 19, 756 (1979), and references contained therein.