Wave-energy density and wave-momentum density of each species of a collisionless plasma

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Expressions for the wave-energy density and wave-momentum density of each species of a collisionless plasma are derived. The sum of the wave-energy (momentum) densities of all the species and the electromagnetic-energy (momentum) density gives the previously known result for the total wave-energy (momentum) density of a dispersive medium.

I. INTRODUCTION

Expressions[1-9] for the wave-energy density and wave-momentum density of a dispersive medium have been applied to a number of problems. The sign of the wave energy can be used to determine whether coupled waves are explosively unstable.[6] The ponderomotive-force density acting on a medium can be deduced[8,9] from the knowledge of the wave-energy density and wave-momentum density. In addition, Nevins[10] has shown that particle transport, caused by trapped-particle instabilities, can be explained by the increase in the wave-momentum density as the unstable wave grows.

In a collisionless plasma, the quantities of concern for each of the above questions can be divided into contributions from the separate plasma species. One may wish to know which species causes a wave to have negative energy, how much of the ponderomotive-force density acts on the ions and how much acts on the electrons, or the relative amounts of ion and electron transport caused by an instability. To answer these questions by the methods of the previous analyses,[6,8-10] one first needs to know the wave-energy density and wave-momentum density of the individual species.

The purpose of this paper is to derive the contributions of the separate species to the wave-energy density and wave-momentum density of the medium. This we do under the following assumptions: (1) The electric field is small. Hence, linear theory is valid. (2) The electric field has the wave-packet form; i.e., it is the product of a slowly varying amplitude with a plane wave. (3) Dissipation is small. (4) The plasma is collisionless. Upon obtaining these expressions for the wave-energy and wave-momentum density, we note that they and the electromagnetic contributions sum to give the previously known[3] formulas for the total wave-energy density and wave-momentum density of the medium.

II. WAVE-ENERGY DENSITY AND WAVE-MOMENTUM DENSITY OF THE SEPARATE SPECIES OF A COLLISIONLESS PLASMA

We consider a homogeneous, multispecies plasma under the influence of a small electric field. In this case, the current response of each species is given by a separate conductivity

$$\vec{J}^s(\vec{k},\omega)=\underline{\sigma}^s(\vec{k},\omega)\cdot\vec{E}(\vec{k},\omega)\,. \tag{1}$$

In addition, we follow Bers[3] in assuming that the electric field is the product of a slowly varying amplitude and a plane wave

$$\vec{E}(\vec{x},t)=\vec{\mathscr{E}}(\vec{x},t)\exp(i\vec{k}_0\cdot\vec{x}-i\omega_0 t)+\text{c.c.} \tag{2}$$

Furthermore, we assume dissipation to be small. Finally, we assume the plasma to be collisionless, so that a given species can obtain energy and momentum only from the macroscopic electromagnetic fields. Under these conditions we derive local conservation laws for the energy and momentum of each species, thereby obtaining the wave-energy density and wave-momentum density of the separate species of a collisionless plasma.

In order to derive these local conservation laws, we must first find local relations between the electric field and the current density of species s. To do this, we Fourier transform Eqs. (1) and (2) to obtain

$$\vec{J}^s(\vec{x},t)=\frac{1}{(2\pi)^4}\int d^3k\,d\omega\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}\underline{\sigma}^s(\vec{k},\omega)\cdot\vec{E}(\vec{k},\omega) \tag{3}$$

and

$$\begin{aligned}\vec{E}(\vec{k},\omega)&=\int d^3x\,dt\,e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\vec{E}(\vec{x},t)\\&=\vec{\mathscr{E}}(\vec{k}-\vec{k}_0,\omega-\omega_0)+\vec{\mathscr{E}}^*(\vec{k}+\vec{k}_0,\omega+\omega_0)\,.\end{aligned} \tag{4}$$

[In this expression, $\vec{\mathscr{E}}^*(\vec{k},\omega)$ denotes the Fourier transform of $\vec{\mathscr{E}}^*(\vec{x},t)$, not the complex conjugate of

$\vec{\mathcal{E}}(\vec{k}, \omega)$.] Insertion of Eq. (4) into Eq. (3) now yields

$$\vec{J}^s(\vec{x}, t) = \frac{1}{(2\pi)^4} \int d^3k\, d\omega\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \underline{\sigma}^s(\vec{k}, \omega) \cdot \vec{\mathcal{E}}(\vec{k} - \vec{k}_0, \omega - \omega_0) + \text{c.c.} \tag{5}$$

This integral is calculated by changing the integration variables to $(\vec{\kappa}, \Omega) \equiv (\vec{k} - \vec{k}_0, \omega - \omega_0)$ and expanding $\underline{\sigma}(\vec{k}, \omega)$ about the point $(\vec{k}_0, \omega_0)$. The result is

$$\vec{J}^s(\vec{x}, t) = e^{i(\vec{k}_0\cdot\vec{x}-\omega_0 t)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(i\frac{\partial}{\partial t}\frac{\partial}{\partial \omega_0} - i\frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{k}_0} \right)^n [\underline{\sigma}^s(\vec{k}_0, \omega_0) \cdot \vec{\mathcal{E}}(\vec{x}, t)] + \text{c.c.} \tag{6}$$

At this point, we invoke the assumption that $\vec{\mathcal{E}}(\vec{x}, t)$ varies slowly. This allows us to neglect the higher derivatives in Eq. (6), thereby obtaining

$$\vec{J}^s(\vec{x}, t) = e^{i(\vec{k}_0\cdot\vec{x}-\omega_0 t)} \left(1 + i\frac{\partial}{\partial t}\frac{\partial}{\partial \omega_0} - i\frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{k}_0} \right) [\underline{\sigma}^s(\vec{k}_0, \omega_0) \cdot \vec{\mathcal{E}}(\vec{x}, t)] + \text{c.c.} \tag{7}$$

a local relation between $\vec{J}^s$ and $\vec{E}$.

By the same procedure, we can obtain a relation between the external charge density and the electric field, using $\rho^s = \vec{k} \cdot \vec{J}^s/\omega$, and a relation between the magnetic field and the electric field, using $\vec{B} = c\vec{k} \times \vec{E}/\omega$:

$$\rho^s(\vec{x}, t) = e^{i(\vec{k}_0\cdot\vec{x}-\omega_0 t)} \left(1 + i\frac{\partial}{\partial t}\frac{\partial}{\partial \omega_0} - i\frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{k}_0} \right) [\vec{k}_0 \cdot \underline{\sigma}^s(\vec{k}_0, \omega_0) \cdot \vec{\mathcal{E}}(\vec{x}, t)/\omega_0] + \text{c.c.} \tag{8}$$

and

$$\vec{B}(\vec{x}, t) = e^{i(\vec{k}_0\cdot\vec{x}-\omega_0 t)} \left(1 + i\frac{\partial}{\partial t}\frac{\partial}{\partial \omega_0} - i\frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{k}_0} \right) [c\vec{k}_0 \times \vec{\mathcal{E}}(\vec{x}, t)/\omega_0] + \text{c.c.} \tag{9}$$

With these expressions in hand, we proceed to calculate the average rate at which species s gains energy. In the absence of collisions, species s gains energy only from the electric field. Hence, the average rate at which energy is transferred to species s per unit volume is given by $\langle \vec{E}(\vec{x}, t) \cdot \vec{J}^s(\vec{x}, t) \rangle$. (The brackets refer to the time-averaged part of the quantity.) From Eqs. (2) and (3) we find the relation for this quantity

$$\begin{aligned}\langle \vec{E} \cdot \vec{J}^s \rangle = {} & 2\vec{\mathcal{E}}^*(\vec{x}, t) \cdot \underline{\sigma}_h^s(\vec{k}_0, \omega_0) \cdot \vec{\mathcal{E}}(\vec{x}, t) - \left(\frac{\partial}{\partial t}\frac{\partial}{\partial \omega_0} - \frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{k}_0} \right) [\vec{\mathcal{E}}^*(\vec{x}, t) \cdot \underline{\sigma}_a^s(\vec{k}_0, \omega_0) \cdot \vec{\mathcal{E}}(\vec{x}, t)] \\ & + \left[i\vec{\mathcal{E}}^*(\vec{x}, t) \cdot \left(\frac{\partial}{\partial t}\frac{\partial}{\partial \omega_0} - \frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{k}_0} \right) [\underline{\sigma}_h^s(\vec{k}_0, \omega_0) \cdot \vec{\mathcal{E}}(\vec{x}, t)] + \text{c.c.} \right],\end{aligned} \tag{10}$$

where we have introduced the Hermitian and anti-Hermitian parts of the conductivity $\underline{\sigma}^s = \underline{\sigma}_h^s + i\underline{\sigma}_a^s$. At this point we invoke our last assumption, that the dissipation is small. This allows us to neglect the last term of Eq. (10) in comparison to the second, thereby obtaining

$$\begin{aligned}\langle \vec{E} \cdot \vec{J}^s \rangle = {} & 2\vec{\mathcal{E}}^*(\vec{x}, t) \cdot \underline{\sigma}_h(\vec{k}_0, \omega_0) \cdot \vec{\mathcal{E}}(\vec{x}, t) \\ & + \frac{\partial}{\partial t}\frac{\partial}{\partial \omega_0} [-\vec{\mathcal{E}}^*(\vec{x}, t) \cdot \underline{\sigma}_a^s(\vec{k}_0, \omega_0) \cdot \vec{\mathcal{E}}(\vec{x}, t)] \\ & + \frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{k}_0} [\vec{\mathcal{E}}^*(\vec{x}, t) \cdot \underline{\sigma}_a^s(\vec{k}_0, \omega_0) \cdot \vec{\mathcal{E}}(\vec{x}, t)].\end{aligned} \tag{11}$$

Equation (11) is an energy-conservation law for species s. The left-hand side of Eq. (11) is the rate at which energy is transferred to species s per unit volume. The first term on the right-hand side of Eq. (11) is the rate at which species s dissipates energy per unit volume. If we neglect this term and integrate Eq. (11) over space and time, we find that the total energy received by species s is given by the spatial integral of

$$W^s = -(\partial/\partial\omega_0)[\vec{\mathcal{E}}^*(\vec{x}, t) \cdot \underline{\sigma}_a^s(\vec{k}_0, \omega_0) \cdot \vec{\mathcal{E}}(\vec{x}, t)]. \tag{12}$$

Hence, W^s is the wave-energy density of species s. Finally, the last term of Eq. (11) is the divergence of the energy flux of species s

$$\vec{F}_W^s = (\partial/\partial\vec{k}_0)[\vec{\mathcal{E}}^*(\vec{x}, t) \cdot \underline{\sigma}_a^s(\vec{k}_0, \omega_0) \cdot \vec{\mathcal{E}}(\vec{x}, t)]. \tag{13}$$

In order to compare Eq. (12) to previous work, which we will do momentarily, we rewrite Eq. (12) in terms of the susceptibility $\underline{\chi}^s \equiv 4\pi i\underline{\sigma}^s/\omega$:

$$W^s = \frac{1}{4\pi}\frac{\partial}{\partial\omega_0}[\vec{\mathcal{E}}^*(\vec{x}, t) \cdot \omega_0\underline{\chi}_h^s(\vec{k}_0, \omega_0) \cdot \vec{\mathcal{E}}(\vec{x}, t)]. \tag{14}$$

By a similar analysis, we find the rate at which momentum is transferred to species s per unit volume:

$$\langle \rho^s \vec{E} + \vec{J}^s \times \vec{B}/c \rangle = (\vec{k}_0/2\pi)\vec{\mathcal{E}}^* \cdot \underline{\chi}_a^s \cdot \vec{\mathcal{E}} + \frac{\partial}{\partial t}\frac{1}{4\pi}\left(\frac{\vec{k}_0}{\omega_0^2}\frac{\partial}{\partial\omega_0}[\omega_0^2\vec{\mathcal{E}}^* \cdot \underline{\chi}_h^s \cdot \vec{\mathcal{E}}] - \frac{1}{\omega_0}\vec{\mathcal{E}}\vec{\mathcal{E}}^* \cdot \underline{\chi}_h^s \cdot \vec{k}_0 - \frac{1}{\omega_0}\vec{\mathcal{E}}^*\vec{k}_0 \cdot \underline{\chi}_h^s \cdot \vec{\mathcal{E}}\right)$$
$$- \frac{\partial}{\partial \vec{x}} \cdot \frac{1}{4\pi}\left(\frac{\partial}{\partial \vec{k}_0}[\vec{k}_0\vec{\mathcal{E}}^* \cdot \underline{\chi}_h^s \cdot \vec{\mathcal{E}}] - 2\underline{I}\vec{\mathcal{E}}^* \cdot \underline{\chi}_h^s \cdot \vec{\mathcal{E}} + \vec{\mathcal{E}}^* \cdot \underline{\chi}_h^s\vec{\mathcal{E}} + \underline{\chi}_h^s \cdot \vec{\mathcal{E}}\vec{\mathcal{E}}^*\right). \tag{15}$$

(The arguments of the functions are the same as in the previous equations.) We therefore conclude that the wave-momentum density of species s is given by

$$\vec{G}^s(\vec{x}, t) = \frac{1}{4\pi}\left(\frac{\vec{k}_0}{\omega_0^2}\frac{\partial}{\partial\omega_0}[\omega_0^2\vec{\mathcal{E}}^* \cdot \underline{\chi}_h^s \cdot \vec{\mathcal{E}}] - \frac{1}{\omega_0}\vec{\mathcal{E}}\vec{\mathcal{E}}^* \cdot \underline{\chi}_h^s \cdot \vec{k}_0 - \frac{1}{\omega_0}\vec{\mathcal{E}}^*\vec{k}_0 \cdot \underline{\chi}_h^s \cdot \vec{\mathcal{E}}\right), \tag{16}$$

and the wave-momentum flux tensor is given by

$$\underline{F}_G^s(\vec{x}, t) = -\frac{1}{4\pi}\left(\frac{\partial}{\partial \vec{k}_0}[\vec{k}_0\vec{\mathcal{E}}^* \cdot \underline{\chi}_h^s \cdot \vec{\mathcal{E}}] - 2\underline{I}\vec{\mathcal{E}}^* \cdot \underline{\chi}_h^s \cdot \vec{\mathcal{E}} + \vec{\mathcal{E}}^* \cdot \underline{\chi}_h^s\vec{\mathcal{E}} + \underline{\chi}_h^s \cdot \vec{\mathcal{E}}\vec{\mathcal{E}}^*\right). \tag{17}$$

To correlate the present results with previous work, we note the following. If one sums the wave-energy density, as given by Eq. (14), of all the species and the electromagnetic energy density $\langle E^2 + B^2\rangle/8\pi$, one finds that the total wave-energy density is given by

$$W = \frac{1}{4\pi}\frac{\partial}{\partial\omega_0}[\vec{\mathcal{E}}^*(\vec{x}, t) \cdot \omega_0\underline{D}_h(\vec{k}_0, \omega_0) \cdot \vec{\mathcal{E}}(\vec{x}, t)], \tag{18}$$

where $\underline{D}(\vec{k}, \omega) = \underline{I}(1 - k^2c^2/\omega^2) + \vec{k}\vec{k}c^2/\omega^2 + \sum\underline{\chi}^s(\vec{k}, \omega)$ is the dispersion tensor. This result is equivalent to the result of Bers (Ref. 3, p. 128) for the total energy density of a dispersive medium. Similarly, if one sums the wave-momentum density, as given by Eq. (16), of all the species and the electromagnetic momentum density $\langle \vec{E} \times \vec{B}\rangle/4\pi c$, one finds that the total wave-momentum density is given by

$$\vec{G} = \frac{1}{4\pi}\left(\frac{\vec{k}_0}{\omega_0^2}\frac{\partial}{\partial\omega_0}[\omega_0^2\vec{\mathcal{E}}^* \cdot \underline{D}_h \cdot \vec{\mathcal{E}}] - \frac{1}{\omega_0}\vec{\mathcal{E}}\vec{\mathcal{E}}^* \cdot \underline{D}_h \cdot \vec{k}_0 - \frac{1}{\omega_0}\vec{\mathcal{E}}^*\vec{k}_0 \cdot \underline{D}_h \cdot \vec{\mathcal{E}}\right). \tag{19}$$

This result is equivalent to the result of Bers (Ref. 3, p. 132) for the total momentum density of a dispersive medium. Analogous statements apply to the energy flux density and the momentum flux density.

III. AN ILLUSTRATION

As an illustration of these ideas we consider the generation of longitudinal drift waves in an electron-ion, low-β plasma with density gradient in the x direction, with magnetic field in the z direction, and with both species having low thermal velocities. In this case the longitudinal susceptibility of species s is given by[11]

$$\chi_s = \frac{\omega_s^2 k_y \kappa}{k^2\Omega_s\omega} - \frac{\omega_s^2}{\omega^2}\left(1 - \frac{\omega_{ns}}{\omega}\right), \tag{20}$$

where ω_s is the plasma frequency of species s, Ω_s is the gyrofrequency (including the sign) of species s, $\omega_{ns} \equiv \kappa k_y T_s c/e_s B_0$ is the drift frequency and $\kappa \equiv n^{-1}dn/dx$. Using Eq. (14) we note that the sign of the energy density of species s must equal the sign of the quantity

$$\frac{\partial}{\partial\omega}(\omega\chi_s) = \frac{\omega_s^2}{\omega^2}\left(1 - \frac{2\omega_{ns}}{\omega}\right). \tag{21}$$

Hence, the wave energy of species s is negative when

$$1 < 2\omega_{ns}/\omega \tag{22}$$

holds.

To be specific, let us examine the case $k_y > 0$ and, thus $\omega_{ni} > 0$ and $\omega_{ne} < 0$ hold. In this case, the electrons have negative wave energy for

$$2\omega_{ne} < \omega < 0 \tag{23}$$

and the ions have negative wave energy for

$$2\omega_{ni} > \omega > 0. \tag{24}$$

Hence, unstable waves with negative phase velocity $\omega/k_y < 0$ are due to the negative wave energy of the electrons. Unstable waves with positive phase velocity are due to the negative wave energy of the ions.

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