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Infrared radiation in potential scattering

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Some time ago Bloch and Nordsieck, working in a model in which the spectrum of the radiation field is cut off beyond some low frequency ω_1, showed that the transition probability for scattering, summed over final photon states, is approximately that which would be obtained if the interaction of the projectile with the radiation field were neglected entirely. Here, within the context of a nonrelativistic treatment of the scattering process, the sum rule is generalized through the inclusion of corrections of first order in ω_1. These corrections can be interpreted in terms of a simple classical picture. In the course of the derivation, a low-frequency approximation for the transition amplitude is obtained which contains as special cases the perturbative result of Low for single-photon bremsstrahlung and the more recently derived approximation for scattering in a low-frequency laser field.

I. INTRODUCTION

Since charged-particle scattering is accompanied by the emission of an infinite number of low-frequency photons, a nonperturbative treatment of the interaction between the projectile (an electron, say) and the low-frequency modes of the radiation field is required in the construction of the asymptotic states. The appropriate asymptotic solutions obtained by Bloch and Nordsieck[1] in their fundamental treatment of this problem are built up from what are now referred to as coherent states of the radiation field.[2] Since hard photons can be accounted for by ordinary perturbation theory, the essential features of the infrared problem can be studied in a model in which the electron-field interaction vanishes unless the photon frequency lies below some small limiting value, call it ω_1. For mathematical convenience one restricts the frequency range to $\omega_0 < \omega < \omega_1$, with $\omega_0 \to 0$ at the end of the calculation. The scatterer is represented by a local, short-range potential. Working in this model, and assuming ω_1 to be sufficiently small, Bloch and Nordsieck derived a simple factorized expression for the transition amplitude. One factor involves the initial and final coherent states of the radiation field, while the other factor is just the amplitude for scattering in the absence of any interaction with the field. (The restriction to the Born approximation in Ref. 1 was later removed by Nordsieck.[3]) This result is not by itself particularly useful, since the field-dependent factor vanishes in the limit $\omega_0 \to 0$. It leads, however, to a cross section which, when summed over final states of the radiation field, is nonvanishing in that limit and is given by the field-free scattering cross section.

The Bloch-Nordsieck approximation for the cross section represents the leading term in an expansion in powers of the limiting frequency ω_1. Here, using a nonrelativistic description of the scattering process, we extend this low-frequency approximation by including the first-order correction term. It turns out that the noteworthy feature of the Bloch-Nordsieck sum rule, namely, that it involves the cross section for scattering in the *absence* of the field, is retained in the extended version. The improved approximation for the transition amplitude derived here represents a generalization of the nonrelativistic version of Low's theorem[4] for low-frequency bremsstrahlung. As will be shown below, Low's result is obtained by expanding the present result in powers of the charge and retaining only the leading term.

It seems likely that the results obtained here in a nonrelativistic model can be extended to the full relativistic problem, allowing for the computation of infrared corrections to scattering processes to higher order in the photon frequency than has been obtained up to now. This expectation is in line with a suggestion made earlier by Brown and Goble.[5]

In Sec. II we set up the basis for the subsequent analysis by reviewing the properties of the asymptotic solutions and establishing a time-independent formulation of the scattering problem in terms of a Lippmann-Schwinger integral equation, appropriately modified to account for the infrared radiation. It should be noted that the asymptotic solutions would have to be modified if one wished to include potentials which are Coulombic at great distances. The necessary procedure for introducing such modifications has been outlined by Kulish and Faddeev[6] in a relativistic treatment. (Long-range Coulomb effects have recently been studied in the context of the problem of nonrelativistic scattering in an *external* radiation field.[7]) In Sec. III the improved low-frequency approximation for the transition amplitude is derived. The sum rule is ob-

tained in Sec. IV and interpreted there in terms of a picture in which the collision is thought of as taking place instantaneously, the radiation being treated classically. The methods used in these derivations are very similar to those employed earlier in the derivation of low-frequency approximations for scattering in an intense external radiation field.[8,9] In fact the results obtained here, when applied to the case where one of the modes of the radiation field in the initial state is highly populated, provides a generalization of the earlier work on scattering in a laser field, with effects of spontaneous emission and photon depletion now properly accounted for.

II. FORMULATION OF THE SCATTERING PROBLEM

The Hamiltonian is taken to be

$$H = H_0 + V , \tag{2.1}$$

where V represents the local, short-range interaction between the electron and the target and H_0 has the form

$$H_0 = p^2/2\mu + H_F + H' . \tag{2.2}$$

The free-field Hamiltonian is

$$H_F = \sum_{\vec{k}\lambda} \hbar\omega_k a^\dagger_{\vec{k}\lambda} a_{\vec{k}\lambda} , \tag{2.3}$$

the sum extending over the modes (assumed for convenience to be discrete at this stage of the calculation) of the radiation field. Here $\omega_k = kc$, k being the wave number. The eigenvalue equation for H_F will be expressed in condensed notation as

$$H_F |n\rangle = E_n |n\rangle , \tag{2.4}$$

where n represents the sequence of occupation numbers $\{n(\vec{k}_1\vec{\lambda}_1), n(\vec{k}_2\vec{\lambda}_2), \ldots\}$ and

$$E_n = \sum_{\vec{k}\lambda} \hbar\omega_k n(\vec{k}\lambda) . \tag{2.5}$$

The dipole approximation will be assumed for the electron-field interaction H' (corresponding to the neglect of electron recoil effects and introducing errors of order v/c), and the A^2 term will be ignored. We then have

$$H' = -(e/\mu c)\vec{p}\cdot\vec{A} , \tag{2.6}$$

with

$$\vec{A} = \sum_{\vec{k}\lambda} \left(\frac{2\pi\hbar c^2}{\omega_k L^3}\right)^{1/2} \vec{\lambda}(a_{\vec{k}\lambda} + a^\dagger_{\vec{k}\lambda}) . \tag{2.7}$$

Here L^3 is the quantization volume. We work in a basis in which the polarization vector $\vec{\lambda}$ is real.

The Schrödinger equation in the asymptotic domain, where V is negligible, is

$$i\hbar \frac{d}{dt} |\Phi(t)\rangle = H_0 |\Phi(t)\rangle . \tag{2.8}$$

Following the standard procedure of time-dependent scattering theory,[10] we determine particular solutions of the form

$$|\Phi_{n\vec{p}}(t)\rangle = \exp(-iE_{n\vec{p}}\, t/\hbar) |\psi_{n\vec{p}}\rangle , \tag{2.9}$$

where $|\psi_{n\vec{p}}\rangle$ is a modified plane-wave state reducing to $|n\rangle|\vec{p}\rangle$ as the interaction is switched off, and satisfying

$$H_0 |\psi_{n\vec{p}}\rangle = E_{n\vec{p}} |\psi_{n\vec{p}}\rangle . \tag{2.10}$$

Wave packets may then be constructed by superposition of these particular solutions. Suppose we represent $|\psi_{n\vec{p}}\rangle$ as

$$|\psi_{n\vec{p}}\rangle = |\vec{p}\rangle W_{\vec{p}} |n\rangle , \tag{2.11}$$

where $W_{\vec{p}}$ is a unitary operator acting on the photon states alone, the electron momentum being conserved in the dipole approximation. Writing

$$E_{n\vec{p}} = p^2/2\mu + E_n + \Delta_{\vec{p}} , \tag{2.12}$$

we see that the eigenvalue equation (2.10) is equivalent to the operator relation

$$[H_F, W_{\vec{p}}] = \left[-\sum_{\vec{k}\lambda} \hbar\omega_k \rho_{\vec{k}\lambda} (a_{\vec{k}\lambda} + a^\dagger_{\vec{k}\lambda}) + \Delta_{\vec{p}} \right] W_{\vec{p}} , \tag{2.13}$$

with

$$\hbar\omega_k \rho_{\vec{k}\lambda} \equiv (2\pi\hbar e^2/\mu^2\omega_k L^3)^{1/2}\, \vec{p}\cdot\vec{\lambda} . \tag{2.14}$$

Now the left-hand side of Eq. (2.13) can be reduced to

$$[H_F, W_{\vec{p}}] = \sum_{\vec{k}\lambda} \hbar\omega_k (a^\dagger_{\vec{k}\lambda} [a_{\vec{k}\lambda}, W_{\vec{p}}] + [a^\dagger_{\vec{k}\lambda}, W_{\vec{p}}] a_{\vec{k}\lambda}) . \tag{2.15}$$

Equation (2.13) will be satisfied, then, provided that

$$[a_{\vec{k}\lambda}, W_{\vec{p}}] = -\rho_{\vec{k}\lambda} W_{\vec{p}} , \tag{2.16a}$$

$$[a^\dagger_{\vec{k}\lambda}, W_{\vec{p}}] = -\rho_{\vec{k}\lambda} W_{\vec{p}} \tag{2.16b}$$

(the two relations being consistent by virtue of the unitarity of $W_{\vec{p}}$), and $\Delta_{\vec{p}}$ is chosen as

$$\Delta_{\vec{p}} = -\sum_{\vec{k}\lambda} \hbar\omega_k \rho^2_{\vec{k}\lambda} . \tag{2.17}$$

Equations (2.16) are easily seen to have the solution

$$W_{\vec{p}} = \exp\left(\sum_{\vec{k}\lambda} \rho_{\vec{k}\lambda} (a_{\vec{k}\lambda} - a^\dagger_{\vec{k}\lambda}) \right) . \tag{2.18}$$

It is not difficult to construct the matrix representation $\langle n'|W_{\vec{p}}|n\rangle$ and to verify that the result is equivalent to that obtained by Bloch and Nordsieck, who employed a different representation of the field operators. Actually, we shall not require this explicit evaluation; the commutation relations (2.16) along with the normalization condition $W_0 = 1$ will be sufficient for our purposes. For example,

the useful relation

$$W^{\dagger}_{\vec{p}'}W_{\vec{p}} = W_{\vec{p}-\vec{p}'} \tag{2.19}$$

may be established by verifying that both sides satisfy the same commutation relations. The orthonormality and completeness relations satisfied by the states $|\psi_{n\vec{p}}\rangle$ follow directly from Eq. (2.19).[11]

The familiar level shift $\Delta_{\vec{p}}$ shown in Eq. (2.17) may be interpreted in terms of a classical picture. The work done on an electron accelerated from rest to momentum $\vec{p}$ is $p^2/2\mu$. However, the energy available to the electron is less than this by the amount of energy $R_{\vec{p}}$ radiated in the process. The observed kinetic energy, $E_{\vec{p}}$, is then $p^2/2\mu - R_{\vec{p}}$. Since in the low-frequency limit $R_{\vec{p}} = |\Delta_{\vec{p}}|$, we have

$$E_{\vec{p}} = p^2/2\mu + \Delta_{\vec{p}}\,. \tag{2.20}$$

With the asymptotic states determined, it is now a straightforward matter to define the scattering matrix and to set up an integral equation of the Lippmann-Schwinger type for the evaluation of the scattering amplitude.[10] (Mathematical questions relating to the existence of the scattering operator have been studied previously.[12]) The scattering amplitude is given by the matrix element

$$T_{n'\vec{p}';n\vec{p}} = \langle\psi_{n'\vec{p}'}|T(E_{n\vec{p}})|\psi_{n\vec{p}}\rangle\,, \tag{2.21}$$

where the operator $T(E)$ satisfies

$$T(E) = V + VG_0(E)T(E)\,. \tag{2.22}$$

Here we have introduced the Green's function $G_0(E) = (E - H_0)^{-1}$ for which we have the eigenfunction expansion

$$G_0(E_{n\vec{p}}) = \sum_{n''}\int d^3p''\,\frac{|\psi_{n''\vec{p}''}\rangle\langle\psi_{n''\vec{p}''}|}{E_{n\vec{p}} - E_{n''\vec{p}''}}\,, \tag{2.23}$$

the sum running over all values of the occupation numbers $n''(\vec{k}_1\vec{\lambda}_1)$, $n''(\vec{k}_2\vec{\lambda}_2)$, ... The presence of a small positive imaginary contribution to the energy denominator will be understood. The Born approximation is

$$\begin{aligned}\langle\psi_{n'\vec{p}'}|V|\psi_{n\vec{p}}\rangle &= \langle n'|\langle\vec{p}'|W^{\dagger}_{\vec{p}'}VW_{\vec{p}}|\vec{p}\rangle|n\rangle \\ &= V(\vec{p}' - \vec{p})\langle n'|W_{\vec{p}-\vec{p}'}|n\rangle\,,\end{aligned} \tag{2.24}$$

where $V(\vec{p}' - \vec{p}) = \langle\vec{p}'|V|\vec{p}\rangle$ is the Born amplitude for the field-free scattering by the local potential V.

III. LOW-FREQUENCY APPROXIMATION

The expression (2.17) for the level shift is to be evaluated by the usual rule

$$L^{-3}\sum_{\vec{k}} \to (2\pi)^{-3}\int d^3k\,,$$

the integration limits on k being ω_0/c and ω_1/c, from which we see that $\Delta_{\vec{p}}$ is proportional to $\omega_1 - \omega_0$. For ω_1 sufficiently small, we may treat the level shift, as well as the field energy E_n, as quantities of first order. The energy denominator in Eq. (2.23) is

$$D = p^2/2\mu + E_n + \Delta_{\vec{p}} - p''^2/2\mu - E_{n''} - \Delta_{\vec{p}''}\,. \tag{3.1}$$

We then have

$$D^{-1} \cong \frac{1}{p^2/2\mu - p''^2/2\mu} + \frac{\Delta_{\vec{p}''} - \Delta_{\vec{p}} + E_{n''} - E_n}{(p^2/2\mu - p''^2/2\mu)^2}\,, \tag{3.2}$$

correct to first order. To begin with, we drop the first-order correction term and define the approximate Green's function

$$\bar{G}_0(E_{n\vec{p}}) = \sum_{n''}\int d^3p''\,\frac{|\psi_{n''\vec{p}''}\rangle\langle\psi_{n''\vec{p}''}|}{p^2/2\mu - p''^2/2\mu}\,. \tag{3.3}$$

This can be evaluated with the aid of Eqs. (2.11) and (2.19) as

$$\bar{G}_0(E_{n\vec{p}}) = \int d^3p''\,\frac{|\vec{p}''\rangle\langle\vec{p}''|}{p^2/2\mu - p''^2/2\mu}\,. \tag{3.4}$$

Associated with this Green's function is the approximate scattering amplitude

$$\begin{aligned}&\langle\psi_{n'\vec{p}'}|\bar{T}(E_{n\vec{p}})|\psi_{n\vec{p}}\rangle \\ &\quad = t(p^2/2\mu;\vec{p}',\vec{p})\langle n'|W_{\vec{p}-\vec{p}'}|n\rangle\,,\end{aligned} \tag{3.5}$$

where t is the field-free amplitude satisfying

$$\begin{aligned}t(p^2/2\mu;\vec{p}',\vec{p}) = V(\vec{p}' - \vec{p}) + \int d^3p''\,V(\vec{p}' - \vec{p}'') \\ \times\frac{1}{p^2/2\mu - p''^2/2\mu}\,t(p^2/2\mu;\vec{p}'',\vec{p})\,.\end{aligned} \tag{3.6}$$

The energy-conservation condition $E_{n'\vec{p}'} = E_{n\vec{p}}$ reduces, in this low-frequency limit, to $p'^2/2\mu = p^2/2\mu$, so that $t(p^2/2\mu;\vec{p}',\vec{p})$ represents the physical (on-shell) scattering amplitude.

The low-frequency approximation (3.5) is in agreement with that obtained by Nordsieck.[3] To derive an improved approximation we include the first-order correction term in Eq. (3.2). This leads to a correction term, to be added on to the lowest-order approximation (3.5), of the form

$$\begin{aligned}C_{n'\vec{p}';n\vec{p}} = \int d^3p''\,t(p^2/2\mu;\vec{p}',\vec{p}'') \\ \times\frac{\Gamma}{(p^2/2\mu - p''^2/2\mu)^2}\,t(p^2/2\mu;\vec{p}'',\vec{p})\,,\end{aligned} \tag{3.7}$$

with

$$\begin{aligned}\Gamma = \sum_{n''}\langle n'|W_{\vec{p}''-\vec{p}'}|n''\rangle[\Delta_{\vec{p}''} - \Delta_{\vec{p}} + E_{n''} - E_n] \\ \times\langle n''|W_{\vec{p}-\vec{p}''}|n\rangle\,.\end{aligned} \tag{3.8}$$

The contribution coming from the terms in the square brackets which are independent of n'' is readily evaluated, using closure and the property (2.19), as

$$\Gamma_1 = (\Delta_{\vec{p}''} - \Delta_{\vec{p}} - E_n)\langle n'|W_{\vec{p}-\vec{p}''}|n\rangle . \tag{3.9}$$

[We have also used $W^{\dagger}_{\vec{p}''} = W_{-\vec{p}''}$ which follows by setting $\vec{p}=0$ in (2.19), with $W_0 = 1$.] To evaluate the remaining contribution involving $E_{n''}$ we make use of the eigenvalue equation (2.4) and closure to obtain

$$\Gamma_2 = \langle n'|W_{\vec{p}''-\vec{p}''}H_F W_{\vec{p}-\vec{p}''}|n\rangle . \tag{3.10}$$

If we commute H_F to the left, for example, we find, using Eq. (2.13), that

$$\Gamma_2 = \langle n'|\Big(E_{n''} + \sum_{\vec{k}\lambda}\hbar\omega_k[(\rho''_{\vec{k}\lambda} - \rho'_{\vec{k}\lambda})(a_{\vec{k}\lambda} + a^{\dagger}_{\vec{k}\lambda}) + (\rho''_{\vec{k}\lambda} - \rho'_{\vec{k}\lambda})^2]\Big)W_{\vec{p}-\vec{p}''}|n\rangle , \tag{3.11}$$

a prime on $\rho_{\vec{k}\lambda}$ indicating the replacement of $\vec{p}$ by $\vec{p}'$ in Eq. (2.14). The expression for $\Gamma = \Gamma_1 + \Gamma_2$ then becomes

$$\Gamma = \mathcal{E} + \vec{p}''\cdot\vec{v}, \tag{3.12}$$

with

$$\mathcal{E} = \langle n'|\Big(E_{n''} - E_n + \sum_{\vec{k}\lambda}\hbar\omega_k[-\rho'_{\vec{k}\lambda}(a_{\vec{k}\lambda} + a^{\dagger}_{\vec{k}\lambda}) + \rho^2_{\vec{k}\lambda} + \rho'^2_{\vec{k}\lambda}]\Big)W_{\vec{p}-\vec{p}''}|n\rangle \tag{3.13}$$

and

$$\vec{v} = \langle n'|\sum_{\vec{k}\lambda}\left(\frac{2\pi\hbar e^2}{\mu^2\omega_k L^3}\right)^{1/2}\vec{\lambda}(a_{\vec{k}\lambda} + a^{\dagger}_{\vec{k}\lambda} - 2\rho'_{\vec{k}\lambda}) \times W_{\vec{p}-\vec{p}''}|n\rangle . \tag{3.14}$$

An expression of the form (3.7) with Γ given by (3.12) can be evaluated in terms of the amplitude t and its derivatives with respect to certain scalar variables. Thus, we express the amplitude $t(e;\vec{q}', \vec{q})$, for arbitrary values of the energy and momentum variables, as $t(\nu, \tau, \xi, \xi')$, where

$$\nu = \tfrac{1}{2}(q^2/2\mu + q'^2/2\mu);\quad \tau = (q'-q)^2,$$
$$\xi = e - q^2/2\mu;\quad \xi' = e - q'^2/2\mu . \tag{3.15}$$

It has been shown[9,13] that to first order in the small quantities $\mathcal{E}$ and $\vec{v}$ Eq. (3.7) can be expressed as

$$C_{n\vec{p}';n\vec{p}} = \tfrac{1}{2}(\vec{p}+\vec{p}')\cdot\vec{v}\frac{\partial t}{\partial\nu} - (\mathcal{E} + \vec{p}\cdot\vec{v})\frac{\partial t}{\partial\xi} - (\mathcal{E} + \vec{p}'\cdot\vec{v})\frac{\partial t}{\partial\xi'} , \tag{3.16}$$

where the derivatives are evaluated for

$$\nu = \tfrac{1}{2}(p'^2/2\mu + p^2/2\mu),\quad \tau = (\vec{p}'-\vec{p})^2,\quad \xi = \xi' = 0 . \tag{3.17}$$

We observe that the coefficient $\mathcal{E} + \vec{p}\cdot\vec{v}$ vanishes. This may be verified by reexpressing the contribution to $\mathcal{E}$ in Eq. (3.13) which involves the term $E_{n''} - E_n$ as a commutator of H_F with $W_{\vec{p}-\vec{p}''}$, and then using Eq. (2.13).

The amplitude $t(p^2/2\mu;\vec{p}', \vec{p})$ appearing in the lowest-order term (3.5) can be expanded, to first order, as

$$t(p^2/2\mu;\vec{p}', \vec{p}) = t + (p^2/2\mu - p'^2/2\mu)\frac{\partial t}{\partial\xi'} , \tag{3.18}$$

where t and $\partial t/\partial\xi'$ are evaluated for values of the scalar variables shown in Eq. (3.17). From energy conservation, we have

$$(p^2/2\mu - p'^2/2\mu) = E_{n''} - E_n + \Delta_{\vec{p}''} - \Delta_{\vec{p}} . \tag{3.19}$$

Combining the lowest-order term with the correction term (3.16), we obtain the improved low-frequency approximation

$$T_{n\vec{p}';n\vec{p}} \cong \langle n'|W_{\vec{p}-\vec{p}'}|n\rangle t + \tfrac{1}{2}(\vec{p}+\vec{p}')\cdot\vec{v}\frac{\partial t}{\partial\nu} . \tag{3.20}$$

Remarkably, the coefficients of the off-shell derivatives $\partial t/\partial\xi$ and $\partial t/\partial\xi'$ both vanish so that the result can be evaluated from a knowledge of the physical field-free scattering amplitude over a small range of energies for a fixed value of the momentum-transfer variable.

To establish the connection between the result just derived and Low's well-known approximation for the bremsstrahlung amplitude,[4] we specialize to the case where the initial state $|n\rangle$ of the radiation field is the vacuum, and the final state $|n'\rangle$ corresponds to the presence of a single photon of wave number $\vec{k}$ and polarization $\vec{\lambda}$. Working to first order in the electronic charge we have

$$W_{\vec{p}-\vec{p}'} \cong 1 + \sum_{\vec{k}\lambda}(\rho_{\vec{k}\lambda} - \rho'_{\vec{k}\lambda})(a_{\vec{k}\lambda} - a^{\dagger}_{\vec{k}\lambda}) ,$$

so that

$$\langle n'|W_{\vec{p}-\vec{p}'}|n\rangle \cong \rho'_{\vec{k}\lambda} - \rho_{\vec{k}\lambda} .$$

An evaluation of $\vec{v}$ to first order in the charge gives

$$\vec{v} \cong \left(\frac{2\pi\hbar e^2}{\mu^2\omega_k L^3}\right)^{1/2}\vec{\lambda} .$$

Equation (3.20) then reduces to

$$T_{\vec{k}\vec{\lambda}\vec{p}';0\vec{p}} \cong \left(\frac{2\pi\hbar e^2}{\mu^2\omega_k L^3}\right)^{1/2}\vec{\lambda}\cdot\left(\frac{(\vec{p}'-\vec{p})}{\hbar\omega_k}t + \tfrac{1}{2}(\vec{p}'+\vec{p})\frac{\partial t}{\partial\nu}\right),$$

which is Low's result.

IV. SUM RULE

For simplicity, we specialize in the following to the case where the initial photon state is the vacuum. The differential cross section is given by

$$d\sigma_{n'} = (2\pi)^4 \frac{\mu\hbar^2}{p} \int d^3p'\, \delta(E_{n'\vec{p}'} - E_{0\vec{p}}) \times |T_{n'\vec{p}';0\vec{p}}|^2 . \tag{4.1}$$

Of greater physical interest is the sum over final photon states

$$\sum_{n'} d\sigma_{n'} \equiv (2\pi)^4(\mu\hbar^2/p)\, dQ , \tag{4.2}$$

with

$$dQ = \sum_{n'} \int d^3p'\, \delta(E_{\vec{p}'} - E_{\vec{p}} + E_{n'}) |T_{n'\vec{p}';0\vec{p}}|^2 . \tag{4.3}$$

In the low-frequency approximation of Eq. (3.20) we have, to first order,[14]

$$|T_{n'\vec{p}';0\vec{p}}|^2 \cong |\langle n'|W_{\vec{p}-\vec{p}'}|0\rangle|^2 |t|^2 + \langle 0|W^\dagger_{\vec{p}-\vec{p}'}|n'\rangle \vec{v}\cdot\tfrac{1}{2}(\vec{p}+\vec{p}')\frac{\partial}{\partial\nu}|t|^2 . \tag{4.4}$$

We write $dQ = dQ_1 + dQ_2$, where dQ_1 and dQ_2 represent the contributions coming from the first and second terms, respectively, on the right-hand side of Eq. (4.4). The δ function in Eq. (4.3) can be represented as

$$\delta(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{ixs}\, ds ,$$

which makes the n' dependence tractable. In evaluating dQ_1, we encounter the sum

$$S_1 = \sum_{n'} \exp(iE_{n'}s)\langle 0|W^\dagger_{\vec{p}-\vec{p}'}|n'\rangle\langle n'|W_{\vec{p}-\vec{p}'}|0\rangle = \langle 0|W^\dagger_{\vec{p}-\vec{p}'}\exp(iH_F s)W_{\vec{p}-\vec{p}'}|0\rangle . \tag{4.5}$$

Retaining only the first two terms in the expansion of the exponential and making use of Eqs. (2.16), we find

$$S_1 = 1 + is R_{\vec{p}'\vec{p}} , \tag{4.6}$$

with

$$R_{\vec{p}'\vec{p}} = \sum_{\vec{k}\lambda} \hbar\omega_k(\rho'_{\vec{k}\lambda} - \rho_{\vec{k}\lambda})^2 . \tag{4.7}$$

We may write, to first order,

$$S_1 = \exp(isR_{\vec{p}'\vec{p}}) . \tag{4.8}$$

The s integration can now be performed with the result

$$dQ_1 = \int d^3p'\, \delta(E_{\vec{p}'} - E_{\vec{p}} + R_{\vec{p}'\vec{p}})\, |t|^2 . \tag{4.9}$$

The energy-conserving δ function appearing in Eq. (4.9) has a simple classical interpretation. The term $R_{\vec{p}'\vec{p}}$ represents the energy which would be radiated by the electron as the result of an instantaneous collision which changes its momentum from $\vec{p}$ to $\vec{p}'$. The electron kinetic energy [which includes the level shift according to Eq. (2.20)] is not conserved, but rather is diminished by the radiation energy $R_{\vec{p}'\vec{p}}$.

Since the second term on the right-hand side of Eq. (4.4) is of first order, the argument of the δ function need only be correct to zero order in the evaluation of dQ_2. Choosing the argument as in Eq. (4.9), we have

$$dQ_2 = \int d^3p'\, \delta(E_{\vec{p}'} - E_{\vec{p}} + R_{\vec{p}'\vec{p}}) S_2 \frac{\partial}{\partial\nu} |t|^2 , \tag{4.10}$$

where, inserting the expression (3.14) for $\vec{v}$,

$$S_2 = \sum_{n'} \sum_{\vec{k}\lambda} \tfrac{1}{2}\hbar\omega_k(\rho_{\vec{k}\lambda} + \rho'_{\vec{k}\lambda})\langle 0|W^\dagger_{\vec{p}-\vec{p}'}|n'\rangle \times \langle n'|(a_{\vec{k}\lambda} + a^\dagger_{\vec{k}\lambda} - 2\rho'_{\vec{k}\lambda})W_{\vec{p}-\vec{p}'}|0\rangle . \tag{4.11a}$$

This may be evaluated using closure and Eqs. (2.16) to give

$$S_2 = \sum_{\vec{k}\lambda} \tfrac{1}{2}\hbar\omega_k(\rho_{\vec{k}\lambda} + \rho'_{\vec{k}\lambda}) \times \langle 0|[a_{\vec{k}\lambda} + a^\dagger_{\vec{k}\lambda} + 2(\rho'_{\vec{k}\lambda} - \rho_{\vec{k}\lambda}) - 2\rho'_{\vec{k}\lambda}]|0\rangle = -\sum_{\vec{k}\lambda} \hbar\omega_k(\rho_{\vec{k}\lambda} + \rho'_{\vec{k}\lambda})\rho_{\vec{k}\lambda} . \tag{4.11b}$$

The addition of dQ_2 to dQ_1 given in (4.9) has the effect of shifting the energy variable to

$$\bar{\nu} = \tfrac{1}{2}(p'^2/2\mu + p^2/2\mu) + S_2 . \tag{4.12a}$$

This can be written, using the result (4.11b) for S_2 along with the value of $p'^2/2\mu$ determined by the δ function in Eqs. (4.9) and (4.10), as

$$\bar{\nu} = E_{\vec{p}} - R_{\vec{p}} . \tag{4.12b}$$

Here $R_{\vec{p}} = |\Delta_{\vec{p}}|$ is the energy which would be radiated classically by a current which vanishes for $t > 0$, while for $t < 0$ it is that generated by an electron moving with momentum $\vec{p}$. The effective energy $\bar{\nu}$ which the electron can deliver to the target is less than the kinetic energy $E_{\vec{p}}$ by the radiation energy $R_{\vec{p}}$.

Combining the expressions for dQ_1 and dQ_2, we find the sum rule

$$dQ = \int d^3p' \, \delta(E_{\vec{p}'} - E_{\vec{p}} + R_{\vec{p}'\vec{p}}) \, |t[\overline{\nu}, (\vec{p}' - \vec{p})^2]|^2 . \tag{4.13}$$

Thus, the total probability of a transition $\vec{p} \to \vec{p}'$ of an electron, independent of the number of photons radiated, can be determined from a knowledge of the transition probability which would be obtained if the interaction with the radiation field were neglected entirely. The result generalizes that obtained by Bloch and Nordsieck through the inclusion of corrections of first order in the cutoff frequency ω_1. These corrections have the effect of introducing a small shift in the energy variable, as well as changing the energy conservation condition in a manner which admits of a simple classical interpretation.

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