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Bound states of four identical bosons in two dimensions

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Within the framework of integral equations, the bound-state energies of tetramers in two spatial dimensions are determined for simple potential models. The equations are solved by using unitary pole expansions for the subamplitudes. Evidence is given for the coincidence of the thresholds of the binding of the n-particle system with $n = 2$, 3, and 4. For the class of potentials studied, it is found that the ratio between the tetramer and trimer binding energies is, to a good approximation, given by 2.9. Using this result, an estimate is given for the ^{4}He tetramer binding energy.

I. INTRODUCTION

In recent years the possibility of the existence of small molecular clusters of helium atoms in three spatial dimensions has been studied experimentally[1,2] and theoretically.[3-5] In particular, there are theoretical investigations indicating that some trimer bound states exhibit the remarkable property that their spatial extent can be very large as compared to the range of the interatomic potential.[4,5] In view of the experimental possibility of detecting bound states in monolayers of quantum gases, it is interesting to study theoretically the bound-state problem of multiparticle systems in two dimensions. In a previous paper[6] it was shown that two peculiar phenomena present in three spatial dimensions, the so-called Thomas and Efimov effects, do not occur in two dimensions. Furthermore, for reasonable interaction parameters of the interatomic potential a trimer bound state of ^{4}He atoms is found which has a binding energy of the order of 0.1 K,[7] this being significantly larger than obtained for the same potential in three spatial dimensions.

The bound-state problem of four identical bosons in two dimensions is considered in this paper for simple interatomic potential models. Since virtually nothing is known in this case about the bound-state properties of the ground state of tetramers, my results may serve as a guideline for what to expect. Similarly, as in the four-nucleon problem,[8] it is also found in this case that a common linear relationship exists between the binding energies of the trimer and tetramer bound states for a certain class of two-particle interactions.[9] From this the binding energy of the tetramer of ^{4}He atoms in two dimensions is estimated to be in excess of 0.3 K. The basic question about the relative positions of the thresholds of the multiparticle ground states can be studied. From this work we find evidence that the thresholds of the n-particle states ($n = 2, 3, 4$) coincide in two spatial dimensions.

Instead of solving the Schrödinger equation directly for the four-particle bound state, the solution is obtained in this paper within the framework of linear integral equations for the scattering amplitude. This approach has the advantage that the boundary conditions are automatically satisfied. Although there exists by now many ways of expressing integral equations,[10] the most commonly used in actual calculations are of the Faddeev-Yakubovsky type[11,12] and the quasiparticle equations of Alt, Grassberger, and Sandhas.[13] These have the nice property that they are free from nonphysical spurious solutions in the scattering region. The integral equations of the Faddeev-Yakubovsky type for four particles in two dimensions are briefly described in Sec. II. It is assumed that the two-particle scattering amplitude admits a separable expansion and only the zero angular momentum parts of the amplitudes are kept. Section III deals with the separable unitary-pole expansions (UPE) of the various subamplitudes occurring in the four-particle equations. The resulting one-dimensional integral equations for the four-particle system are written down explicitly. Using these equations, I study the bound-state properties of tetramers numerically for various simple potential models. For the interatomic potential I have taken pure S-wave separable interactions. On the other hand, the realistic interatomic potentials constructed up to date are in general taken to be local and to have a repulsion at short distances. As a first step for more realistic calculations, I also consider in this paper simple local-potential models of a Gaussian form with the possibility of a repulsive core. The results are presented in Sec. IV, and some concluding remarks are made in Sec. V.

II. FOUR-PARTICLE SYSTEM

In this section I summarize the integral equations for the bound states of four identical bo-

sons. Let us consider four spinless bosons with mass m which are moving nonrelativistically in two spatial dimensions. Their mutual interaction is described by S-wave pair forces, so that the total Hamiltonian is given by

$$H=\sum_{i=1}^{4}\frac{\vec{p}_i^{\,2}}{2m}+\sum_{\alpha}V_\alpha, \tag{1}$$

$\vec{p}_i$ being the momentum operator of the ith particle and V_α the potential between a particle pair $\alpha=i,j$. The starting point in the integral equation approach is the off-shell four-particle scattering amplitude T, for which we can write an equation of the Faddeev form

$$T^\alpha=t_\alpha+\sum_{\beta\neq\alpha}t_\alpha G_0(z)T^\beta, \tag{2}$$

with

$$T=\sum_{\alpha}T^\alpha. \tag{3}$$

In Eq. (2) G_0 is the free Green's function

$$G_0(z)=(z-H_0)^{-1}, \tag{4}$$

with H_0 the kinetic-energy operator and z the energy variable. Furthermore, t_α is the two-particle T matrix, satisfying the Lippmann-Schwinger equation

$$t_\alpha=V_\alpha+V_\alpha G_0(z)t_\alpha. \tag{5}$$

In order to get mathematically meaningful integral equations an additional step is needed. With

$$M^\alpha=T^\alpha-t_\alpha, \tag{6}$$

we decompose M^α into

$$M^\alpha=\sum_{k\in\alpha}A_k+B, \tag{7}$$

where the amplitudes A_k and B are characterized by means of the initial collision sequences which take place between the particles. From Eq. (2) it follows that, in all the terms of the Neumann-series solution for M^α, the first collision is always between the particle pair α. The amplitude A_k contains all those scattering contributions in which the second collision is between a third particle k not belonging to the pair α with one of the particles from the pair, while B contains the terms in which the second collision is between the other pair $\beta\neq\alpha$. In the case of identical bosons, the various amplitudes A_k can simply be related to each other by permutations of the particles, leading to a single function A. As is known,[14] the amplitudes A and B satisfy for the four-particle bound state a coupled set of homogeneous equations. This can be written formally as

$$\begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}K_{11}&K_{12}\\0&K_{22}\end{pmatrix}G_0(z)\begin{pmatrix}A\\B\end{pmatrix}, \tag{8}$$

with z being minus the tetramer binding energy $E_4>0$ and where K_{11} and K_{12} are related to the off-shell three-particle T matrix and K_{22} to the amplitude describing the scattering of two noninteracting pairs of particles. It is convenient to define two sets of relative momenta

$$\begin{aligned}\vec{k}&=(1/2\sqrt{m})(\vec{p}_1-\vec{p}_2),\\ \vec{p}&=(1/2\sqrt{3m})(\vec{p}_1+\vec{p}_2-2\vec{p}_3),\\ \vec{q}&=(1/2\sqrt{6m})(\vec{p}_1+\vec{p}_2+\vec{p}_3-3\vec{p}_4)\end{aligned} \tag{9}$$

to be used in case of the amplitude A and

$$\begin{aligned}\vec{k}&=(1/2\sqrt{m})(\vec{p}_1-\vec{p}_2),\\ \vec{p}&=(1/2\sqrt{m})(\vec{p}_3-\vec{p}_4),\\ \vec{q}&=(1/2\sqrt{2m})(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\end{aligned} \tag{10}$$

for the amplitude B. The kinetic energy of four particles has in the overall center-of-momentum system for both sets of momenta the form

$$H_0=p^2+q^2+k^2. \tag{11}$$

In the following we suppose that the two-particle T matrix can be written in the two-particle Hilbert space as

$$\langle\vec{k}|t(E)|\vec{k}'\rangle=\sum_{l,l'}g_l(k)d_{ll'}(E)g_{l'}(k'), \tag{12}$$

with E the two-particle energy variable. Furthermore, in the analysis of Eq. (8) we confine ourselves to the zero angular momentum states and

TABLE I. Dependence of trimer and tetramer binding energies E_3 and E_4 on the number of separable terms in the UPE expansion of the two-particle T matrix of the local potential, Eq. (22), for various strengths of the coupling constants λ_i. N_λ^+ and N_λ^- denote the number of terms with positive and negative eigenvalues λ.

λ_1	λ_2	N_λ^+	N_λ^-	E_2 (10^{-2})	E_3 (10^{-2})	E_4 (10^{-1})
0.8	0	1	0	0.82	7.42	2.32
		2	0		7.44	2.33
		3	0		7.44	2.33
3.3	10	1	0	1.24	8.26	2.31
		2	0		8.28	2.32
		1	1		8.12	2.25
		2	1		8.14	2.26
5.0	20	1	0	1.62	9.53	2.57
		2	0		9.57	2.59
		3	0		9.58	2.59
		1	1		8.97	2.34
		1	2		8.96	2.34
		2	1		9.03	2.35
		2	2		9.02	2.35

assume in addition that all nonzero angular momentum contributions of the subsystems may be neglected. Then the amplitudes can be written in the plane-wave representation as

$$A(\vec{k},\vec{p},\vec{q}) = \sum_{l,l'} g_l(k) d_{ll'}(-E_4 - p^2 - q^2) a_{l'}(p,q), \quad (13)$$

$$B(\vec{k},\vec{p},\vec{q}) = \sum_{l,l'} g_l(k) d_{ll'}(-E_4 - p^2 - q^2) b_{l'}(p,q). \quad (14)$$

The standard procedure of a partial-wave reduction of Eq. (8) leads with Eqs. (13) and (14) to a coupled set of two-dimensional integral equations for a_l and b_l. They are of the form

$$a_l(p,q) = \sum_{l',l''} \int_0^\infty q'\,dq' \int_0^{2\pi} d\phi\, K_{ll'}(p,q|Q_1,q') d_{l'l''}(-E_4 - Q_2^2 - q'^2) a_{l''}(Q_2,q')$$
$$+ \sum_{l',l''} \int_0^\infty q'\,dq' \int_0^{2\pi} d\phi\, L_{ll'}(p,q|S_1,q') d_{l'l''}(-E_4 - S_2^2 - q'^2) b_{l''}(S_2,q'), \quad (15)$$

$$b_l(p,q) = \sum_{l',l''} \int_0^\infty q'\,dq' \int_0^{2\pi} d\phi\, M_{ll'}(pq|S_1,q') d_{l'l''}(-E_4 - S_2^2 - q'^2) a_{l''}(S_2,q'), \quad (16)$$

with

$$Q_1^2 = \tfrac{9}{8}q'^2 + \tfrac{1}{8}q^2 + \tfrac{3}{4}qq'\cos\phi, \quad (17)$$
$$Q_2^2 = \tfrac{9}{8}q^2 + \tfrac{1}{8}q'^2 + \tfrac{3}{4}qq'\cos\phi,$$
$$S_1^2 = \tfrac{3}{2}q'^2 + \tfrac{1}{2}q^2 + \sqrt{3}\,qq'\cos\phi, \quad (18)$$
$$S_2^2 = \tfrac{3}{2}q^2 + \tfrac{1}{2}q'^2 + \sqrt{3}\,qq'\cos\phi.$$

The explicit expressions of the kernels in Eqs. (15) and (16) are given in Appendix A.

III. UPE EXPANSION OF SUBAMPLITUDES

In our study of the binding of trimers in two dimensions, we made use of the simple separable potentials

$$V(k,k') = -(\lambda/2\pi) g(k) g(k'), \quad (19)$$

with

$$g(k) = (k^2 + \beta^2)^{-m}, \quad (20)$$

where $m = 1, 2$. In the following the unit of the energy is chosen to be such that $\beta = 1$. The solution of Eq. (5) for the two-particle T matrix has the form of Eq. (12) with

$$d(z) = \frac{-\lambda}{2\pi}\left(1 + \lambda \int_0^\infty \frac{g(k)^2}{z - k^2} k\,dk\right)^{-1}. \quad (21)$$

In addition to these interactions, we also consider in this paper potentials of a local type. They are acting only in the S-wave state and also contain a repulsive part. In the coordinate representation they are of the Gaussian form

$$V(r) = -\lambda_1 e^{-r^2} + \lambda_2 e^{-4r^2}, \quad (22)$$

with $\lambda_i \geq 0$. These potentials can easily be transformed to the momentum space, leading to

$$V(k,k') = -(\lambda_1/4\pi)\exp[-\tfrac{1}{4}(k^2 + k'^2)] I_0(\tfrac{1}{2}kk')$$
$$+ (\lambda_2/16\pi)\exp[-\tfrac{1}{16}(k^2 + k'^2)] I_0(\tfrac{1}{8}kk'), \quad (23)$$

where I_0 is the modified Bessel function of zeroth order. To get the separable representation, Eq. (12), for the potentials, Eq. (23), I have used the UPE expansion.[15] Let g_l be the eigenfunction of the homogeneous Lippmann-Schwinger equation at the two-particle bound-state energy $z = -E_2$ with eigenvalue λ_l:

$$\lambda_l g_l(k) = 2\pi \int_0^\infty k'\,dk' \frac{V(k,k')}{-E_2 - k'^2} g_l(k'). \quad (24)$$

The eigenfunctions form an orthogonal set. Let g_l further be real and normalized as

$$\int_0^\infty k\,dk \frac{g_l(k) g_{l'}(k)}{E_2 + k^2} = \delta_{ll'}. \quad (25)$$

We then approximate

$$V(k,k') = -\frac{1}{2\pi} \sum_{l=1}^{N_\lambda} \lambda_l g_l(k) g_l(k'), \quad (26)$$

with λ_l being the N_λ eigenvalues which are the largest in magnitude. For this rank-N_λ potential the two-particle T matrix can readily be obtained from Eq. (5). It has the form of Eq. (12) with $d_{ll'}$ a matrix of rank N_λ satisfying

$$\sum_{l''=1}^{N_\lambda} \left(\delta_{ll''} + \lambda_l \int_0^\infty k\,dk \frac{g_l(k) g_{l''}(k)}{z - k^2}\right) d_{l''l'}(z)$$
$$= -\frac{1}{2\pi}\lambda_l \delta_{ll'}. \quad (27)$$

For the special case that $z = -E_2$ we have, in view of Eq. (25),

$$d_{ll'}(-E_2) = -\delta_{ll'}(\lambda_l/2\pi) 1/(1 - \lambda_l). \quad (28)$$

We now turn to the four-particle equations (15) and (16). Although the solution of this type of equations can in principle be constructed directly with the use of Padé approximants,[16,17] for simple central potentials it is numerically less time consuming to find the solution by applying separable expansion techniques to the subamplitudes in the kernel of the four-particle equations. In this work we have used the UPE method. Encouraging results have been obtained recently in the study of the four-nucleon system with these techniques.[18,19] Proceeding in the same way as in the two-particle case, we define

$$\xi_n \hat{w}_{nl}(p) = 2\pi \sum_{l',l''} \int_0^\infty p'\,dp'\, W_{ll'}(p,p'|z_1) \times d_{l'l''}(z_1 - p'^2)\hat{w}_{nl''}(p') \tag{29}$$

and

$$\eta_n \hat{v}_{nl}(p) = 2\pi \sum_{l',l''} \int_0^\infty p'\,dp'\, V_{ll'}(p,p'|z_2) \times d_{l'l''}(z_2 - p'^2)\hat{v}_{nl''}(p'), \tag{30}$$

with $W_{ll'}$ and $V_{ll'}$ given by Eqs. (A2) and (A9).

For the value of z_1 we have chosen the trimer bound-state energy $z_1 = -E_3$ and for z_2 twice the dimer bound-state energy $z_2 = -2E_2$. Expanding the subamplitudes $w_{ll'}$ and $v_{ll'}$ in terms of the eigenfunctions, given by Eqs. (29) and (30), we obtain the separable representations

$$w_{ll'}(p,p'|z) = \sum_{n,n'} \hat{w}_{nl}(p)\Theta_{nn'}(z)\hat{w}_{n'l'}(p') \tag{31}$$

and

$$v_{ll'}(p,p'|z) = \sum_{n,n'} \hat{v}_{nl}(p)\Delta_{nn'}(z)\hat{v}_{n'l'}(p'). \tag{32}$$

The expressions of the propagators $\Theta_{nn'}$ and $\Delta_{nn'}$ are given in Appendix B. Using Eqs. (31), (32), (A5), (A6), and (A8), we see that the solution of Eqs. (15) and (16) has the form

$$a_l(p,q) = \sum_{n,n'} \hat{w}_{nl}(p)\Theta_{nn'}(z - q^2)\hat{a}_{n'}(q), \tag{33}$$

$$b_l(p,q) = \sum_{n,n'} \hat{v}_{nl}(p)\Delta_{nn'}(z - q^2)\hat{b}_{n'}(q), \tag{34}$$

where $\hat{a}_n$ and $\hat{b}_n$ satisfy a coupled set of one-dimensional integral equations

$$\hat{a}_n(q) = \sum_{n',n''} \int_0^\infty q'\,dq'\, X_{nn'}(q,q')\Theta_{n'n''}(z - q^2)\hat{a}_{n''}(q') + \sum_{n',n''} \int_0^\infty q'\,dq'\, Y_{nn'}(q,q')\Delta_{n'n''}(z - q'^2)\hat{b}_{n''}(q'), \tag{35}$$

$$\hat{b}_n(q) = 2 \sum_{n',n''} \int_0^\infty q'\,dq'\, Z_{nn'}(q,q') \times \Theta_{n'n''}(z - q'^2)\hat{a}_{n''}(q'), \tag{36}$$

with

$$X_{nn'}(q,q') = \left(\frac{3}{2\sqrt{2}}\right)^2 \sum_{l,l'} \int_0^{2\pi} d\phi\, \hat{w}_{nl}(Q_1) \times d_{ll'}(\hat{z}_1)\hat{w}_{n'l'}(Q_2), \tag{37}$$

$$Y_{nn'}(q,q') = \tfrac{3}{2} \sum_{l,l'} \int_0^{2\pi} d\phi\, \hat{w}_{nl}(S_1) \times d_{ll'}(\hat{z}_2)\hat{v}_{n'l'}(S_2), \tag{38}$$

$$Z_{nn'}(q,q') = Y_{n'n}(q',q). \tag{39}$$

Here we have defined the shifted energies

$$\hat{z}_1 = -E_4 - \tfrac{9}{8}(q^2 + q'^2 + \tfrac{2}{3}qq'\cos\phi), \tag{40}$$

$$\hat{z}_2 = -E_4 - \tfrac{3}{2}(q^2 + q'^2 + \tfrac{2}{3}\sqrt{3}\,qq'\cos\phi). \tag{41}$$

IV. NUMERICAL RESULTS

In Sec. III we have shown that with the UPE method coupled four-particle integral equations in one continuous variable can be obtained. Obviously the separable expansions of the subamplitudes are only useful if they converge rapidly enough. In practice reliable answers can be obtained for central potentials by keeping a limited

TABLE II. Dependence of tetramer binding energy E_4 on the number of separable terms used in the UPE expansion of the subamplitudes $w_{ll'}$ and $v_{ll'}$ for the two-particle potential, Eq. (22), with $\lambda_1 = 5$, $\lambda_2 = 20$.

N_ξ	N_η	E_4 (10^{-1})
1	0	2.57
2	1	2.56
3	1	2.54
4	1	2.54
1	2	2.65
1	3	2.70
1	4	2.70
2	2	2.65
3	3	2.73
4	4	2.75
5	5	2.75

TABLE III. The dimer, trimer, and tetramer binding energies E_2, E_3, and E_4 for various values of the coupling constant λ of the separable potential, Eq. (19), with $m=1,2$.

λ	m	E_2 (10^{-2})	E_3 (10^{-2})	E_4 (10^{-1})
0.06	1	0.19	1.72	0.48
0.08	1	0.73	5.30	1.54
0.10	1	1.73	10.6	2.94
0.08	2	0.32	2.32	0.67
0.10	2	0.74	4.63	1.28
0.12	2	1.35	7.51	2.00
0.14	2	2.11	10.8	2.79

number of terms, say two or three, in the expansion. In Tables I and II are shown some results for the trimer and tetramer binding energies E_3 and E_4 for the local potentials of the Gaussian type. The calculations are done by discretizing the integral equations and solving the resulting matrix equations. Gaussian quadratures are used with 16 mesh points in the occurring integrations. The accuracy was checked by varying the number and distributions of the integration points. It was estimated to be of the order of two percent. Table I shows the dependence of the binding energies on the number of terms in the UPE expansion of the two-particle T matrix. Only one term is kept in the subamplitudes $v_{ll'}$ and $w_{ll'}$. From this table we see that the convergence rate for both E_3 and E_4 is extremely fast. To calculate the binding energies to within one percent, we have only to include at most two eigenfunctions belonging to the largest

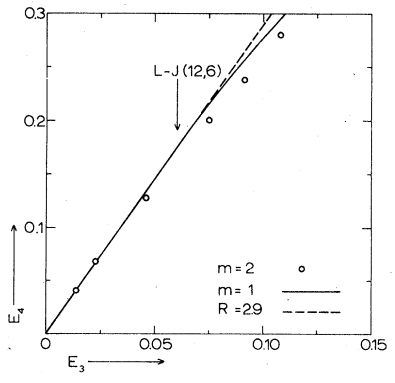

FIG. 1. E_3,E_4 plot for the attractive separable potentials, Eq. (19), with $m=1,2$. Each point corresponds to the calculated trimer and tetramer binding energies E_3 and E_4 for a given two-particle potential. The value of E_3 for the Lennard-Jones (6,12) potential from Ref. 6 is also indicated together with the straight line $R=E_4/E_3=2.9$.

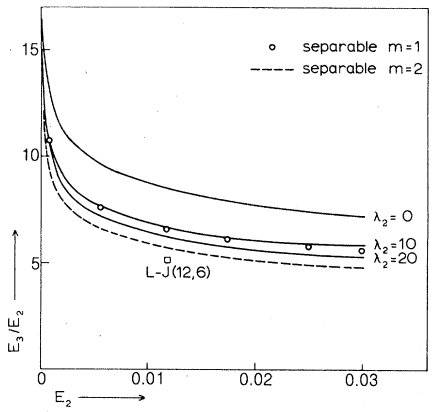

FIG. 2. The ratio of trimer and dimer binding energies E_3/E_2 as a function of the dimer energy E_2 for the various separable and Gaussian potentials.

positive and negative eigenvalue. As to the quality of a one-term separable approximation to the subamplitude $w_{ll'}$, we see from Table II that it is excellent, while three terms are needed for $v_{ll'}$ to reach convergence.

For the separable potentials, Eq. (19), the tetramer binding energy has been calculated as a function of the coupling constant λ. Under the approximation used here only one term is kept in the subamplitudes $v_{ll'}$ and $w_{ll'}$. Some results are given in Table III. Following Ref. 9 we have plotted in Fig. 1 the calculated E_3-E_4 values. For both potentials $m=1,2$ the results are close to each other. They form a linear band in this E_3-E_4 plot, the deviation from the linear relation $R=E_4/E_3=2.9$ being larger at larger values of E_3. Also for vanishing small trimer binding the tetramer becomes unbound at the same time. To relate this to the threshold behavior of the ground-state energy of the dimer, the ratio

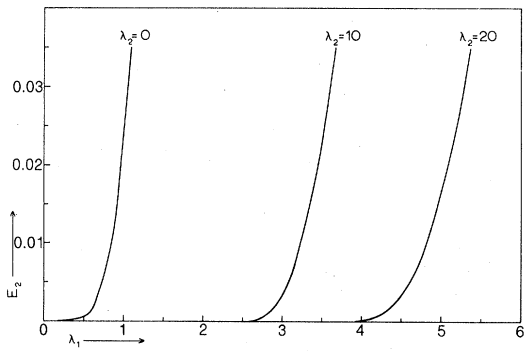

FIG. 3. Dimer binding energy E_2 as a function of the coupling constant λ_1 for various strengths of the repulsive part in the two-particle potential, Eq. (22).

TABLE IV. The dimer, trimer, and tetramer binding energies E_2, E_3, and E_4 for various values of the coupling constants λ of the local potential, Eq. (22).

λ_1	λ_2	E_2 (10^{-2})	E_3 (10^{-1})	E_4 (10^{-1})
0.6	0	0.15	0.18	0.57
1.0	0	2.33	1.75	5.16
3.0	10	0.33	0.28	0.81
3.2	10	0.85	0.61	1.72
3.4	10	1.72	1.08	2.97
4.6	20	0.48	0.35	1.02
4.8	20	0.95	0.62	1.71
5.2	20	2.53	1.37	3.60

E_3/E_2 is plotted in Fig. 2 as a function of E_2. [In Fig. 2 from Ref. 6, the coupling constant Γ for the separable potential with $m=2$ is in contrast with what is stated defined there in terms of s_2 through Eq. (19) with $\Gamma=\pi g^2/\beta^4$.] For $E_2 \to 0$ the ratio tends to a finite value. Hence within the numerical accuracy the binding energies of the n-particle system with $n=2, 3, 4$ all vanish simultaneously.

To study the effect of locality and repulsion in the two-particle interaction, the potentials of Eq. (22) are considered. In the calculations we take for the strengths of the repulsion in the potential $\lambda_2=0$, 10, and 20. The dependence of the dimer energy on the coupling constant λ_1 is shown in Fig. 3. Some results for the binding energies E_2, E_3, and E_4 are given in Table IV. Also in this case, as is seen from Fig. 2, the ratio E_3/E_2 reaches for $E_2 \to 0$ the value of 16.1—the zero-range result of the separable potential.[6] In Fig. 4 are plotted all the results of E_3 and E_4 obtained for these Gaussian potentials. They again form independent of the repulsion parameter λ_2 a narrow band around the linear function $R=2.9$. The remarkable linear relationship between E_3 and E_4 has also recently been verified in three spatial dimensions for a class of realistic interatomic potentials.[20]

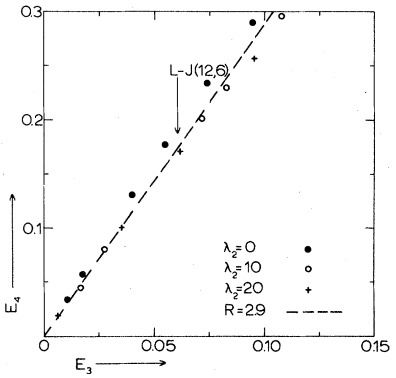

FIG. 4. The same as Fig. 1, but for the Gaussian potentials.

V. CONCLUDING REMARKS

In summary, I have presented results for the bound-state energies of tetramers in two dimensions within the framework of integral equations. In particular, we have demonstrated the practical applicability of the UPE expansion in this case. Similarly, as in the four-nucleon system, a linear relationship is found between the trimer and tetramer binding energies in a certain region of E_3, a relationship which is independent of the two-particle interaction used. If we conjecture that this result also holds for a more general class of interactions, an estimate may be obtained for the tetramer binding energies for more realistic interactions. Taking the value of the trimer binding energy $E_3=0.11$ K obtained from the variational calculations[7] for the Lennard-Jones (6, 12) potential with the Boer-Michels ^{4}He parameters, I estimate the tetramer binding energy from $R=2.9$ to be $E_4=0.32$ K. The value of R is expected to increase if we take more terms into account in the separable expansion of $v_{ll'}$ and $w_{ll'}$. Furthermore, since studies of other realistic potential models indicate more trimer binding, the tetramer binding energy may be considerably larger than the value estimated above.

Although simple potential models have been used, the calculations described in this paper can in principle be extended to the more realistic interactions. Because of the singular nature of some of the interatomic potentials currently in use, the two-body eigenfunctions needed for the UPE cannot be constructed directly from the Lippmann-Schwinger equation. A possible answer in that case would be through the explicit solution of the two-particle Schrödinger equation along the same lines as discussed by Huber and Lim.[21] It should be interesting to carry out such a calculation to see at least whether my conjecture about the linear relationship is valid.

APPENDIX A

The kernels of Eq. (15) are expressed in terms of the three-particle off-shell amplitude $w_{ll'}$ which satisfies the Faddeev equation

$$w_{ll'}(p,p'|z) = W_{ll'}(p,p'|z) + 2\pi \sum_{l'',l'''} \int_0^\infty p''\,dp''\, W_{ll''}(p,p''|z) \times d_{l''l'''}(z-p''^2) w_{l'''l'}(p'',p'|z), \quad \text{(A1)}$$

with an effective potential

$$W_{ll'}(p,p'|z) = \frac{1}{\pi}\left(\frac{2}{\sqrt{3}}\right)^2 \int_0^{2\pi} d\phi \frac{g_l(\bar{p}_1)g_{l'}(\bar{p}_2)}{z - \frac{4}{3}(p^2+p'^2+pp'\cos\phi)} \quad \text{(A2)}$$

and

$$\bar{p}_1^2 = \tfrac{1}{3}p^2 + \tfrac{4}{3}p'^2 + \tfrac{4}{3}pp'\cos\phi\,, \quad \text{(A3)}$$

$$\bar{p}_2^2 = \tfrac{4}{3}p^2 + \tfrac{1}{3}p'^2 + \tfrac{4}{3}pp'\cos\phi\,. \quad \text{(A4)}$$

They are given by

$$K_{ll'}(pq|p'q') = \left(\frac{3}{2\sqrt{2}}\right)^2 w_{ll'}(p,p'|-E_4-q^2) \quad \text{(A5)}$$

and

$$L_{ll'}(pq|p'q') = \tfrac{3}{2}w_{ll'}(p,p'|-E_4-q^2)\,. \quad \text{(A6)}$$

Furthermore, for the kernel $M_{ll'}$ in Eq. (16) we have

$$M_{ll'}(pq|p'q') = 3v_{ll'}(p,p'|-E_4-q^2)\,, \quad \text{(A7)}$$

where $v_{ll'}$ is the off-shell scattering amplitude of two noninteracting pairs, given by

$$v_{ll'}(p,p'|z) = V_{ll'}(p,p'|z) + 2\pi \sum_{l'',l'''} \int_0^\infty p''dp'' V_{ll''}(p,p''|z) \times d_{l''l'''}(z-p''^2)v_{l'''l'}(p'',p'|z)\,, \quad \text{(A8)}$$

with

$$V_{ll'}(p,p'|z) = g_l(p')g_{l'}(p)/(z-p^2-p'^2)\,. \quad \text{(A9)}$$

APPENDIX B

In this appendix I give the explicit equations for the propagators $\Theta_{nn'}$ and $\Delta_{nn'}$ in Eqs. (31) and (32). Consider the three-particle subamplitude $w_{ll'}$ satisfying the Faddeev equation (A1). Suppose that the eigenfunctions $\hat{w}_{nl}$ from Eq. (29) are real and normalized as

$$\sum_{l,l'} \int_0^\infty p\,dp\,\hat{w}_{nl}(p)d_{ll'}(z_1-p^2)\hat{w}_{n'l'}(p) = -\delta_{nn'}\,. \quad \text{(B1)}$$

Then we expand the effective interaction $W_{ll'}$ in the eigenfunctions $\hat{w}_{nl}$:

$$W_{ll'}(p,p'|z) = \sum_{n,n'} \hat{w}_{nl}(p)\Theta^0_{nn'}(z)\hat{w}_{n'l'}(p')\,. \quad \text{(B2)}$$

Using (B1), we obtain

$$\Theta^0_{nn'}(z) = \sum_{\{l_i\}} \int_0^\infty p\,dp \int_0^\infty p'\,dp'\,\hat{w}_{nl_1}(p)d_{l_1l_2}(z_1-p^2) \times W_{l_2l_3}(p,p'|z)d_{l_3l_4}(z_1-p'^2)\hat{w}_{n'l_4}(p')\,. \quad \text{(B3)}$$

Inserting (B2) into (A1) yields the separable representation (31) with $\Theta_{nn'}$ satisfying a set of linear equations

$$\Theta_{nn'}(z) = \Theta^0_{nn'}(z) + \sum_{n'',n'''} \Theta^0_{nn''}(z)S_{n''n'''}(z)\Theta_{n'''n'}(z)\,, \quad \text{(B4)}$$

with

$$S_{nn'}(z) = 2\pi \sum_{l,l'} \int_0^\infty p\,dp\,\hat{w}_{nl}(p)d_{ll'}(z-p^2)\hat{w}_{n'l'}(p)\,. \quad \text{(B5)}$$

As in the two-particle case, we have for $z = z_1$

$$\Theta_{nn'}(z_1) = -\delta_{nn'}(\xi_n/2\pi)1/(1-\xi_n)\,. \quad \text{(B6)}$$

The representation (32) for the scattering amplitudes of two noninteracting pairs can be obtained in the same way by using the eigenfunctions $\hat{v}_{nl}$. As a result we have

$$\Delta_{nn'}(z) = \Delta^0_{nn'}(z) + \sum_{n'',n'''} \Delta^0_{nn''}(z)R_{n''n'''}(z)\Delta_{n'''n'}(z)\,, \quad \text{(B7)}$$

with

$$\Delta^0_{nn'}(z) = \sum_{\{l_i\}} \int_0^\infty p\,dp \int_0^\infty p'\,dp'\,\hat{v}_{nl_1}(p)d_{l_1l_2}(z_2-p^2) \times V_{l_2l_3}(p,p'|z)d_{l_3l_4}(z_2-p'^2)\hat{v}_{n'l_4}(p') \quad \text{(B8)}$$

and

$$R_{nn'}(z) = 2\pi \sum_{l,l'} \int_0^\infty p\,dp\,\hat{v}_{nl}(p)d_{ll'}(z-p^2)\hat{v}_{n'l'}(p)\,. \quad \text{(B9)}$$

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