Theory of emission of radiation from an assembly of N two-level atoms

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Interaction with a multimode quantized radiation field of an assembly of N two-level atoms contained in a small volume is considered for the special cases of (i) highly excited, (ii) weakly excited, and (iii) superradiant atomic assemblies, and solutions of equations of motion are obtained in closed form. For (i) and (ii) we show that, if the atomic assembly is in the Dicke state $|r,m\rangle$, the spontaneously emitted radiation is Lorentzian, and its spectral width and the numerical value of the Lamb shift are $r+|m|$ times that for a single excited atom. For (i) the sign of Lamb shift is opposite to that for (ii) and for a single excited atom. Change in the frequency shift gives chirping in the pulse radiation emitted spontaneously by an assembly of atoms. We obtain the reduced density operator for a single mode and for the complete spontaneous emitted radiation, and discuss its characteristic function, weight function in diagonal representation, coherence functions, and normalized variance. We find that the emitted radiation is chaotic if all atoms are excited or if the assembly is weakly excited and is in thermal equilibrium. For a superradiant assembly, the radiation is coherent; it is only amplitude coherent if the initial atomic density operator is nondiagonal. We also study stimulated emission and scattering for the cases (i) and (ii). For case (iii) these phenomena do not take place.

I. INTRODUCTION

After the appearance of Dicke's paper[1] on spontaneous emission of radiation by an assembly of atoms, cooperative effects and superradiance were studied by several authors.[2-11] Some experimental observations have also been reported.[12] However, the study is largely incomplete as, mostly, only the number of photons (or of intensity)[2-5] and, in the name of coherence properties of radiation, the variance of photons (or of intensity)[2-5] and self-correlation[4] have been studied. There have been some attempts[6-8] to study coherence properties of radiation but the authors have taken only a single resonant mode of radiation. For a single-mode resonant radiation closed form of atom plus radiation state has been obtained by Smithers and Lu[3] and by the present authors[9] for a few special cases of the initial atomic state. All these studies involving only a single mode of radiation do not give any information about the intercorrelation of modes, line-width and frequency shift.

Frequency shifts in emission from a small atomic assembly was studied quantum mechanically by Fain[13] and also by Arecchi and Kim[14] who studied linewidth also. Stroud, Eberly, Lama, and Mandel[11] studied these in detail using the neoclassical theory[15] and reported chirping in emission. The results of all these authors are different. Banfi and Bonifacio[16] have considered a pencil-shaped atomic assembly and studied frequency shift and linewidth.

In the present paper, we shall consider atoms in a small volume interacting with a multimode radiation field. We shall solve the equations of motion for a few special cases and then study not only the linewidth, frequency shift, and emission rate but also the coherence properties of the emitted radiation.

II. HAMILTONIAN

Consider an assembly of N identical two-level atoms interacting with the radiation field. If the atoms are assumed to be contained in a volume whose linear dimensions[17] are much smaller than the wavelength of the resonant radiation, and if they interact with each other only through their interaction with the common radiation field, the Hamiltonian in the dipole and rotating wave approximation is given by[18]

$$H=\omega_0 R_3+\sum_k \omega_k a_k^\dagger a_k+\sum_k (\beta_k a_k R_+ + \mathrm{H.c.})\,, \qquad (2.1)$$

where ω_0 is the absorption frequency of the atoms, ω_k, a_k, and $a_k^\dagger$ are the frequency, annihilation operator, and creation operator for radiation in the k mode which has wave vector $\vec{k}$ and the polarization $\vec{\epsilon}_k$, R_3 and $R_\pm$ are the collective atom operators, β_k is the coupling constant given by

$$\beta_k = ie\omega_0(2\pi/\omega_k V)^{1/2}(\vec{\epsilon}_k\cdot\vec{x}_{ul})e^{i\vec{k}\cdot\vec{x}_0}\,, \qquad (2.2)$$

V is the volume used for normalization, $e\vec{x}_{ul}$ is the transition dipole moment between the upper state u and the lower state l of any atom, $\vec{x}_0$ is the mean position of the atoms, and H.c. stands for Hermitian conjugate.

In the boson formalism,[19] we associate a boson mode to the upper level and another to the lower level. If the boson annihilation operators for these modes are b and c, respectively, the Dicke operator's, R_+, R_-, and R_3, and the Dicke state[1] $|r,m\rangle$ correspond, respectively, to the boson operators $b^\dagger c$, $bc^\dagger$, and $\frac{1}{2}(b^\dagger b - c^\dagger c)$, and the boson state $|n_b, n_c\rangle$, where $n_b = r+m$ and $n_c = r-m$. The Hamiltonian then takes the form

$$H = \tfrac{1}{2}\omega_0(b^\dagger b - c^\dagger c) + \sum_k \omega_k a_k^\dagger a_k + \sum_k (\beta_k a_k b^\dagger c + \text{H.c.}) . \tag{2.3}$$

It is difficult to solve the nonlinear Heisenberg equations of motion obtained from this trilinear Hamiltonian. However, in special cases, if one of the operators is replaced by a c number, the resulting bilinear Hamiltonian gives linear equations of motion which can be exactly solved. Some special cases[20,21] are those of (i) highly excited assembly of atoms, for which $m \simeq r \gg 1$ and hence $n_b \gg n_c$ and the b mode can be treated classically, (ii) weakly excited assembly of atoms, for which $-m \simeq r \gg 1$ and hence $n_c \gg n_b$ and the c mode can be treated classically, and (iii) superradiant assembly of atoms, for which $m \simeq 0$ and hence $n_b \sim n_c \gg 1$ and both b and c mode can be treated classically. We shall treat these cases one by one in Secs. III and IV.

III. SOLUTIONS OF EQUATIONS OF MOTION AND LINEWIDTHS, LAMB SHIFTS, AND RADIATION RATES FOR SPONTANEOUS EMISSION

Case A: Highly excited assembly of atoms ($m \simeq r \gg 1$)

We saw for this case the b mode can be treated classically. The term $\frac{1}{2}\omega_0 b^\dagger b$ in H in Eq. (2.3) suggests that b should be replaced by $\xi \exp(-\frac{1}{2}i\omega_0 t)$, where ξ is a c number, satisfying $|\xi|^2 = n_b = r+m$. The resulting bilinear Hamiltonian,[22]

$$H = -\tfrac{1}{2}\omega_0 c^\dagger c + \sum_k \omega_k a_k^\dagger a_k + \sum_k [\beta_k \xi^* \exp(\tfrac{1}{2}i\omega_0 t) a_k c + \text{H.c.}] , \tag{3.1}$$

resembles with that for parametric amplification.[23] We note that

$$\left[H, c^\dagger c - \sum_k a_k^\dagger a_k\right] = 0 . \tag{3.2}$$

We shall use this fact later in this section to find the spectral width of the emitted radiation.

If we define new operators $\hat{a}_k$ and $\hat{c}$ by

$$\hat{a}_k \equiv a_k \exp(i\omega_k t) , \quad \hat{c} \equiv c \exp(-\tfrac{1}{2}i\omega_0 t) , \tag{3.3}$$

Hamiltonian (3.1) leads to the equations of motion

$$i\dot{\hat{a}}_k = \beta_k^* \xi \exp(i\Delta_k t)\hat{c}^\dagger , \qquad i\dot{\hat{c}}^\dagger = -\sum_k \beta_k \xi^* \exp(-i\Delta_k t)\hat{a}_k , \tag{3.4}$$

where

$$\Delta_k \equiv \omega_k - \omega_0 . \tag{3.5}$$

Coupled equations (3.4) can be solved by using the Laplace transforms

$$\begin{bmatrix} A_k(p) \\ C^\dagger(p) \end{bmatrix} = \int_0^\infty dt\, e^{-pt} \begin{bmatrix} \hat{a}_k \\ \hat{c}^\dagger \end{bmatrix} .$$

If we find Laplace transforms of Eq. (3.4) and eliminate $A_k(p)$ between them, we get

$$\left(p - \sum_k \frac{|\beta_k \xi|^2}{p + i\Delta_k}\right) C^\dagger(p) = c_s^\dagger + \sum_k \frac{i\beta_k \xi^*}{p + i\Delta_k} a_{ks} , \tag{3.6}$$

where c_s and a_{ks} are the values of $\hat{c}$ and $\hat{a}_k$ at $t=0$, i.e., they are Schrödinger representation operators, which coincide with the Heisenberg representation operators at $t=0$.

We now take the Weisskopf-Wigner-type approximation[24]

$$\sum_k (p + i\Delta_k)^{-1} |\beta_k \xi|^2 = \mu_A + i\epsilon_A , \tag{3.7}$$

where μ_A and ϵ_A are independent of p and give the damping constant and Lamb shift, respectively. For a single excited atom, the Weisskopf-Wigner theory takes[24]

$$\sum_k (p + i\Delta_k)^{-1} |\beta_k|^2 = \mu + i\epsilon , \tag{3.8}$$

and leads to the frequency of emission $\omega_0 + \epsilon$. Hence, the damping constant and Lamb shift for a highly excited assembly of atoms is $|\xi|^2 = r+m$ times that for a single excited atom.

On using (3.7), Eq. (3.6) leads to

$$\hat{c}^\dagger = c_s^\dagger \exp(\mu_A t + i\epsilon_A t) + \text{terms in } a_{ks} . \tag{3.9}$$

Equations (3.3) and (3.4) then give

$$a_k = a_{ks} \exp(-i\omega_k t) + \gamma_{kA} c_s^\dagger + \sum_l \lambda_{kl} a_{ls} , \tag{3.10}$$

where

$$\gamma_{kA} = \beta_k^* \xi \, \frac{\exp(-i\omega_k t) - \exp(\mu_A t - i\Omega_A t)}{\omega_k - \Omega_A - i\mu_A} , \tag{3.11}$$

$$\lambda_{kl} = \sum_l \frac{\beta_k^* \beta_l |\xi|^2}{\omega_l - \Omega_A - i\mu_A} \left(\frac{\exp(-i\omega_k t) - \exp(\mu_A t - i\Omega_A t)}{\omega_k - \Omega_A - i\mu_A} - \frac{\exp(-i\omega_k t) - \exp(-i\omega_l t)}{\omega_k - \omega_l} \right) , \tag{3.12}$$

$$\Omega_A \equiv \omega_0 - \epsilon_A . \tag{3.13}$$

To study spontaneous emission from a highly excited assembly of atoms in the Dicke state $|r,m\rangle$, we note that initially the occupancy in the c mode is $r-m$ and no photons are present, and hence we can write the initial state as $|i\rangle = |r-m, \text{vac}\rangle$. The number of photons in the k mode at time t is then

$$n_k(t) = \langle i|a_k^\dagger a_k|i\rangle = |\gamma_{kA}|^2 = \frac{|\beta_k|^2(r+m)(r-m+1)}{(\omega_k-\Omega_A)^2+\mu_A^2}[1+\exp(2\mu_A t) - 2\exp(\mu_A t)\cos(\omega_k - \Omega_A)t] . \tag{3.14}$$

This shows that the emitted radiation has a Lorentzian line shape with the mean frequency $\Omega_A = \omega_0 - \epsilon_A$ and half-intensity-half-width μ_A. Comparison with the emission frequency $\Omega = \omega_0 + \epsilon$ for a single atom tells that the directions of frequency shifts in these two cases are different. The total number of photons is

$$\sum_k n_k = \frac{2}{3\mu_A} e^2\omega_0^2\Omega_A |\vec{x}_{ul}|^2 \times (r+m)(r-m+1)(e^{2\mu_A t}-1) . \tag{3.15}$$

The rate of emission $(d/dt)\sum_k n_k$, obviously increases exponentially with time.

We can find μ_A by noting that Eqs. (3.3) and (3.9) give the occupancy in c mode at time t as

$$n_c(t) = \langle i|c^\dagger c|i\rangle = (r-m+1)e^{2\mu_A t} - 1 . \tag{3.16}$$

Using (3.2), (3.15), and (3.16), we then find that

$$\mu_A = \tfrac{2}{3} e^2\omega_0^2\Omega_A|\vec{x}_{ul}|^2(r+m) , \tag{3.17}$$

$$\sum_k n_k = (r-m+1)(e^{2\mu_A t}-1) . \tag{3.18}$$

Case B: Weakly excited assembly of atoms ($-m \simeq r \gg 1$)

For this case, the c mode can be treated classically. Term $-\frac{1}{2}\omega_0 c^\dagger c$ in H in Eq. (2.3) tells that c should be replaced by $\eta\exp(\frac{1}{2}i\omega_0 t)$, where η is a c number satisfying $|\eta|^2 = n_c = r-m$. H takes the form,

$$H = \tfrac{1}{2}\omega_0 b^\dagger b + \sum_k \omega_k a_k^\dagger a_k + \sum_k [\beta_k \eta \exp(\tfrac{1}{2}i\omega_0 t) a_k b^\dagger + \text{H.c.}] . \tag{3.19}$$

On proceeding in a way similar to that of case A and using the Weisskopf-Wigner-type approximation,

$$\sum_k (p+i\Delta_k)^{-1}|\beta_k\eta|^2 = \mu_B + i\epsilon_B = (r-m)(\mu+i\epsilon) , \tag{3.20}$$

it is found that

$$\hat{b} = b_s \exp(-\mu_B t - i\epsilon_B t) + \text{terms in } a_{ks} , \tag{3.21}$$

$$a_k = a_{ks}\exp(-i\omega_k t) + \gamma_{kB} b_s + \sum_l \lambda_{kl} a_{ls} , \tag{3.22}$$

$$\gamma_{kB} = \beta_k^* \eta \frac{\exp(-\mu_B t - i\Omega_B t) - \exp(-i\omega_k t)}{\omega_k - \Omega_B + i\mu_B} , \tag{3.23}$$

$$\lambda_{kl} = -\frac{\beta_k^*\beta_l|\eta|^2}{\omega_l - \Omega_B + i\mu_B}\left(\frac{\exp(-i\omega_k t) - \exp(-\mu_B t - i\Omega_B t)}{\omega_k - \Omega_B - i\mu_B} - \frac{\exp(-i\omega_k t) - \exp(-i\omega_l t)}{\omega_k - \omega_l}\right) . \tag{3.24}$$

$$\Omega_B \equiv \omega_0 + \epsilon_B . \tag{3.25}$$

These give

$$n_k(t) = \frac{|\beta_k|^2(r-m)(r+m)}{(\omega_k-\Omega_B)^2+\mu_B^2} \times [1+e^{-2\mu_B t} - 2e^{-\mu_B t}\cos(\omega_k - \Omega_B)t] , \tag{3.26}$$

$$\sum_k n_k(t) = (r+m)(1-e^{-\mu_B t}) . \tag{3.27}$$

Thus we see that (i) the line shape is Lorentzian, (ii) the linewidth and Lamb shifts are $r-m$ times that for a single atom, (iii) the Lamb shift is in the same direction as that for a single excited atom (i.e., opposite to that for case A), and (iv) the rate of emission $(d/dt)\sum_k n_k(t)$ decreases exponentially with time.

Since, in cases A and B, magnitude and direction of the Lamb shift depends on m, it appears that in a pulse of radiation emitted by an assembly of atoms the frequency will change as m decreases, i.e., the radiation will be chirped. Chirping in emitted radiation has been obtained neoclassically by Stroud, Eberly, Lama, and Mandel. Their results agree[25] with our results, since $|m| \simeq r$.

Case C: Superradiant assembly of atoms ($r \gg m$)

In this case both b and c modes can be treated classically. Replacing b and c by $\xi\exp(-\frac{1}{2}i\omega_0 t)$ and $\eta\exp(\frac{1}{2}i\omega_0 t)$ respectively, we get the Hamiltonian

$$H = \sum_k \omega_k a_k^\dagger a_k + \sum_k (\beta_k \xi^* \eta e^{i\omega_0 t} a_k + \text{H.c.}) . \tag{3.28}$$

This gives the exact solution

$$a_k = a_{ks} e^{-i\omega_k t} + \gamma_{kC} . \tag{3.29}$$

$$\gamma_{kC} = \beta_k^* \xi \eta^* (\omega_k - \omega_0)^{-1}(e^{-i\omega_k t} - e^{-i\omega_0 t}) . \tag{3.30}$$

which leads to

$$n_k(t) = 8\pi e^2\omega_0^2\omega_k^{-1}V^{-1}|\vec{\epsilon}\cdot\vec{X}_{ul}|^2(r^2-m^2)$$
$$\times(\omega_k-\omega_0)^{-2}\sin^2\tfrac{1}{2}(\omega_k-\omega_0)t\,, \qquad (3.31)$$

$$\sum_k n_k(t) = \tfrac{4}{3}\omega_0^3 e^2|\vec{X}_{ul}|^2(r^2-m^2)t\,. \qquad (3.32)$$

We note that the radiation rate $(d/dt)\sum_k n_k(t)$ is constant in time and proportional to square of the number of atoms, as is expected for superradiant emission.

IV. COHERENCE PROPERTIES OF SPONTANEOUSLY EMITTED RADIATION

Our solutions (3.9), (3.10), (3.21), (3.22), and (3.29) of the equations of motion contain complete information about the system. We can use these to find the reduced density operator for any mode of radiation or for the complete radiation field. Our procedure is to first find the normally ordered characteristic function[26,27]

$$\chi_{Nk}(z,t) = \mathrm{Tr}[\rho(0)\exp(za_k^\dagger)\exp(-z^*a_k)], \qquad (4.1)$$

where $\rho(0)$ is the initial density operator of the system. The reduced density operator for the k mode, ρ_k^R, is then given by the Weyl representation by[26,27]

$$\rho_k^R = \pi^{-2}\int d^2z\,\chi_{Nk}(-z,t)e^{-|z|^2}e^{za_{ks}^\dagger}e^{-z^*a_{ks}}\,, \qquad (4.2)$$

and in the diagonal representation by[27,28]

$$\rho_k^R = \int d^2\alpha\; P_k(\alpha,t)|\alpha\rangle_{kk}\langle\alpha|\,, \qquad (4.3)$$

where[27]

$$P_k(\alpha,t) = \pi^{-2}\int d^2z\,\chi_{Nk}(z,t)\exp(z\alpha^*-z^*\alpha) \qquad (4.4)$$

and $|\alpha\rangle_k$ is the coherent state for the k mode defined by[28]

$$a_{ks}|\alpha\rangle_k = \alpha|\alpha\rangle_k\,. \qquad (4.5)$$

The occupation-number-space matrix elements of the density operator ${}_k\langle m|\rho_k^R|n\rangle_k$ can be obtained directly from Eq. (4.3). The coherence functions,

$$\Gamma_k^{(m,n)} \equiv \mathrm{Tr}[\rho(0)a_k^{\dagger m}a_k^n] = \mathrm{Tr}(\rho_k^R a_{ks}^{\dagger m}a_{ks}^n)\,. \qquad (4.6)$$

can be obtained either by using the identity

$$\Gamma_k^{(m,n)} = (\partial/\partial z)^m(-\partial/\partial z^*)^n\chi_{Nk}(z,t)|_{z=z^*=0}\,, \qquad (4.7)$$

or by using (4.3) and (4.4).

To find ρ^R, the density operator for the complete radiation, we first obtain the complete normally ordered characteristic function,

$$\chi_N(\{z_k\},t) = \mathrm{Tr}\left[\rho(0)\exp\left(\sum_k z_k a_k^\dagger\right)\exp\left(-\sum_k z_k^* a_k\right)\right]\,. \qquad (4.8)$$

ρ^R is then given in the Weyl representation by

$$\rho^R = \prod_k\left(\pi^{-2}\int d^2z_k\,e^{-|z_k|^2}e^{za_{ks}^\dagger}e^{-z^*a_{ks}}\right)\chi_{Nk}(\{-z_k\},t)\,, \qquad (4.9)$$

and in the diagonal representation by

$$\rho^R = \int\prod_k d^2\alpha_k P(\{\alpha_k\},t)|\{\alpha_k\}\rangle\langle\{\alpha_k\}|\,, \qquad (4.10)$$

where the composite weight function $P(\{\alpha_k\},t)$ and the coherent state $|\{\alpha_k\}\rangle$ are given by

$$P(\{\alpha_k\},t) = \prod_k\left(\pi^{-2}\int d^2z_k\exp(-z_k\alpha_k^*+z_k^*\alpha_k)\right)\chi_N(\{z_k\},t)\,,$$
$$a_{ls}|\{\alpha_k\}\rangle = \alpha_l|\{\alpha_k\}\rangle\,. \qquad (4.11)$$

Case A: Highly excited assembly of atoms ($m \simeq r \gg 1$)

Initially, if $m=r-R$, we have R particles in the c-mode and no photons, present, then the initial state can be written as $|i\rangle = |R,\mathrm{vac}\rangle$. Equations (4.1) and (3.10) and the identity,

$$e^Ae^B = e^Be^Ae^{[A,B]} \qquad (4.12)$$

for operators A and B satisfying $[A,[A,B]] = [B,[A,B]] = 0$, give

$$\chi_{Nk}(z,t) = \exp(-|z\gamma_{kA}|^2)L_R(|z\gamma_{kA}|^2)\,, \qquad (4.13)$$

where

$$L_R(x) = \sum_n [n!^2(R-n)!]^{-1}R!(-x)^n \qquad (4.14)$$

is the Laguerre polynomial.

On writing $\chi_{Nk}(z,t)$ in the form

$$\chi_{Nk}(z,t) = (R!)^{-1}|\gamma_{kA}|^{-2R}(-1)^R\left(\frac{\partial}{\partial z}\right)^R\left(\frac{-\partial}{\partial z^*}\right)^R e^{-|z\gamma_{kA}|^2}\,, \qquad (4.15)$$

Equation (4.4) gives

$$P_k(\alpha,t) = (\pi R!)^{-1}|\gamma_{kA}|^{-2(R+1)}|\alpha|^{2R}\exp(-|\alpha/\gamma_{kA}|^2)\,. \qquad (4.16)$$

We note that the radiation is chaotic if $R=0$, i.e., if all atoms are initially excited. For other cases, the light is not exactly chaotic.

Occupation-number-space matrix elements and coherence functions are seen to be

$$_k\langle m|\rho_k^R|n\rangle_k=\delta_{mn}(R!n!)^{-1}(n+R)!|\gamma_{kA}|^n \times(1+|\gamma_{kA}|^2)^{-n-R-1}, \quad (4.17)$$

$$\Gamma_k^{(m,n)}=\delta_{mn}(R!)^{-1}(R+n)!|\gamma_{kA}|^{2n}. \quad (4.18)$$

The normalized variance of intensity of radiation in this mode is

$$(\Gamma_k^{(1,1)})^{-2}[\Gamma_k^{(2,2)}-(\Gamma_k^{(1,1)})^2]=1+R. \quad (4.19)$$

This is greater than the value for the chaotic light. Thus, if an assembly of excited atoms radiate, the radiation is chaotic initially, but as soon as m decreases the spontaneously emitted radiation no longer remains chaotic and the fluctuations increase.

For the reduced-density operator of the complete radiation, we get

$$\chi_N(\{z_k\},t)=\int d^2z\,\delta^2\Big(z-\sum_k z_k\gamma_{kA}^*\Big)e^{-|z|^2}L_R(|z|^2), \quad (4.20)$$

$$P(\{\alpha_k\},t)=(\pi R!)^{-1}\int d^2\beta|\beta|^{2R}e^{-|\beta|^2} \times\prod_k\delta^2(\alpha_k-\beta\gamma_{kA}). \quad (4.21)$$

Appearance of δ functions in Eq. (4.21) show that the modes are completely coupled in the sense that the intermode correlations are completely determined from the correlation in a single mode, i.e.,

$$\mathrm{Tr}\Big(\rho(0)\prod_k a_k^{\dagger m_k}a_k^{n_k}\Big)=\prod_k\left[\left(\frac{\gamma_{kA}^*}{\gamma_{1A}^*}\right)^{m_k}\left(\frac{\gamma_{kA}}{\gamma_{1A}}\right)^{n_k}\right] \times\mathrm{Tr}[\rho(0)a_1^{\dagger\,\Sigma_k n_k}a_1^{\Sigma_k n_k}]. \quad (4.22)$$

Case B: Weakly excited assembly of atoms ($-m\simeq r\gg 1$)

For this case, if $m=-r+R$ initially we have R particles in the b mode and hence $\rho(0)=|R,\mathrm{vac}\rangle\langle R,\mathrm{vac}|$. Inthis case, Eqs. (3.22), (4.1), (4.2), and (4.6) give

$$\chi_{Nk}(z,t)=L_R(|z\gamma_{kB}|^2), \quad (4.23)$$

$$_k\langle m|\rho_k^R|n\rangle_k =\delta_{mn}[n!(R-n)!]^{-1}R!|\gamma_{kB}|^{2n}(1-|\gamma_{kB}|^2)^{R-n}, \quad (4.24)$$

$$\Gamma_k^{(m,n)}=\delta_{mn}[R!/(R-n)!]|\gamma_{kB}|^{2n}. \quad (4.25)$$

We do not get a nonsingular compact $P_k(\alpha)$ in this case as the number of photons is limited to R (the initial occupancy of the b mode) and hence the field is a quantum one. This is also indicated by a negative value of the variance of intensity, we have

$$\Gamma_k^{(2,2)}-(\Gamma_k^{(1,1)})^2=-|\gamma_{kB}|^4, \quad (4.26)$$

The complete normally ordered characteristic function is given by

$$\chi_N(\{z_k\},t)=L_R\left(\left|\sum_k z_k\gamma_{kB}\right|^2\right). \quad (4.27)$$

A nonsingular compact $P(\{\alpha_k\},t)$ does not exist. ρ^R is, however, known in the Weyl representation and can be used to study any intermode correlation.

An important and interesting case is that of an assembly of atom in thermal equilibrium. If the temperature is so low and the number of atoms is so large that $r\omega_0\gg kT$, the reduced atomic density operator ρ^A gives

$$\langle r,m|\rho^A|r,m\rangle=(1-e^{-x})e^{-mx},\quad x=\omega_0/kT. \quad (4.28)$$

Equations (4.23), (4.24), and (4.25) then reduce to

$$\chi_{Nk}(z,t)=\exp(-n_k|z|^2),$$
$$_k\langle m|\rho_k^R|n\rangle_k=\delta_{mn}(1+n_k)^{-n-1}n_k^n, \quad (4.29)$$
$$\Gamma_k^{(m,n)}=\delta_{mn}n!\,n_k^m,$$

where

$$n_k=(e^x-1)^{-1}|\gamma_{kB}|^2. \quad (4.30)$$

These equations show that reduced nature of the emitted light is exactly chaotic with n_k photons.

For the composite coherence functions, in this case, we obtain

$$\chi_N(\{z_k\},t)=\int d^2z\,\delta^2\Big(z-\sum_k z_k\gamma_{kB}\Big)\exp[-(e^x-1)^{-1}|z|^2], \quad (4.31)$$

which leads to

$$P(\{\alpha_k\},t)=\int d^2\alpha\,\pi^{-1}(e^x-1)\exp[-(e^x-1)|\alpha|^2] \times\prod_k\delta^2(\alpha_k-\gamma_{kB}\alpha). \quad (4.32)$$

Appearance of the δ functions gives the complete correlation of modes.

Case C: Superradiant assembly of atoms ($r\gg m$)

For this case, the results are

$$\chi_{Nk}(z,t)=\exp(z\gamma_{kC}^*-z^*\gamma_{kC}); \quad (4.33)$$

$$P_k(\alpha,t)=\delta^2(\alpha-\gamma_{kC}). \quad (4.34)$$

χ_N and P are products of all χ_{Nk} and P_k. Since γ_{kC} [given by Eq. (3.30)] is proportional to $\xi\eta^*$, we see that if ξ and η have definite phases the superradiant states emit coherent radiation. If, however, the phases of ξ and η are random, the above equations modify to

$$\chi_{Nk}(z,t)=J_0(2|z\gamma_{kC}|), \quad (4.35)$$

$$P_k(\alpha,t)=(2\pi|\gamma_{kC}|)^{-1}\delta(|\alpha|-|\gamma_{kC}|). \quad (4.36)$$

The emitted light in this case is not exactly coherent but only amplitude coherent.[29] It has a diagonal density operator. Bialynicka-Birula[8] also showed that the radiation emitted spontaneously from a superradiant assembly of atoms is amplitude coherent but, as the author considered only a single mode of radiation, his amplitude is proportional to t and hence the number of photons to t^2. Our radiation field has total number of photons proportional to t given by Eq. (3.32).

V. INDUCED EMISSION AND SCATTERING

In Sec. IV, we took zero for the initial number of photons in each mode. We now consider the case where some photons are present initially and study induced emission and scattering. For the sake of simplicity, we assume that only one mode (the incident mode i) is initially populated with n_0 photons and that the initial density operator is diagonal. For cases A and B, Eqs. (3.10) and (3.22) give

$$n_k(t)=n_0\delta_{ki}+n_k^{sp}(t)+2\mathrm{Re}\lambda_{ki}\delta_{ki}n_0+|\lambda_{ki}|^2 n_0\,, \quad (5.1)$$

where $n_k^{sp}(t)=|\gamma_{kA}|^2(r+m+1)$ for case A and $|\gamma_{kB}|^2(r-m)$ for case B. Here the first term gives the initial photons and the second term gives spontaneous emission. The third and fourth terms which are proportional to n_0 represent, respectively, the pure induced emission and scattering. For case C, Eq. (3.29) leads to $n_k(t)=n_0\delta_{ki}+|\alpha_{kC}|^2$. In this case we see that no induced emission or scattering takes place. The former is explained by the fact that the population in the upper and lower levels is the same and so induced emission is anulled by equal absorption. The latter is in agreement with similar results of Ajai and Prakash.[30]

If we consider the i mode only, Eq. (5.1) gives

$$n_i(t)=n_0+2\mathrm{Re}\lambda_{ii}n_0+|\lambda_{ii}|^2 n_0\,. \quad (5.2)$$

While the second term gives "pure" induced emission, the third term may be called forward scattering and the total effect may be called "net" induced emission. We shall first discuss below this total effect for the cases A and B. Let us consider the incident frequency ω_i to be well within the narrow band of frequencies emitted by the atoms. For case A, it means $|\omega_i-\Omega_A|\ll\mu_A$. For small values of t, such that $t \ll$ the relaxation time μ_A^{-1}, $\lambda_{ii}=\frac{1}{2}|\beta_i\xi|^2t^2=\frac{1}{2}(t/\tau_A)^2$; the characteristic time τ_A is given by

$$\tau_A^2=|\beta_i\xi|^{-2}=[2\pi\omega_0 e^2|\vec{\epsilon}_i\cdot\vec{x}_{ul}|^2(r+m)/V]^{-1}\,, \quad (5.3)$$

and is inversely proportional to square root of the number density of atoms. Further, since

$$\mu_A^2\tau_A^2=(3\pi)^{-1}V\omega_0^3(\omega_A/\omega_0)|\vec{\epsilon}\cdot\vec{x}_{ul}|^{-2}|\vec{x}_{ul}|^2\ll 1 \quad (5.4)$$

(because $V\omega_0^3\ll 1$ in view of our assumption of a small volume V and $\mu_A\ll\omega_0$), we find that $\tau_A \ll$ the relaxation time μ_A^{-1}. For small time $t\ll\mu_A^{-1}$, we can discuss separately the cases $t\ll\tau_A$ and $\tau_A\ll t\ll\mu_A^{-1}$. The net induced emission is

$$(2\mathrm{Re}\lambda_{ii}+|\lambda_{ii}|^2)n_0=\tfrac{1}{4}\tau_A^{-4}t^2(t^2+4\tau_A^2)n_0\,. \quad (5.5)$$

For $t\ll\tau_A$, the pure induced emission predominates and it is proportional to t^2 and to τ_A^{-2} (i.e., to the number density of atoms). For $\tau_A\ll t\ll\mu_A^{-1}$, the forward scattering predominates and it is proportional to t^4 and to τ_A^{-4} (i.e., to the square of the number density of atoms). For large times, such that $t\gg\mu_A^{-1}$, our theory shall not give correct results as the assumption of high excitation will not hold for all times.

For case B also, we can define a characteristic time τ_B given by

$$\tau_B^2=|\beta_i\eta|^{-2}=[2\pi\omega_0 e^2|\vec{\epsilon}_i\cdot\vec{x}_{ul}|^2(r-m)/V]^{-1} \quad (5.6)$$

and show that $\mu_B\tau_B\ll 1$. The "net" induced emission is then

$$(2\mathrm{Re}\lambda_{ii}+|\lambda_{ii}|^2)n_0=\tfrac{1}{4}\tau_B^{-4}t^2(t^2-4\tau_B^2)\,. \quad (5.7)$$

For $t<2\tau_B$, we note that the "net" effect is negative and is due to a large absorption of incident photon by the unexcited atoms. For $t>2\tau_B$, however, the "net" effect is positive and is predominated by forward scattering. The absorption is proportional to t^2 and to τ_B^{-2} (i.e., to the number density of atoms) the forward scattering is proportional to t^4 and to τ_B^{-4} (i.e., to the square of the number density of atoms).

Let us now consider the photon scattered into other modes also. For scattering at an angle θ, we obtain using the usual procedure, the differential scattering cross section

$$\frac{d\sigma}{d\Omega}=e^4|\vec{\epsilon}_i\cdot\vec{x}_{ul}|^2|\vec{x}_{ul}|^2(r+m)^2[(\omega_i-\Omega_A)^2+\mu_A^2]^{-1}\times\omega_0^4(e^{2\mu_A t}+1)\cos^2\theta\,, \quad (5.8)$$

$$\frac{d\sigma}{d\Omega}=e^4|\vec{\epsilon}_i\cdot\vec{x}_{ul}|^2|\vec{x}_{ul}|^2(r-m)^2[(\omega_i-\Omega_B)^2+\mu_B^2]^{-1}\times\omega_0^4(e^{-2\mu_B t}+1)\cos^2\theta \quad (5.9)$$

for the cases A and B. We note that the cross sections are proportional to the square of the total number of atoms. The resonant increase is given by the factors $[(\omega_i - \Omega_{A,B})^2 + \mu_{A,B}^2]^{-1}$. For the case A the cross section increases exponentially with time, while for B, the cross section decreases and stabilizes to one half of its initial value.

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